Constant Angle Surfaces in $\mathbb{S}^3(1) \times \mathbb{R}$*

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Abstract

In this article we study surfaces in $\mathbb{S}^3(1) \times \mathbb{R}$ for which the $\mathbb{R}$-direction makes a constant angle with the normal plane. We give a complete classification for such surfaces with parallel mean curvature vector.

1 Introduction

In recent years, there has been done some research about surfaces in a 3-dimensional Riemannian product of a surface $\mathbb{M}^2(c) \times \mathbb{R}$ ([1, 9, 11, 14], etc.), where $\mathbb{M}^2(c)$ is the simply-connected 2-dimensional space form of constant curvature c, in particular $\mathbb{M}^2(c) = \mathbb{R}^2,\ \mathbb{H}^2,\ \mathbb{S}^2$ for $c = 0,\ -1,\ 1$ respectively.

Recently, constant angle surfaces were studied in product spaces $\mathbb{M}^2(c) \times \mathbb{R}$ (see [3, 4, 5, 6, 12, 13]), where the angle was considered between the unit normal of the surface M and the tangent direction to $\mathbb{R}$. For example, F. Dillen et al. gave the complete classification for constant angle surfaces in $\mathbb{S}^2 \times \mathbb{R}$ in [4]. The problem of constant angle surfaces was also investigated in the 3-dimensional Heisenberg group (see [8]) and in Minkowski space (see [10]). In [15], R. Tojeiro gave a complete description of all hypersurfaces in the product spaces $\mathbb{S}^n \times \mathbb{R}$ and $\mathbb{H}^n \times \mathbb{R}$ that have flat normal bundle when regarded as submanifolds with codimension two of the underlying flat spaces $\mathbb{R}^{n+2} \supset \mathbb{S}^n \times \mathbb{R}$ and $\mathbb{L}^{n+2} \supset \mathbb{H}^n \times \mathbb{R}$. In [7], helix submanifolds in Euclidean space were studied by solving the Eikonal equation. The applications of constant angle surfaces in the theory of

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liquid crystals and of layered fluids were considered by P. Cermelli and A. J. Di Scala in [2].

In this article we study surfaces in $\mathbb{S}^3(1) \times \mathbb{R}$ for which the $\mathbb{R}$-direction makes a constant angle with the normal plane. In Section 2, we first review some basic equations for constant angle surfaces in $\mathbb{S}^3(1) \times \mathbb{R}$. In Section 3, we will prove that the constant angle surfaces in $\mathbb{S}^3(1) \times \mathbb{R}$ with parallel mean curvature vector are minimal (see Theorem 1). In Section 4, we will give a complete classification for minimal constant angle surfaces in $\mathbb{S}^3(1) \times \mathbb{R}$ (see Theorem 3).

2 Preliminaries

Let $\widetilde{M} = \mathbb{S}^3(1) \times \mathbb{R}$ be the Riemannian product of $\mathbb{S}^3(1)$ and $\mathbb{R}$ with the standard metric $\langle , \rangle$ and the Levi-Civita connection $\widetilde{\nabla}$. We denote by t the (global) coordinate on $\mathbb{R}$ and hence $\partial_t = \frac{\partial}{\partial t}$ is the unit vector field in the tangent bundle $T(\mathbb{S}^3(1) \times \mathbb{R})$ that is tangent to the $\mathbb{R}$-direction.

For $p \in \mathbb{S}^3(1) \times \mathbb{R}$, the Riemann-Christoffel curvature tensor $\tilde{R}$ of $\mathbb{S}^3(1) \times \mathbb{R}$ is given by

$$\langle \tilde{R}(X,Y)Z,W\rangle = \langle X_{\mathbb{S}^3(1)}, W_{\mathbb{S}^3(1)}\rangle\langle Y_{\mathbb{S}^3(1)}, Z_{\mathbb{S}^3(1)}\rangle - \langle X_{\mathbb{S}^3(1)}, Z_{\mathbb{S}^3(1)}\rangle\langle Y_{\mathbb{S}^3(1)}, W_{\mathbb{S}^3(1)}\rangle,$$

where $\tilde{R}(X,Y) = [\widetilde{\nabla}_X, \widetilde{\nabla}_Y] - \widetilde{\nabla}_{[X,Y]}; X,Y,Z,W \in T_p(\mathbb{S}^3(1) \times \mathbb{R})$ and $X_{\mathbb{S}^3(1)} = X - \langle X, \partial_t\rangle\partial_t$ is the projection of X to the tangent space of $\mathbb{S}^3(1)$.

Now consider a surface M in $\mathbb{S}^3(1) \times \mathbb{R}$. We can decompose ∂_t as

$$\partial_t = \sin\theta T + \cos\theta\xi, \tag{2.1}$$

where θ is the angle between ξ and ∂_t, ξ is a unit normal vector to M and T is a unit tangent vector to M.

For a constant angle surface M in $\mathbb{S}^3(1) \times \mathbb{R}$, we mean a surface for which the angle function θ is constant on M. There are two trivial cases, $\theta = 0$ and $\theta = \frac{\pi}{2}$. The condition $\theta = 0$ means that ∂_t is always normal, so we get a surface $\Sigma^2 \times \{t_0\}$, where Σ^2 is a surface in $\mathbb{S}^3(1)$. In the second case, ∂_t is always tangent. This corresponds to the Riemannian product of a curve in $\mathbb{S}^3(1)$ and $\mathbb{R}$.

From now on, in the rest of this paper, we only consider the constant angle surface M with constant angle $\theta \in (0, \frac{\pi}{2})$. We extend $\{T, \xi\}$ to an orthonormal frame $\{T, Q, \xi, \eta\}$ on $\mathbb{S}^3(1) \times \mathbb{R}$, where T, Q are tangent to M and ξ, η are normal to M. Since ∂_t is a parallel vector field in $\mathbb{S}^3(1) \times \mathbb{R}$, we can obtain from (2.1) that, for any $X \in TM$,

$$0 = \widetilde{\nabla}_X\partial_t = \sin\theta\nabla_X T + \sin\theta h(X,T) - \cos\theta A_\xi X + \cos\theta\nabla^\perp_X\xi, \tag{2.2}$$

where we use the formulas of Gauss and Weingarten, h is the second fundamental form of M, A_ξ is the shape operator associated to ξ, and $\nabla^\perp$ is the normal connection.

