Approximate Connes-amenability of dual Banach algebras

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Abstract

We introduce the notions of approximate Connes-amenability and approximate strong Connes-amenability for dual Banach algebras. Then we characterize these two types of algebras in terms of approximate normal virtual diagonals and approximate $\sigma WC-$virtual diagonals. We investigate these properties for von Neumann algebras, measure algebra and the algebra of p-pseudomeasures on locally compact groups. In particular we show that a von Neumann algebra is approximately Connes-amenable if and only if it has an approximate normal virtual diagonal. This is the "approximate" analog of the main result of Effros in [10].

We show that in general the concepts of approximate Connes-amenability and Connes-amenability are distinct, but for measure algebras these two concepts coincide. Moreover cases where approximate Connes-amenability of $\mathcal{A}^{**}$ implies approximate Connes-amenability or approximate amenability of $\mathcal{A}$ are also discussed.

1 introduction

The concept of amenability for Banach algebras was introduced and studied for the first time by B. E. Johnson in [21]. Since then several variants of this concept have appeared in the literature each, as a kind of cohomological triviality.

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In [23], Johnson, Kadison, and Ringrose introduced a notion of amenability for von Neumann algebras which modified Johnson's original definition for Banach algebras in the sense that it takes the dual space structure of a von Neumann algebra into account. This notion of amenability was later called Connes-amenability by A. Ya. Helemskii [19], due to the seminal work of A. Connes [1,2]. See also [17,33]. Johnson in [22] showed that a Banach algebra $\mathcal{A}$ is amenable if and only if it has a virtual diagonal. A von Neumann algebraic analogue of this result was discovered by Haagerup [17]; See also [10], where the author introduces the notion of normal virtual diagonal and presents another proof of Haagerup's result. Runde extended the notion of Connes-amenability to the larger class of dual Banach algebras [26] and studied certain concrete Banach algebras in the subsequent papers [28, 29, 30]. In particular he showed that existence of normal virtual diagonals implies Connes-amenability but the converse is no longer valid for arbitrary dual Banach algebras.

In all of the above mentioned concepts, all bounded derivations from a given Banach algebra $\mathcal{A}$ into certain Banach $\mathcal{A}$-bimodules are required to be exactly inner. Gourdeau provided the following characterization of amenability; A Banach algebra $\mathcal{A}$ is amenable if and only if any bounded derivation from $\mathcal{A}$ into any Banach $\mathcal{A}$-bimodule is approximately inner, or equivalently weakly approximately inner [15, Proposition 2.1]. Motivated by Gourdeau's result, Ghahramani and Loy [13] introduced several approximate notions of amenability by requiring that all bounded derivations from a given Banach algebra $\mathcal{A}$ into certain Banach $\mathcal{A}$-bimodules to be approximately inner. However in contrast to Gourdeau's result, they removed the boundedness assumption on the net of implementing elements. In the same paper and the subsequent one [14], the authors showed the distinction between each of these concepts and the corresponding classical notions and investigated properties of algebras in each of these new classes. At the beginning, Ghahramani and Loy asked which of the standard results on amenability work for the approximate concepts [See 13, page 233]; A question which identified the main direction of [11, 13, 14] and the present paper.

Motivated by the above question and [10], we introduce and study approximate Connes-amenability and approximate strong Connes-amenability. In Section 2 we present the definition and some basic properties of approximate Connes-amenability. An example presented at the beginning of section 2, shows the distinction of Connes-amenability and approximate Connes-amenability. In Section 3 we introduce approximate strong Connes-amenability, approximate normal virtual diagonals and approximate $\sigma WC-$virtual diagonals. Then we show that a dual Banach algebra is approximately Connes-amenable [respectively, approximately strongly Connes-amenable] if and only if it has an approximate $\sigma WC-$virtual diagonal [respectively, approximately normal virtual diagonal]. In Section 4 which is the main part of this paper, we prove that a von Neumann algebra is approximately Connes-amenable if and only if it has an approximate normal virtual diagonal. This is the "approximate" analog of the main result of Effros [10]. In Section 5 we show that for a locally compact group G, the measure algebra $M(G)$ of G is Connes-amenable if and only if it is approximately Connes-amenable if and only if it has an approximate normal virtual diagonal. This strengthens the main result of [29]. Section 6 is devoted to investigating approx-