Comparing the tangent part and the normal part in (2.2), we have

$$\begin{cases} \nabla_X T = \cot\theta A_\xi X, \\ h(X,T) = -\cot\theta\nabla^\perp_X\xi. \end{cases} \tag{2.3}$$

From (2.3), we have

$$\langle A_\xi X, T\rangle = \langle A_\xi T, X\rangle = 0, \quad \forall X \in TM,$$

that is,

$$A_\xi T = 0.$$

Therefore, we can suppose the shape operators with respect to ξ and η are, respectively,

$$A_\xi = \begin{pmatrix} 0 & 0 \\ 0 & \lambda \end{pmatrix}, \quad A_\eta = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & \beta_3 \end{pmatrix}, \tag{2.4}$$

where λ, β_j $(j = 1,2,3)$ are smooth functions defined on the surface M.

From (2.3) and (2.4), we obtain that

$$\begin{cases} \nabla_T T = \nabla_T Q = 0, \\ \nabla_Q T = \lambda \cot\theta Q, \\ \nabla_Q Q = -\lambda \cot\theta T, \end{cases} \tag{2.5}$$

$$\begin{cases} h(T,T) = \beta_1 \eta, \\ h(T,Q) = \beta_2 \eta, \\ h(Q,Q) = \lambda\xi + \beta_3 \eta, \end{cases} \tag{2.6}$$

$$\begin{cases} \nabla^\perp_T \xi = -\tan\theta\, \beta_1 \eta, \\ \nabla^\perp_T \eta = \tan\theta\, \beta_1 \xi, \\ \nabla^\perp_Q \xi = -\tan\theta\, \beta_2 \eta, \\ \nabla^\perp_Q \eta = \tan\theta\, \beta_2 \xi. \end{cases} \tag{2.7}$$

Now we can take coordinates (x, y) on M with $\partial_x = \beta T$, $\partial_y = \alpha Q$ where β, α are positive functions. From (2.5) and the condition $[\partial_x, \partial_y] = 0$, we find that

$$\begin{aligned} \beta_y &= 0, \\ \alpha_x &= \alpha\beta\lambda\cot\theta. \end{aligned} \tag{2.8}$$

Equation (2.8) implies that, after a change of the x-coordinate, we can assume $\beta = 1$ and thus the metric takes the form

$$ds^2 = dx^2 + \alpha^2(x,y)dy^2.$$

The Gauss and Ricci equation are, respectively, given by

$$\begin{aligned} (\widetilde{R}(T,Q)T)^\top &= R(T,Q)T + A_{h(T,T)}Q - A_{h(Q,T)}T, \\ (\widetilde{R}(T,Q)\eta)^\perp &= R^\perp(T,Q)\eta + h(A_\eta T, Q) - h(A_\eta Q, T), \end{aligned}$$

where

$$\begin{aligned} \widetilde{R}(X,Y)Z =& \big(\langle Y,Z\rangle - \langle Y,\partial_t\rangle\langle Z,\partial_t\rangle\big)X - \big(\langle X,Z\rangle - \langle X,\partial_t\rangle\langle Z,\partial_t\rangle\big)Y \\ & - \big(\langle Y,Z\rangle\langle X,\partial_t\rangle - \langle X,Z\rangle\langle Y,\partial_t\rangle\big)\partial_t, \forall X,Y,Z \in T(\mathbb{S}^3(1)\times\mathbb{R}) \\ R^\perp(T,Q)\eta =& \big(\nabla^\perp_T\nabla^\perp_Q - \nabla^\perp_Q\nabla^\perp_T - \nabla^\perp_{[T,Q]}\big)\eta. \end{aligned}$$

The Codazzi equations are

$$(\widetilde{R}(T,Q)T)^{\perp} = (\nabla^{\perp}_{T}h)(Q,T) - (\nabla^{\perp}_{Q}h)(T,T),$$
$$(\widetilde{R}(T,Q)Q)^{\perp} = (\nabla^{\perp}_{T}h)(Q,Q) - (\nabla^{\perp}_{Q}h)(T,Q),$$

where $(\nabla^{\perp}_{X}h)(Y,Z) = \nabla^{\perp}_{X}\big(h(Y,Z)\big) - h(\nabla_X Y,Z) - h(Y,\nabla_X Z)$ for any $X,Y,Z \in TM$.

By a direct computation with (2.5)–(2.7), the equations of Gauss, Ricci and Codazzi yield

$$\lambda^2\cot^2\theta + \lambda_x\cot\theta + \cos^2\theta + \beta_1\beta_3 - \beta_2^2 = 0, \tag{2.9}$$

$$\frac{(\beta_2)_y}{\alpha} + \lambda\cot\theta\sec^2\theta\beta_1 - \lambda\cot\theta\beta_3 - (\beta_3)_x = 0, \tag{2.10}$$

$$\frac{(\beta_1)_y}{\alpha} - 2\lambda\cot\theta\beta_2 - (\beta_2)_x = 0. \tag{2.11}$$

In fact, the Codazzi equations imply all three equations above, while the Gauss and Ricci equations coincide with (2.9) and (2.11) respectively.

3 Constant angle surfaces with parallel mean curvature vector

In this section, we will discuss the constant angle surface M with parallel mean curvature vector in $\mathbb{S}^3(1)\times\mathbb{R}$. In fact, we have

Theorem 1. *If M is a constant angle surface in $\mathbb{S}^3(1)\times\mathbb{R}$ with parallel mean curvature vector $\vec{H}$, then $\vec{H}=0$, that is, M is a minimal surface in $\mathbb{S}^3(1)\times\mathbb{R}$.*

Proof. Since the mean curvature vector $\vec{H}$ of M is parallel, that is, $\nabla^{\perp}\vec{H}=0$, from (2.7), we have

$$\lambda_x = -(\beta_1+\beta_3)\beta_1\tan\theta, \tag{3.1}$$
$$(\beta_1)_x + (\beta_3)_x = \lambda\beta_1\tan\theta, \tag{3.2}$$

and

$$\lambda_y = -\alpha(\beta_1+\beta_3)\beta_2\tan\theta, \tag{3.3}$$
$$(\beta_1)_y + (\beta_3)_y = \alpha\lambda\beta_2\tan\theta. \tag{3.4}$$

From (2.9) and (3.1), we get

$$\beta_1^2 + \beta_2^2 = \cot^2\theta(\lambda^2+\sin^2\theta).$$

Thus we can set

$$\begin{cases} \beta_1 = \cot\theta\sqrt{\lambda^2+\sin^2\theta}\cos\gamma, \\ \beta_2 = \cot\theta\sqrt{\lambda^2+\sin^2\theta}\sin\gamma, \end{cases} \tag{3.5}$$

for some function γ on M.

Since $\beta_1^2+\beta_2^2 = \cot^2\theta(\lambda^2+\sin^2\theta) > 0$, taking the derivatives of (3.5), we obtain

$$(\beta_1)_x = -\beta_2\gamma_x + \frac{\lambda\lambda_x}{\beta_1^2+\beta_2^2}\beta_1\cot^2\theta, \tag{3.6}$$

$$(\beta_1)_y = -\beta_2\gamma_y + \frac{\lambda\lambda_y}{\beta_1^2+\beta_2^2}\beta_1\cot^2\theta, \tag{3.7}$$

$$(\beta_2)_x = \beta_1\gamma_x + \frac{\lambda\lambda_x}{\beta_1^2+\beta_2^2}\beta_2\cot^2\theta, \tag{3.8}$$

$$(\beta_2)_y = \beta_1\gamma_y + \frac{\lambda\lambda_y}{\beta_1^2+\beta_2^2}\beta_2\cot^2\theta. \tag{3.9}$$

Using (3.1)–(3.3), (3.6) and (3.9), from (2.10) we get

$$\frac{\beta_1}{\alpha}\gamma_y - \beta_2\gamma_x = 2\lambda\beta_3\cot\theta. \tag{3.10}$$

Using (3.1), (3.3), (3.7) and (3.8), from (2.11) we get

$$\frac{\beta_2}{\alpha}\gamma_y + \beta_1\gamma_x = -2\lambda\beta_2\cot\theta. \tag{3.11}$$

From (3.10) and (3.11) we have

$$\begin{cases} \gamma_x = \dfrac{-2\lambda\cot\theta}{\beta_1^2+\beta_2^2}\beta_2(\beta_1+\beta_3), \\ \gamma_y = \dfrac{2\alpha\lambda\cot\theta}{\beta_1^2+\beta_2^2}(\beta_1\beta_3-\beta_2^2). \end{cases} \tag{3.12}$$

Putting (3.12) into (3.6)–(3.9), from (3.1), (3.3) and (3.4), we have

$$\begin{aligned}
\lambda_{xy} &= -\tan\theta\Big[(\beta_1)_y(\beta_1+\beta_3)+\beta_1(\beta_1+\beta_3)_y\Big] \\
&= -\tan\theta\left\{(\beta_1+\beta_3)\Big[-\beta_2\gamma_y - \frac{\alpha\lambda\cot\theta}{\beta_1^2+\beta_2^2}\beta_1\beta_2(\beta_1+\beta_3)\Big] + \alpha\lambda\beta_1\beta_2\tan\theta\right\} \\
&= \tan\theta\left\{(\beta_1+\beta_3)\frac{\alpha\lambda\cot\theta}{\beta_1^2+\beta_2^2}\Big[2\beta_2(\beta_1\beta_3-\beta_2^2)+\beta_1\beta_2(\beta_1+\beta_3)\Big] - \alpha\lambda\beta_1\beta_2\tan\theta\right\} \\
&= \beta_2(\beta_1+\beta_3)\frac{\alpha\lambda}{\beta_1^2+\beta_2^2}(3\beta_1\beta_3-2\beta_2^2+\beta_1^2) - \alpha\lambda\beta_1\beta_2\tan^2\theta.
\end{aligned}$$

Similarly, we also obtain

$$\begin{aligned}
\lambda_{yx} &= -\tan\theta\Big[\alpha_x\beta_2(\beta_1+\beta_3)+\alpha\beta_2(\beta_1+\beta_3)_x+\alpha(\beta_2)_x(\beta_1+\beta_3)\Big] \\
&= -\tan\theta\Big[\alpha\lambda\cot\theta\beta_2(\beta_1+\beta_3)+\alpha\lambda\beta_1\beta_2\tan\theta - \alpha\frac{\lambda\cot\theta}{\beta_1^2+\beta_2^2}3\beta_1\beta_2(\beta_1+\beta_3)^2\Big] \\
&= \beta_2(\beta_1+\beta_3)\frac{\alpha\lambda}{\beta_1^2+\beta_2^2}(3\beta_1\beta_3+2\beta_1^2-\beta_1^2) - \alpha\lambda\beta_1\beta_2\tan^2\theta.
\end{aligned}$$

Since $\alpha > 0$, from the integrability condition $\lambda_{xy} = \lambda_{yx}$, we have

$$\lambda\beta_2(\beta_1 + \beta_3) = 0. \tag{3.13}$$

We claim that $\lambda(p) = 0$ for any $p \in M$. Then from (3.1) and (3.3) we get $\beta_1 + \beta_3 = 0$ since β_1 and β_2 cannot be zero simultaneously. Hence M is minimal in $\mathbb{S}^3(1) \times \mathbb{R}$.

To prove the claim, we discuss the equation (3.13) in two cases.

Case 1. $\beta_2 \neq 0$ at some point $p \in M$.

In this case, there exists a neighborhood U of p such that $\lambda(\beta_1 + \beta_3) = 0$ in U. If $\lambda(p) \neq 0$, then there exists a neighborhood $V \subset U$ such that $\beta_1 + \beta_3 = 0$ in V. This contradicts (3.4). Hence $\lambda(p) = 0$.

Case 2. $\beta_2 = 0$ at some point $p \in M$.

First we assume that there exists a neighborhood U of p such that $\beta_2 = 0$ in U. Then we get, in U,

$$(\beta_1)_x = -\lambda \cot\theta(\beta_1 - \beta_3)$$

from (2.10) and (3.2). On the other hand, from (3.6) and (3.1) we have, in U,

$$(\beta_1)_x = -\lambda \cot\theta(\beta_1 + \beta_3).$$

If $\lambda(p) \neq 0$, there exists a neighborhood $V \subset U$ such that $\lambda \neq 0$ in V. Then $\beta_3 = 0$ in V. Hence, $\beta_1 = 0$ in V from (2.10). This contradicts $\beta_1^2 + \beta_2^2 > 0$. Hence $\lambda(p) = 0$.

Otherwise, there exists a sequence $\{q_i\}_{i=1}^{\infty}$ approaching p such that $\beta_2(q_i) \neq 0$. Then $\lambda(q_i)(\beta_1 + \beta_3)(q_i) = 0$. By taking the limit, $\lambda(p)(\beta_1 + \beta_3)(p) = 0$. If $\lambda(p) \neq 0$, then $(\beta_1 + \beta_3)(p) = 0$. From (3.13), there exists a neighborhood U of p such that $\lambda \neq 0$ in U, which implies $\beta_2(\beta_1 + \beta_3) = 0$ in U. Taking derivatives with respect to x and y, using (3.1)–(3.4), (3.8), (3.9) and (3.12), we get

$$-\frac{\lambda\beta_1\beta_2(\beta_1 + \beta_3)^2 \cot\theta}{\beta_1^2 + \beta_2^2} + \lambda\beta_1\beta_2 \tan\theta = 0, \tag{3.14}$$

$$\frac{2\alpha\lambda\beta_1(\beta_1 + \beta_3)(\beta_1\beta_3 - \beta_2^2)\cot\theta}{\beta_1^2 + \beta_2^2} + \alpha\lambda\beta_2^2 \tan\theta = 0. \tag{3.15}$$