imate Connes-amenability of $PM_p(G)$ and $VN(G)$ for arbitrary locally compact groups. In the last section we show that under certain conditions approximate Connes-amenability of $\mathcal{A}^{**}$ implies approximate Connes-amenability or approximate amenability of $\mathcal{A}$.
We also should mention that some of our arguments were inspired by their classic analogs mostly from [10, 21, 26, 27, 30].
Before proceeding further we recall some terminology.
Throughout $\mathcal{A}$ is a Banach algebra and $\mathcal{X}$ is a Banach $\mathcal{A}$-bimodule. Also the identity element of $\mathcal{A}$, whenever it exists, is denoted with e. We denote the commutant of $S \subseteq \mathcal{A}$ by S'. The dual space $\mathcal{X}^*$ of $\mathcal{X}$, is an $\mathcal{A}$-module, with module actions

$$\langle \phi.a\ ,\ x\rangle = \langle \phi\ ,\ a.x\rangle \quad , \quad \langle a.\phi\ ,\ x\rangle = \langle \phi\ ,\ x.a\rangle \quad , \phi \in \mathcal{X}^*,\ x \in \mathcal{X},\ a \in \mathcal{A}.$$

Using the natural $\mathcal{A}$-module structure of $\mathcal{A}^*$ the first and second Arens multiplications on $\mathcal{A}^{**}$ that we denote by "." and "$\Box$" respectively, are defined as follows. For every $a \in \mathcal{A}$, $f \in \mathcal{A}^*$, $m, n \in \mathcal{A}^{**}$,

$$\begin{aligned} \langle n.f\ ,\ a\rangle &= \langle n\ ,\ f.a\rangle\,, & \langle f\Box m\ ,\ a\rangle &= \langle m\ ,\ a.f\rangle \\ \langle m.n\ ,\ f\rangle &= \langle m\ ,\ n.f\rangle\,, & \langle m\Box n\ ,\ f\rangle &= \langle n\ ,\ f\Box m\rangle. \end{aligned}$$

The second dual of a Banach algebra, equipped with the first [respectively second] Arens product is a Banach algebra. A Banach algebra $\mathcal{A}$ is called Arens regular whenever these two products coincide on $\mathcal{A}^{**}$. We always consider the second dual of a Banach algebra with the first Arens product.
Throughout "derivation" means "bounded derivation" and the set of all bounded derivations $D : \mathcal{A} \longrightarrow \mathcal{X}$ is denoted by $Z^1(\mathcal{A}, \mathcal{X})$. For $x \in \mathcal{X}$ the map $ad_x(a) = a.x - x.a \quad (a \in \mathcal{A})$ is called the inner derivation induced by x. A derivation $D : \mathcal{A} \longrightarrow \mathcal{X}$ is approximately inner if there exists a net $(x_\alpha) \subseteq \mathcal{X}$ such that for every $a \in \mathcal{A}$, $D(a) = \lim_\alpha(a.x_\alpha - x_\alpha.a)$, the limit being taken in norm. We say that $\mathcal{A}$ is approximately amenable if for any $\mathcal{A}$-bimodule $\mathcal{X}$, every derivation $D : \mathcal{A} \longrightarrow \mathcal{X}^*$ is approximately inner.
$\mathcal{A}$ is called a dual Banach algebra if there is a closed submodule $\mathcal{A}_*$ of $\mathcal{A}^*$ such that $\mathcal{A} = (\mathcal{A}_*)^*$. In general the predual module is not necessarily unique. We will therefore assume that $\mathcal{A}$ always comes with a fixed predual $\mathcal{A}_*$. Measure algebras of locally compact groups and second duals of Arens regular Banach algebras are examples of dual Banach algebras.
Let $\mathcal{A}$ be a dual Banach algebra and $\mathcal{X}$ be a Banach $\mathcal{A}$-bimodule. We call an element $\phi \in \mathcal{X}^*$ a normal element if the maps

$$\mathcal{A} \longrightarrow \mathcal{X}^*, \quad a \longmapsto \begin{cases} a.\phi \\ \phi.a \end{cases}$$

are $\omega^* - \omega^*$ continuous. If every element of $\mathcal{X}^*$ is normal, then we say that $\mathcal{X}^*$ is normal. An element $x \in \mathcal{X}$ is called ω^*-weak continuous if the module maps

$$\mathcal{A} \longrightarrow \mathcal{X}, \quad a \longmapsto \begin{cases} a.x \\ x.a \end{cases}$$