From (3.14) and (3.15), we have, in U,

$$\frac{\alpha\lambda \cot\theta}{\beta_1^2 + \beta_2^2}\beta_1(\beta_1 + \beta_3)(2\beta_1^2\beta_3 - \beta_1\beta_2^2 + \beta_3\beta_2^2) = 0. \tag{3.16}$$

Since $\beta_2(p) = 0$, we can assume $\beta_1(p) > 0$ without loss of generality. Hence $\beta_3(p) < 0$ from $(\beta_1 + \beta_3)(p) = 0$. Then there exists a neighborhood $V \subset U$ such that $\beta_1 > 0, \beta_3 < 0$ in V. Thus in V, we have

$$2\beta_1^2\beta_3 - \beta_1\beta_2^2 + \beta_3\beta_2^2 < 0.$$

Then (3.16) implies that $\beta_1 + \beta_3 = 0$ in V. This contradicts (3.2). Therefore, $\lambda(p) = 0$.

Hence we have proved the claim and completed the proof of Theorem 1. ■

4 Classification of minimal constant angle surfaces

In this section, we consider the minimal constant angle surface M in $\mathbb{S}^3(1) \times \mathbb{R}$.

Lemma 2. *Let M be a minimal constant angle surface in $\mathbb{S}^3(1) \times \mathbb{R}$. Then the shape operators with respect to ξ and η are, respectively,*

$$A_\xi = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix}, \quad A_\eta = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & -\beta_1 \end{pmatrix},$$

where β_1 and β_2 are constants, satisfying $\beta_1^2 + \beta_2^2 = \cos^2\theta$.

Proof. From (2.4) and the minimality of M in $\mathbb{S}^3(1) \times \mathbb{R}$, the shape operator A_ξ associated to ξ is

$$A_\xi = \begin{pmatrix} 0 & 0 \\ 0 & 0 \end{pmatrix} \tag{4.1}$$

Hence, we have

$$\nabla_T T = \nabla_T Q = \nabla_Q T = \nabla_Q Q = 0,$$

which means that M is flat. The coordinates (x, y) on M now can be chosen such that $\partial_x = T, \partial_y = Q$ (i.e. $\alpha = 1$).

From the minimality of M in $\mathbb{S}^3(1) \times \mathbb{R}$, the shape operator A_η becomes

$$A_\eta = \begin{pmatrix} \beta_1 & \beta_2 \\ \beta_2 & -\beta_1 \end{pmatrix}.$$

The equations of Gauss, Ricci, and Codazzi (2.9)–(2.11) are

$$\begin{aligned} &\beta_1^2 + \beta_2^2 = \cos^2\theta, \\ &(\beta_2)_y = -(\beta_1)_x, \\ &(\beta_1)_y = (\beta_2)_x. \end{aligned}$$

The above equations yield that both β_1 and β_2 are constant. ■

Now let us consider $\mathbb{S}^3(1) \times \mathbb{R}$ as a hypersurface in $\mathbb{E}^5$ and denote ∂_t by $(0, 0, 0, 0, 1)$. We obtain the following classification theorem.

Theorem 3. *A surface M immersed in $\mathbb{S}^3(1) \times \mathbb{R}$ is a minimal constant angle surface if and only if the immersion*

$$\begin{aligned} F: \quad M \quad &\rightarrow \quad \mathbb{S}^3(1) \times \mathbb{R} \subset \mathbb{E}^5 \\ (x, y) \quad &\mapsto \quad F(x, y) \end{aligned}$$

is (*up to isometries of* $\mathbb{S}^3(1) \times \mathbb{R}$) *locally given by*

$$\begin{aligned} F(x, y) = (&c_1 \cos(\mu_1 x + \nu_2 y), c_1 \sin(\mu_1 x + \nu_2 y), c_2 \cos(\mu_2 x - \nu_1 y), \\ &c_2 \sin(\mu_2 x - \nu_1 y), x \sin\theta), \end{aligned} \tag{4.2}$$

where $\theta \in (0, \frac{\pi}{2})$ *is the constant angle,* $\nu_1 \in [1, 1+\cos^2\theta]$ *is a constant, and* ν_2, μ_1, μ_2, c_1, c_2 *are nonnegative constants given by*

$$\nu_2^2 = \frac{1+\cos^2\theta - \nu_1^2}{1-\nu_1^2\sin^2\theta}, \ \mu_1^2 = \frac{\nu_1^2\cos^4\theta}{1-\nu_1^2\sin^2\theta}, \ \mu_2^2 = 1+\cos^2\theta - \nu_1^2,$$
$$c_1^2 = \frac{1-\nu_1^2\sin^2\theta}{1+\cos^2\theta-\nu_1^2\sin^2\theta}, \ c_2^2 = \frac{\cos^2\theta}{1+\cos^2\theta-\nu_1^2\sin^2\theta}.$$

Proof. First we prove that the given immersion (4.2) is a minimal constant angle surface in $\mathbb{S}^3(1)\times\mathbb{R}$. To see this, we calculate the tangent vectors

$$\begin{aligned} F_x =&(-\mu_1 c_1 \sin(\mu_1 x+\nu_2 y), \mu_1 c_1\cos(\mu_1 x+\nu_2 y), -\mu_2 c_2\sin(\mu_2 x-\nu_1 y),\\ &\mu_2 c_2\cos(\mu_2 x-\nu_1 y), \sin\theta),\\ F_y =&(-\nu_2 c_1\sin(\mu_1 x+\nu_2 y), \nu_2 c_1\cos(\mu_1 x+\nu_2 y), \nu_1 c_2\sin(\mu_2 x-\nu_1 y),\\ &-\nu_1 c_2\cos(\mu_2 x-\nu_1 y), 0). \end{aligned}$$

The normal N of $\mathbb{S}^3(1)\times\mathbb{R}$ in $\mathbb{E}^5$ is

$$N = (c_1\cos(\mu_1 x+\nu_2 y), c_1\sin(\mu_1 x+\nu_2 y), c_2\cos(\mu_2 x-\nu_1 y), c_2\sin(\mu_2 x-\nu_1 y), 0).$$

Let

$$\begin{aligned} \xi =&(\mu_1 c_1\tan\theta\sin(\mu_1 x+\nu_2 y), -\mu_1 c_1\tan\theta\cos(\mu_1 x+\nu_2 y), \mu_2 c_2\tan\theta\sin(\mu_2 x-\nu_1 y),\\ &-\mu_2 c_2\tan\theta\cos(\mu_2 x-\nu_1 y), \cos\theta),\\ \eta =&(-c_2\cos(\mu_1 x+\nu_2 y), -c_2\sin(\mu_1 x+\nu_2 y), c_1\cos(\mu_2 x-\nu_1 y), c_1\sin(\mu_2 x-\nu_1 y), 0). \end{aligned}$$

We can verify that F_x, F_y, ξ, η, N are orthonormal in $\mathbb{E}^5$. Thus $\{\xi,\eta\}$ is a basis of the normal plane of M in $\mathbb{S}^3(1)\times\mathbb{R}$. Moreover, we have

$$\partial_t = \sin\theta F_x + \cos\theta\xi,$$

which means that the angle between ∂_t and the normal plane is constant θ.

Furthermore, we can calculate the shape operators with respect to ξ and η on M in $\mathbb{S}^3(1)\times\mathbb{R}$ respectively,

$$A_\xi = \begin{pmatrix} 0 & 0\\ 0 & 0\end{pmatrix}, \quad A_\eta = \begin{pmatrix} \beta_1 & \beta_2\\ \beta_2 & \beta_3\end{pmatrix},$$

where

$$\beta_1 = -\beta_3 = \frac{(\nu_1^2-1)\cos\theta}{\sqrt{1-\nu_1^2\sin^2\theta}}, \quad \beta_2 = \frac{\nu_1\cos\theta\sqrt{1+\cos^2\theta-\nu_1^2}}{\sqrt{1-\nu_1^2\sin^2\theta}}.$$

Therefore, M is a minimal surface in $\mathbb{S}^3(1)\times\mathbb{R}$. Moreover, we can see that $(\beta_1)^2 + (\beta_2)^2 = \cos^2\theta$.

Conversely, let us consider M as an immersed surface in $\mathbb{E}^5$ with codimension 3. Denote by D, $\tilde{\nabla}^\perp$ the Euclidean connection and the normal connection of M in

$\mathbb{E}^5$, respectively. For the immersion $F=(F_1,F_2,F_3,F_4,F_5):M\to\mathbb{S}^3(1)\times\mathbb{R}\subset\mathbb{E}^5$, we have three unit normals

$$\begin{aligned}N&=(F_1,F_2,F_3,F_4,0),\\ \xi&=(\xi_1,\xi_2,\xi_3,\xi_4,\cos\theta),\\ \eta&=(\eta_1,\eta_2,\eta_3,\eta_4,0),\end{aligned}$$

where N is normal to $\mathbb{S}^3(1)\times\mathbb{R}$ with the shape operator $\tilde{A}_N$.

For simplicity, we denote the first four components of a vector in $\mathbb{E}^5$ by adding a tilde on it, say $F=(\tilde{F},F_5)$, etc.

Noticing that $\langle T,\partial_t\rangle=(F_5)_x=\sin\theta$, $\langle Q,\partial_t\rangle=(F_5)_y=0$, we can take $F_5=x\sin\theta$ without loss of generality.

For any $X\in T_pM$, we have

$$\begin{aligned}\tilde{\nabla}^{\perp}_X N&=\langle D_X N,\xi\rangle\xi+\langle D_X N,\eta\rangle\eta\\ &=\langle X-\langle X,\partial_t\rangle\partial_t,\xi\rangle\xi+\langle X-\langle X,\partial_t\rangle\partial_t,\eta\rangle\eta\\ &=-\sin\theta\cos\theta\langle X,T\rangle\xi.\end{aligned}$$

By the Weingarten formula, we have

$$\begin{aligned}\tilde{A}_N T&=-D_T N+\tilde{\nabla}^{\perp}_T N\\ &=-(\tilde{F}_x,0)-\sin\theta\cos\theta(\tilde{\xi},\cos\theta),\\ \tilde{A}_N Q&=-D_Q N+\tilde{\nabla}^{\perp}_Q N\\ &=-(\tilde{F}_y,0).\end{aligned}\tag{4.3}$$

Thus the shape operator associated to N is

$$\tilde{A}_N=\begin{pmatrix}-\sin^2\theta & 0\\ 0 & -1\end{pmatrix}.$$

Comparing the first four components of (4.3), we get

$$\xi_i=-\tan\theta(F_i)_x.$$

Taking $(X,Y)=(T,T),(T,Q),(Q,Q)$ in $D_XY=\tilde{\nabla}_XY+\tilde{h}(X,Y)$, and $X=T,Q$ in $D_X\eta=-\tilde{A}_\eta X+\tilde{\nabla}^{\perp}_X\eta$ respectively, we get the PDE system for $i=1,2,3,4$,

$$(F_i)_{xx}=\beta_1\eta_i-\cos^2\theta F_i,\tag{4.4}$$

$$(F_i)_{xy}=\beta_2\eta_i,\tag{4.5}$$

$$(F_i)_{yy}=-\beta_1\eta_i-F_i,\tag{4.6}$$

$$(\eta_i)_x=-\frac{\beta_1}{\cos^2\theta}(F_i)_x-\beta_2(F_i)_y,\tag{4.7}$$

$$(\eta_i)_y=-\frac{\beta_2}{\cos^2\theta}(F_i)_x+\beta_1(F_i)_y,\tag{4.8}$$

where β_1 and β_2 are as in Lemma 2. Obviously, the integrable conditions are all satisfied. Moreover, we have $\xi_i=-\tan\theta(F_i)_x$ and $F_5=x\sin\theta$, $\xi_5=\cos\theta$, $\eta_5=0$.

In the following, we will solve the above PDE system in three cases.

Case 1. $\beta_2 = 0$.

In this case, we can choose the direction of η such that $\beta_1 = \cos\theta > 0$, and then the PDE system becomes

$$(F_i)_{xx} = \cos\theta\eta_i - \cos^2\theta F_i, \tag{4.9}$$
$$(F_i)_{xy} = 0, \tag{4.10}$$
$$(F_i)_{yy} = -\cos\theta\eta_i - F_i, \tag{4.11}$$
$$(\eta_i)_x = -\frac{1}{\cos\theta}(F_i)_x, \tag{4.12}$$
$$(\eta_i)_y = \cos\theta(F_i)_x. \tag{4.13}$$

From (4.10), we know that the solution has a separating form: $F_i(x, y) = f_i(x) + g_i(y)$. Denote $\rho = \sqrt{1 + \cos^2\theta}$. Taking the derivative of (4.9) with respect to x and using (4.12), we get

$$f_i''' = -\rho^2 f_i',$$

and then $f_i'(x) = k_i \cos(\rho x) + l_i \sin(\rho x)$. Taking the same operation with respect to y, we find the solution has the form

$$F_i(x, y) = A_i \cos(\rho x) + B_i \sin(\rho x) + C_i \cos(\rho y) + D_i \sin(\rho y).$$

We can derive from (4.9) that

$$\eta_i(x, y) = -\frac{A_i}{\cos\theta}\cos(\rho x) - \frac{B_i}{\cos\theta}\sin(\rho x) + C_i \cos\theta\cos(\rho y) + D_i \cos\theta\sin(\rho y),$$

and we can also check that (4.11)–(4.13) are all satisfied.