are ω^*-weak continuous. The set of all ω^*-weak continuous elements of $\mathcal{X}$ is denoted by $\sigma WC(\mathcal{X})$. A dual Banach algebra $\mathcal{A}$ is Connes-amenable if for every

normal dual Banach $\mathcal{A}$-module $\mathcal{X}$, every $\omega^* - \omega^*$ continuous derivation $D \in Z^1(\mathcal{A}, \mathcal{X})$ is inner.
A left $\mathcal{A}$-submodule $\mathcal{X}$ of $\mathcal{A}^*$ is called left introverted if for every $\phi \in E$ and $m \in E^*$ the functional $m.\phi \in \mathcal{A}^*$, which is defined by $\langle m.\phi, a\rangle = \langle m, \phi.a\rangle \quad (a \in \mathcal{A})$, lies again in E. This turns E^* into a dual Banach algebra by letting $\langle nm, \phi\rangle = \langle n, m.\phi\rangle \quad (n, m \in E^*, \phi \in E)$. It is known that $WAP(\mathcal{A}^*) = \{\phi \in \mathcal{A}^* : a \longmapsto a.\phi$ is weakly compact, $a \in \mathcal{A}\}$ is a left introverted subspace of $\mathcal{A}^*$ and hence $WAP(\mathcal{A}^*)^*$ with the above product is a dual Banach algebra.

2 Definition and basic properties

Definition. A dual Banach algebra $\mathcal{A}$ is approximately Connes-amenable if for every normal, dual Banach $\mathcal{A}$-bimodule $\mathcal{X}$, every $\omega^* - \omega^*$ continuous derivation $D \in Z^1(\mathcal{A}, \mathcal{X})$ is approximately inner.
The following examples show the distinction between Connes-amenability and approximate Connes-amenability.

Examples 2.1. (i) In this part we present an example of a dual Banach algebra which is approximately Connes-amenable but is not Connes-amenable. Let $\mathbb{N}_\vee$ be the set of natural numbers with the binary operation $(m, n) \mapsto \max\{m, n\}$. Then $\mathbb{N}_\vee$ is a unital, commutative, weakly cancellative semigroup, that is, for every $s, t \in \mathbb{N}_\vee$ the set $\{x \in \mathbb{N}_\vee \ : \ sx = t\}$ is finite. Let $\mathcal{A} = \ell^1(\mathbb{N}_\vee)$. Since $\mathbb{N}_\vee$ is weakly cancellative, then by [7, Theorem 4.6] $\mathcal{A}$ is a dual Banach algebra with respect to the predual $c_0(\mathbb{N}_\vee)$. If $\mathcal{A}$ is Connes-amenable, then by [7, Theorem 5.13] $\mathbb{N}_\vee$ should be a group which is not the case. Thus $\mathcal{A}$ is not Connes-amenable. However as it was shown in [8, Example 10.10], $\mathcal{A}$ is approximately amenable and hence is approximately Connes-amenable.
(ii) In this part we present an example of a dual Banach algebra which is approximately Connes-amenable but is neither Connes-amenable nor approximately amenable. Note that the algebra of the preceding example was approximately amenable. Let G be an amenable, non-discrete, locally compact group and let $\mathcal{A} := \ell^1(\mathbb{N}_\vee) \oplus_1 M(G)$. If $\mathcal{A}$ is Connes-amenable, then so would be its image under the w^*-continuous natural epimorphism $\mathcal{A} \longrightarrow \ell^1(\mathbb{N}_\vee)$ which is a contradiction, since by the preceding example $\ell^1(\mathbb{N}_\vee)$ is not Connes-amenable. Therefore $\mathcal{A}$ is not Connes-amenable. Similarly if $\mathcal{A}$ is approximately amenable, then so is its homomorphic image $M(G)$ under the natural epimorphism $\mathcal{A} \longrightarrow M(G)$ which is not the case by [13, Theorem 3.1]; Therefore $\mathcal{A}$ is not approximately amenable. However, by Proposition 2.3 and Theorem 5.2 below, $\mathcal{A}$ is approximately Connes-amenable.