Since

$$\begin{aligned}
(F_i)_x &= \rho\big(B_i \cos(\rho x) - A_i \sin(\rho x)\big),\\
(F_i)_y &= \rho\big(D_i \cos(\rho y) - C_i \sin(\rho y)\big),\\
\xi_i &= -\rho\tan\theta\big(B_i \cos(\rho x) - A_i \sin(\rho x)\big),
\end{aligned}$$

and F_x, F_y are orthonormal, we have

$$\begin{aligned}
\cos^2\theta = \sum_i ((F_i)_x)^2 &= \rho^2\Big(\sum_i B_i^2 \cos^2(\rho x) + \sum_i A_i^2 \sin^2(\rho x) - \sum_i A_i B_i \sin(2\rho x)\Big),\\
1 = \sum_i ((F_i)_y)^2 &= \rho^2\Big(\sum_i D_i^2 \cos^2(\rho y) + \sum_i C_i^2 \sin^2(\rho y) - \sum_i C_i D_i \sin(2\rho y)\Big),\\
0 = \sum_i (F_i)_x (F_i)_y &= \rho^2\Big(\sum_i B_i D_i \cos(\rho x)\cos(\rho y) + \sum_i A_i C_i \sin(\rho x)\sin(\rho y)\\
&\qquad - \sum_i B_i C_i \cos(\rho x)\sin(\rho y) - \sum_i A_i D_i \sin(\rho x)\cos(\rho y)\Big).
\end{aligned}$$

Since x, y are arbitrary, we have

$$\sum_i A_i^2 = \sum_i B_i^2 = \frac{\cos^2\theta}{\rho^2}, \quad \sum_i C_i^2 = \sum_i D_i^2 = \frac{1}{\rho^2},$$

$$\sum_i A_iB_i=\sum_i C_iD_i=\sum_i B_iD_i=\sum_i A_iC_i=\sum_i B_iC_i=\sum_i A_iD_i=0,$$

and we can check that F_x, F_y, ξ, η are orthonormal. Hence, we have

$$\tilde{F}(x,y)=\frac{\cos\theta}{\rho}\cos(\rho x)\vec{e}_1+\frac{\cos\theta}{\rho}\sin(\rho x)\vec{e}_2+\frac{1}{\rho}\cos(\rho y)\vec{e}_3+\frac{1}{\rho}\sin(\rho y)\vec{e}_4.$$

where $\{\vec{e}_i\}_{i=1}^4$ is a fixed orthonormal basis of $\mathbb{E}^4$. If we choose $\vec{e}_1=(1,0,0,0)$, $\vec{e}_2=(0,1,0,0)$, $\vec{e}_3=(0,0,1,0)$, $\vec{e}_4=(0,0,0,-1)$, the surface is locally given by

$$F(x,y)=\left(\frac{\cos\theta}{\rho}\cos(\rho x),\frac{\cos\theta}{\rho}\sin(\rho x),\frac{1}{\rho}\cos(\rho y),-\frac{1}{\rho}\sin(\rho y),x\sin\theta\right).$$

This is the case $\nu_1=\rho=\sqrt{1+\cos^2\theta}$ (hence $\mu_1=\rho$, $\mu_2=\nu_2=0$, $c_1=\frac{\cos\theta}{\rho}$, $c_2=\frac{1}{\rho}$) in (4.2).

Case 2. $\beta_1=0$.

In this case, we can choose the direction of η such that $\beta_2=\cos\theta>0$. The PDE system becomes

$$(F_i)_{xx}=-\cos^2\theta F_i, \tag{4.14}$$
$$(F_i)_{xy}=\cos\theta\eta_i, \tag{4.15}$$
$$(F_i)_{yy}=-F_i, \tag{4.16}$$
$$(\eta_i)_x=-\cos\theta(F_i)_y, \tag{4.17}$$
$$(\eta_i)_y=-\frac{1}{\cos\theta}(F_i)_x. \tag{4.18}$$

Solving (4.14) and (4.16), we find that the solution has the form

$$\begin{aligned}F_i(x,y)=&A_i\cos(x\cos\theta)\cos y+B_i\cos(x\cos\theta)\sin y\\&+C_i\sin(x\cos\theta)\cos y+D_i\sin(x\cos\theta)\sin y.\end{aligned}$$

We can derive from (4.15) that

$$\begin{aligned}\eta_i=&D_i\cos(x\cos\theta)\cos y-C_i\cos(x\cos\theta)\sin y\\&-B_i\sin(x\cos\theta)\cos y+A_i\sin(x\cos\theta)\sin y,\end{aligned}$$

and we can check that (4.17) and (4.18) are satisfied. Moreover, we have

$$\begin{aligned}(F_i)_x=&\cos\theta(C_i\cos(x\cos\theta)\cos y+D_i\cos(x\cos\theta)\sin y\\&-A_i\sin(x\cos\theta)\cos y-B_i\sin(x\cos\theta)\sin y),\\(F_i)_y=&B_i\cos(x\cos\theta)\cos y-A_i\cos(x\cos\theta)\sin y\\&+D_i\sin(x\cos\theta)\cos y-C_i\sin(x\cos\theta)\sin y,\\\xi_i=&-\sin\theta\big(C_i\cos(x\cos\theta)\cos y+D_i\cos(x\cos\theta)\sin y\\&-A_i\sin(x\cos\theta)\cos y-B_i\sin(x\cos\theta)\sin y\big).\end{aligned}$$

From the fact that F_x, F_y, ξ, η are orthonormal, a similar discussion as in Case 1 yields

$$\begin{aligned}\tilde{F}(x,y)=&\cos(x\cos\theta)\cos y\vec{e}_1+\cos(x\cos\theta)\sin y\vec{e}_2\\&+\sin(x\cos\theta)\cos y\vec{e}_3+\sin(x\cos\theta)\sin y\vec{e}_4,\end{aligned}$$

where $\{\vec{e}_i\}_{i=1}^4$ is a fixed orthonormal basis of $\mathbb{E}^4$. If we choose $\vec{e}_1 = (\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0)$, $\vec{e}_2 = (0, \frac{1}{\sqrt{2}}, 0, -\frac{1}{\sqrt{2}})$, $\vec{e}_3 = (0, \frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}})$, $\vec{e}_4 = (-\frac{1}{\sqrt{2}}, 0, \frac{1}{\sqrt{2}}, 0)$, the surface is locally given by

$$F(x,y) = \Big(\frac{1}{\sqrt{2}} \cos(x\cos\theta + y), \frac{1}{\sqrt{2}} \sin(x\cos\theta + y), \frac{1}{\sqrt{2}} \cos(x\cos\theta - y), \frac{1}{\sqrt{2}} \sin(x\cos\theta - y), x\sin\theta \Big).$$

This is the case $\nu_1 = 1$ $\big($hence $\mu_1 = \mu_2 = \cos\theta, \nu_2 = 1, c_1 = c_2 = \frac{1}{\sqrt{2}}\big)$ in (4.2).

Case 3. $\beta_1\beta_2 \neq 0$.

Taking the derivative of equation (4.4) with respect to x, and using equation (4.7), we get

$$(F_i)_{xxx} = -\frac{\beta_1^2}{\cos^2\theta}(F_i)_x - \cos^2\theta (F_i)_x - \beta_1\beta_2 (F_i)_y.$$

Taking the derivative with respect to x again, and using equations (4.5), (4.4), we get

$$(F_i)_{xxxx} = \left(-\frac{\beta_1^2}{\cos^2\theta} - \beta_2^2 - \cos^2\theta \right) (F_i)_{xx} - \beta_2^2 \cos^2\theta F_i. \tag{4.19}$$

Similarly, taking the derivative of equation (4.6) with respect to y twice, and using equations (4.8), (4.5), (4.6), we get

$$(F_i)_{yyy} = \frac{\beta_1\beta_2}{\cos^2\theta}(F_i)_x - \beta_1^2 (F_i)_y - (F_i)_y,$$

and

$$(F_i)_{yyyy} = \left(-\frac{\beta_2^2}{\cos^2\theta} - \beta_1^2 - 1 \right) (F_i)_{yy} - \frac{\beta_2^2}{\cos^2\theta} F_i. \tag{4.20}$$

The characteristic equation of (4.19) is

$$z^4 + \left(\frac{\beta_1^2}{\cos^2\theta} + \beta_2^2 + \cos^2\theta \right) z^2 + \beta_2^2\cos^2\theta = 0. \tag{4.21}$$

Denote $b_1 = \frac{\beta_1^2}{\cos^2\theta} + \beta_2^2 + \cos^2\theta$ and $c_1 = \beta_2^2\cos^2\theta$. Considering equation (4.21) as a quadratic equation in $u = z^2$, the discriminant is

$$\Delta_1 = b_1^2 - 4c_1 = \frac{\beta_1^4}{\cos^4\theta} + \frac{2\beta_1^2\beta_2^2}{\cos^2\theta} + 2\beta_1^2 + \beta_1^4 > 0.$$

Since $c_1 > 0$, the two negative roots $u = -\mu_1^2$ and $u = -\mu_2^2$ of the equation are

$$-\mu_1^2 = -\frac{1}{2}(b_1 + \sqrt{\Delta_1}), \quad -\mu_2^2 = -\frac{1}{2}(b_1 - \sqrt{\Delta_1}),$$

where we assume $\mu_1 > 0, \mu_2 > 0$.

Similarly, the characteristic equation of (4.20) is

$$w^4 + \left(\frac{\beta_2^2}{\cos^2\theta} + \beta_1^2 + 1 \right) w^2 + \frac{\beta_2^2}{\cos^2\theta} = 0. \tag{4.22}$$

Denote $b_2 = \frac{\beta_2^2}{\cos^2\theta} + \beta_1^2 + 1$ and $c_2 = \frac{\beta_2^2}{\cos^2\theta}$. Considering equation (4.22) as a quadratic equation as above, the discriminant is

$$\Delta_2 = b_2^2 - 4c_2 = \Delta_1 > 0$$

and the two negative roots are

$$-\nu_1^2 = -\frac{1}{2}(b_2 + \sqrt{\Delta_2}), \quad -\nu_2^2 = -\frac{1}{2}(b_2 - \sqrt{\Delta_2}),$$

where we assume $\nu_1 > 0, \nu_2 > 0$.

Now we denote $\Delta = \Delta_1 = \Delta_2$. Since $(F_i)_{xx} + (F_i)_{yy} = -(1+\cos^2\theta)F_i$ and $\mu_1^2 + \nu_2^2 = \mu_2^2 + \nu_1^2 = 1 + \cos^2\theta$, the solution takes the form

$$\begin{aligned} F_i(x,y) = {} & c_1^{(i)} \cos(\mu_1 x)\cos(\nu_2 y) + c_2^{(i)} \cos(\mu_1 x)\sin(\nu_2 y) + c_3^{(i)} \sin(\mu_1 x)\cos(\nu_2 y) \\ & + c_4^{(i)} \sin(\mu_1 x)\sin(\nu_2 y) + c_5^{(i)} \cos(\mu_2 x)\cos(\nu_1 y) + c_6^{(i)} \cos(\mu_2 x)\sin(\nu_1 y) \\ & + c_7^{(i)} \sin(\mu_2 x)\cos(\nu_1 y) + c_8^{(i)} \sin(\mu_2 x)\sin(\nu_1 y). \end{aligned}$$

We can derive η_i from (4.4),

$$\begin{aligned} \eta_i &= \frac{1}{\beta_1}((F_i)_{xx} + \cos^2\theta F_i) \\ &= \frac{\cos^2\theta - \mu_1^2}{\beta_1}(c_1^{(i)} \cos(\mu_1 x)\cos(\nu_2 y) + c_2^{(i)} \cos(\mu_1 x)\sin(\nu_2 y) \\ &\quad + c_3^{(i)} \sin(\mu_1 x)\cos(\nu_2 y) + c_4^{(i)} \sin(\mu_1 x)\sin(\nu_2 y)) \\ &\quad + \frac{\cos^2\theta - \mu_2^2}{\beta_1}(c_5^{(i)} \cos(\mu_2 x)\cos(\nu_1 y) + c_6^{(i)} \cos(\mu_2 x)\sin(\nu_1 y) \\ &\quad + c_7^{(i)} \sin(\mu_2 x)\cos(\nu_1 y) + c_8^{(i)} \sin(\mu_2 x)\sin(\nu_1 y)). \end{aligned} \tag{4.23}$$

On the other hand, from (4.5)

$$\begin{aligned} \eta_i &= \frac{1}{\beta_2}(F_i)_{xy} \\ &= \frac{\mu_1\nu_2}{\beta_2}(c_4^{(i)} \cos(\mu_1 x)\cos(\nu_2 y) - c_3^{(i)} \cos(\mu_1 x)\sin(\nu_2 y) \\ &\quad - c_2^{(i)} \sin(\mu_1 x)\cos(\nu_2 y) + c_1^{(i)} \sin(\mu_1 x)\sin(\nu_2 y)) \\ &\quad + \frac{\mu_2\nu_1}{\beta_2}(c_8^{(i)} \cos(\mu_2 x)\cos(\nu_1 y) - c_7^{(i)} \cos(\mu_2 x)\sin(\nu_1 y) \\ &\quad - c_6^{(i)} \sin(\mu_2 x)\cos(\nu_1 y) + c_5^{(i)} \sin(\mu_2 x)\sin(\nu_1 y)). \end{aligned} \tag{4.24}$$