Proposition 2.2. Suppose that $\mathcal{A}$ is approximately Connes-amenable. Then $\mathcal{A}$ has left and right approximate identities. In particular $\mathcal{A}^2$ is dense in $\mathcal{A}$.
Proof. Let $\mathcal{X}$ be the Banach $\mathcal{A}$-bimodule whose underlying linear space is $\mathcal{A}$ equipped with the module operations $a.x = ax$ and $x.a = 0$, $(a \in \mathcal{A},\ \ x \in \mathcal{X})$.
Obviously $\mathcal{X}$ is a normal dual Banach $\mathcal{A}$-bimodule and the identity map on $\mathcal{A}$ is a $\omega^* - \omega^*$ continuous derivation. Since $\mathcal{A}$ is approximately Connes-amenable,

then there exists a net $(a_\alpha) \subseteq \mathcal{X}$ such that $a = \lim_\alpha aa_\alpha \qquad (a \in \mathcal{A})$.
This means that $\mathcal{A}$ has a right approximate identity. Similarly, one see that $\mathcal{A}$ has a left approximate identity. ∎

Let $(\mathcal{A}, \mathcal{A}_*)$ be a dual Banach algebra. Then its unitization, $\mathcal{A}^\# = \mathcal{A} \oplus_1 \mathbb{C}$ is a dual Banach algebra with predual $\mathcal{A}_* \oplus_\infty \mathbb{C}$, where $\oplus_1$ and $\oplus_\infty$ denote the ℓ^1 and ℓ^∞-direct sums respectively. More generally if $\mathcal{A}$ and $\mathcal{B}$ are dual Banach algebras, then $\mathcal{A} \oplus_1 \mathcal{B}$ is a dual Banach algebra with predual $\mathcal{A}_* \oplus_\infty \mathcal{B}_*$.

Proposition 2.3. Let $\mathcal{A}$ and $\mathcal{B}$ be dual Banach algebras.
(i) $\mathcal{A}$ is approximately Connes-amenable if and only if $\mathcal{A}^\#$ is approximately Connes-amenable.
(ii) Suppose $\mathcal{A}$ and $\mathcal{B}$ are unital. Then $\mathcal{A} \oplus_1 \mathcal{B}$ is approximately Connes-amenable if and only if $\mathcal{A}$ and $\mathcal{B}$ are approximately Connes-amenable.
Proof. (i) Suppose $\mathcal{A}$ is approximately Connes-amenable and $D : \mathcal{A}^\# \longrightarrow \mathcal{X}$ is a $\omega^* - \omega^*$ continuous derivation where $\mathcal{X}$ is a normal dual Banach $\mathcal{A}^\#$-bimodule. By [13, Lemma 2.3], $D = D_1 + ad_\eta$ where $D_1 : \mathcal{A}^\# \longrightarrow e.\mathcal{X}.e$ is a $\omega^* - \omega^*$ continuous derivation and $\eta \in \mathcal{X}$. Since $e.\mathcal{X}.e$ is a normal dual Banach $\mathcal{A}$-bimodule, then $D_1(e) = 0$ and $D_1|_\mathcal{A}$ is approximately inner; whence D is approximately inner. Thus $\mathcal{A}^\#$ is approximately Connes-amenable.
Now suppose $\mathcal{A}^\#$ is approximately Connes-amenable and $D : \mathcal{A} \longrightarrow \mathcal{X}$ is a $\omega^* - \omega^*$ continuous derivation where $\mathcal{X}$ is a normal dual Banach $\mathcal{A}$-bimodule. Set

$$\widetilde{D} : \mathcal{A}^\# \longrightarrow \mathcal{X}, \quad \widetilde{D}(a + \lambda e) = Da \quad (a \in \mathcal{A}, \lambda \in \mathbb{C}).$$

If we define $e.x = x.e = x \quad (e \in \mathcal{A}^\#, x \in \mathcal{X})$, then $\mathcal{X}$ turns into a normal dual Banach $\mathcal{A}^\#$-bimodule and $\widetilde{D}$ is a $\omega^* - \omega^*$ continuous derivation. So $\widetilde{D}$ is approximately inner, and hence so is D.
(ii) If $\mathcal{A} \oplus_1 \mathcal{B}$ is approximately Connes-amenable, then so are $\mathcal{A}$ and $\mathcal{B}$, since the natural projections on $\mathcal{A}$ and $\mathcal{B}$ are $\omega^* - \omega^*$ continuous. Conversely suppose $\mathcal{A}$ and $\mathcal{B}$ are approximately Connes-amenable and $\mathcal{X}$ is a normal dual Banach $\mathcal{A} \oplus_1 \mathcal{B}$-bimodule. Then in the following decomposition