Comparing the first four terms, we find that

$$\frac{\cos^2\theta-\mu_1^2}{\beta_1}c_1^{(i)}=\frac{\mu_1\nu_2}{\beta_2}c_4^{(i)},\quad \frac{\cos^2\theta-\mu_1^2}{\beta_1}c_4^{(i)}=\frac{\mu_1\nu_2}{\beta_2}c_1^{(i)},$$
$$\frac{\cos^2\theta-\mu_1^2}{\beta_1}c_2^{(i)}=\frac{\mu_1\nu_2}{\beta_2}c_3^{(i)},\quad \frac{\cos^2\theta-\mu_1^2}{\beta_1}c_3^{(i)}=\frac{\mu_1\nu_2}{\beta_2}c_2^{(i)}.$$

Since $\mu_1>0,\mu_2>0,\nu_1>0,\nu_2>0$,

$$2(\cos^2\theta-\mu_1^2)=\beta_1^2-\frac{\beta_1^2}{\cos^2\theta}-\sqrt{\Delta}<0,$$

$$2(\cos^2\theta-\mu_2^2)=\beta_1^2-\frac{\beta_1^2}{\cos^2\theta}+\sqrt{\Delta}>0,$$

we have that

$$(c_1^{(i)})^2=(c_4^{(i)})^2,(c_2^{(i)})^2=(c_3^{(i)})^2.$$

Similarly, comparing the last four terms of (4.23) and (4.24), we obtain that

$$(c_5^{(i)})^2=(c_8^{(i)})^2,(c_6^{(i)})^2=(c_7^{(i)})^2.$$

Furthermore, we have for $\beta_1\beta_2>0$,

$$c_1^{(i)}=-c_4^{(i)},\ c_2^{(i)}=c_3^{(i)},\ c_5^{(i)}=c_8^{(i)},\ c_6^{(i)}=-c_7^{(i)};$$

and for $\beta_1\beta_2<0$,

$$c_1^{(i)}=c_4^{(i)},\ c_2^{(i)}=-c_3^{(i)},\ c_5^{(i)}=-c_8^{(i)},\ c_6^{(i)}=c_7^{(i)}.$$

Hence, for $\beta_1\beta_2>0$, we can set

$$F_i(x,y)=A_i\cos(\mu_1x+\nu_2y)+B_i\sin(\mu_1x+\nu_2y)+ C_i\cos(\mu_2x-\nu_1y)+D_i\sin(\mu_2x-\nu_1y).$$

In fact, we can easily verify that the solution above satisfies the PDE system (4.4)–(4.8).

Moreover, using the fact that F_x, F_y, ξ, η are orthonormal, we can derive that

$$\tilde{F}(x,y)=c_1\cos(\mu_1x+\nu_2y)\vec{e}_1+c_1\sin(\mu_1x+\nu_2y)\vec{e}_2 +c_2\cos(\mu_2x-\nu_1y)\vec{e}_3+c_2\sin(\mu_2x-\nu_1y)\vec{e}_4$$

where $\{\vec{e}_i\}_{i=1}^4$ is a fixed orthonormal basis of $\mathbb{E}^4$, c_1,c_2 are positive constants satisfying $c_1^2=\frac{\nu_1^2-1}{\nu_1^2-\nu_2^2}$, $c_2^2=\frac{1-\nu_2^2}{\nu_1^2-\nu_2^2}$. If we choose the natural basis of $\mathbb{E}^4$, the surface is locally given by

$$F(x,y)=\big(c_1\cos(\mu_1x+\nu_2y),c_1\sin(\mu_1x+\nu_2y),c_2\cos(\mu_2x-\nu_1y), c_2\sin(\mu_2x-\nu_1y),x\sin\theta\big).$$

This is the case $1 < \nu_1 < \sqrt{1+\cos^2\theta}$ in (4.2).

Similarly, for $\beta_1\beta_2 < 0$, the surface is locally given by

$$F(x,y) = \big(c_1\cos(\mu_1 x - \nu_2 y), c_1\sin(\mu_1 x - \nu_2 y), c_2\cos(\mu_2 x + \nu_1 y), \\ c_2\sin(\mu_2 x + \nu_1 y), x\sin\theta\big).$$

If we change the coordinate to be $\{x, -y\}$, then this is the case $1 < \nu_1 < \sqrt{1+\cos^2\theta}$ in (4.2).

Here we need to derive the relations among the constants ν_1, ν_2, μ_1, μ_2, c_1, c_2 when $1 < \nu_1 < \sqrt{1+\cos^2\theta}$. In fact, by the definitions of ν_1 and ν_2, we have $\nu_1^2\nu_2^2 = \frac{\beta_2^2}{\cos^2\theta}$ and

$$\begin{aligned}\nu_1^2+\nu_2^2 &= \frac{\beta_2^2}{\cos^2\theta} + \beta_1^2 + 1 \\ &= \nu_1^2\nu_2^2 + \cos^2\theta - \cos^2\theta\nu_1^2\nu_2^2 + 1 \\ &= \nu_1^2\nu_2^2\sin^2\theta + \cos^2\theta + 1.\end{aligned}$$

Since $1+\cos^2\theta < \frac{1}{\sin^2\theta}$ when $\theta \in (0, \frac{\pi}{2})$, we have $\nu_2^2 = \frac{1+\cos^2\theta-\nu_1^2}{1-\nu_1^2\sin^2\theta}$. By a direct computation, we have

$$\begin{aligned}\mu_1^2 &= 1+\cos^2\theta - \nu_2^2 = \frac{\nu_1^2\cos^4\theta}{1-\nu_1^2\sin^2\theta}, \\ \mu_2^2 &= 1+\cos^2\theta-\nu_1^2, \\ c_1^2 &= \frac{\nu_1^2-1}{\nu_1^2-\nu_2^2} = \frac{1-\nu_1^2\sin^2\theta}{1+\cos^2\theta-\nu_1^2\sin^2\theta}, \\ c_2^2 &= \frac{1-\nu_2^2}{\nu_1^2-\nu_2^2} = \frac{\cos^2\theta}{1+\cos^2\theta-\nu_1^2\sin^2\theta}.\end{aligned}$$

Hence we complete the proof of Theorem 3. ■

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