$$\begin{aligned}\mathcal{X} = &\, e_\mathcal{A} \,.\, \mathcal{X} \,.\, e_\mathcal{A} + e_\mathcal{B} \,.\, \mathcal{X} \,.\, e_\mathcal{B} + e_\mathcal{A} \,.\, \mathcal{X} \,.\, e_\mathcal{B} + e_\mathcal{B} \,.\, \mathcal{X} \,.\, e_\mathcal{A} \\ &+ (1 - e_\mathcal{A})(1 - e_\mathcal{B}) \,.\, \mathcal{X} \,.\, e_\mathcal{A} + (1 - e_\mathcal{A})(1 - e_\mathcal{B}) \,.\, \mathcal{X} \,.\, e_\mathcal{B} \\ &+ e_\mathcal{A} \,.\, \mathcal{X} \,.\, (1 - e_\mathcal{A})(1 - e_\mathcal{B}) + e_\mathcal{B} \,.\, \mathcal{X} \,.\, (1 - e_\mathcal{A})(1 - e_\mathcal{B}) \\ &+ (1 - e_\mathcal{A})(1 - e_\mathcal{B}) \,.\, \mathcal{X} \,.\, (1 - e_\mathcal{A})(1 - e_\mathcal{B}).\end{aligned}$$

each summand is a normal dual Banach $\mathcal{A} \oplus_1 \mathcal{B}$ bimodule. With an argument similar to the proof of Proposition 2.7 in [13] one can show that $\mathcal{A} \oplus_1 \mathcal{B}$ is approximately Connes-amenable. ∎

Proposition 2.4. Suppose that $\mathcal{A}$ is a dual Banach algebra with identity. Then $\mathcal{A}$ is approximately Connes-amenable if and only if every $\omega^* - \omega^*$ continuous derivation into every unital normal dual Banach $\mathcal{A}$-bimodule $\mathcal{X}$ is approximately inner.
Proof. Suppose $D \in Z^1(\mathcal{A}, \mathcal{X})$ is a $\omega^* - \omega^*$ continuous derivation into the normal dual Banach bimodule $\mathcal{X}$. By [8, Lemma 2.3], we have $D = D_1 + ad_\eta$ where

$D_1 : \mathcal{A} \longrightarrow e.\mathcal{X}.e$ is a derivation and $\eta \in \mathcal{X}$. Since D is a $\omega^* - \omega^*-$continuous derivation and $\mathcal{X}$ is a normal dual Banach bimodule then D_1 is $\omega^* - \omega^*-$continuous and $e.\mathcal{X}.e$ is normal. So by assumption D_1 is approximately inner, and therefore $\mathcal{A}$ is approximately Connes-amenable. The converse holds obviously. ∎

3 approximate normal virtual diagonals

Throughout this section we assume that $\mathcal{A}$ is a dual Banach algebra with identity. See Remark 3.4 at the end of this section regarding the non-unital case.

Let $L^2(\mathcal{A}, \mathbb{C})$ be the space of all bounded bilinear functionals on $\mathcal{A}$ and $L^2_{\omega^*}(\mathcal{A}, \mathbb{C})$ be the space of separately ω^* continuous elements of $L^2(\mathcal{A}, \mathbb{C})$. Following the terminology of [10, 23], we turn $L^2(\mathcal{A}, \mathbb{C})$ into a Banach $\mathcal{A}$-bimodule through the identification $L^2(\mathcal{A}, \mathbb{C}) \simeq (\mathcal{A}\widehat{\otimes}\mathcal{A})^*$. Then the module actions of $\mathcal{A}$ on $L^2(\mathcal{A}, \mathbb{C})$ are as follow.

$$(a.F)(b,c) = F(b,ca), \quad (F.a)(b,c) = F(ab,c), \quad a,b,c \in \mathcal{A}, \quad F \in L^2(\mathcal{A}, \mathbb{C}).$$

Clearly, $L^2_{\omega^*}(\mathcal{A}, \mathbb{C})$ is a Banach $\mathcal{A}$-submodule of $L^2(\mathcal{A}, \mathbb{C})$. Moreover we have a natural $\mathcal{A}$-bimodule map

$$\theta : \mathcal{A} \otimes \mathcal{A} \longrightarrow L^2_{\omega^*}(\mathcal{A}, \mathbb{C})^*, \quad \theta(a \otimes b)(F) = F(a,b).$$

Since $\mathcal{A}_* \otimes \mathcal{A}_* \subseteq L^2_{\omega^*}(\mathcal{A}, \mathbb{C})$ and $\mathcal{A}_* \otimes \mathcal{A}_*$ separates points of $\mathcal{A} \otimes \mathcal{A}$, then θ is one-to-one. We will identify $\mathcal{A} \otimes \mathcal{A}$ with its image, writing

$$\mathcal{A} \otimes \mathcal{A} \subseteq L^2_{\omega^*}(\mathcal{A}, \mathbb{C})^*.$$

The map $\Delta_{\mathcal{A}}$ is defined as follows.

$$\Delta_{\mathcal{A}} : \mathcal{A}\widehat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}, \qquad a \otimes b \longmapsto ab \quad (a,b \in \mathcal{A}).$$

Since multiplication in a dual Banach algebra is separately $\omega^* - \omega^*$-continuous, we have

$$\Delta^*_{\mathcal{A}}(\mathcal{A}_*) \subset L^2_{\omega^*}(\mathcal{A}, \mathbb{C}).$$

So the restriction of $\Delta^{**}_{\mathcal{A}}$ to $L^2_{\omega^*}(\mathcal{A}, \mathbb{C})^*$ turns into a Banach $\mathcal{A}$-bimodule homomorphism

$$\Delta_{\omega^*} : L^2_{\omega^*}(\mathcal{A}, \mathbb{C})^* \longrightarrow \mathcal{A}.$$

Suppose $F \in L^2_{\omega^*}(\mathcal{A}, \mathbb{C})$ and $M \in L^2_{\omega^*}(\mathcal{A}, \mathbb{C})^*$. We use the notation,

$$\int F(a,b)dM(a,b) = \int FdM := \langle M, F \rangle.$$

More generally given a dual Banach space $\mathcal{X}^*$ and a bounded bilinear function $F : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{X}^*$ such that $a \longrightarrow F(a,b)$ and $b \longrightarrow F(a,b)$ are $\omega^* - \omega^*$-continuous, $\int FdM \in \mathcal{X}^*$ is defined by

$$\langle \int FdM, x \rangle = \int \langle F(a,b), x \rangle dM(a,b) \quad (x \in \mathcal{X}).$$

Sometimes we also use the term $\int F(a,b)dM(a,b)$ for $\int FdM$.

Definition. A net (M_α) in $L^2_{\omega^*}(\mathcal{A},\mathbb{C})^*$ is called an approximate normal, virtual diagonal for $\mathcal{A}$ if for every $a \in \mathcal{A}$

$$a.M_\alpha - M_\alpha.a \longrightarrow 0 \quad and \quad \Delta_{\omega^*}(M_\alpha) \longrightarrow e,$$

the limits being taken in norm.
It is well known that every dual Banach algebra with a normal virtual diagonal is Connes-amenable [26]. In the following theorem we extend this result to approximate Connes-amenability.
Theorem 3.1. If $\mathcal{A}$ has an approximate normal, virtual diagonal $\{M_\alpha\}$, then $\mathcal{A}$ is approximately Connes-amenable.
Proof. Suppose $\mathcal{X}$ is a normal dual Banach $\mathcal{A}$-bimodule with predual $\mathcal{X}_*$ and $D \in Z^1(\mathcal{A},\mathcal{X})$ is $\omega^* - \omega^*$-continuous. Since $\mathcal{A}$ has an identity, by Proposition 2.4 we can assume that $\mathcal{X}$ is unital. Since the bilinear map

$$F : \mathcal{A} \times \mathcal{A} \longrightarrow \mathcal{X}, \quad F(a,b) = Da.b$$

is separately $\omega^* - \omega^*$ continuous, then by the preceding remark we may define

$$\phi_\alpha = \int F(a,b)dM_\alpha(a,b) = \int Da.b\, dM_\alpha \in \mathcal{X}.$$

For $c \in \mathcal{A}$, $x \in \mathcal{X}_*$ we have

$$\langle c.\phi_\alpha, x\rangle = \langle \phi_\alpha, x.c\rangle = \int \langle c.Da.b, x\rangle dM_\alpha(a,b) = \langle \int c.Da.b\, dM_\alpha(a,b), x\rangle.$$

Therefore

$$c.\phi_\alpha = \int c.Da.b\, dM_\alpha(a,b) \tag{1}$$

and similarly

$$\phi_\alpha.c = \int Da.bc\, dM_\alpha(a,b). \tag{2}$$

So if we define $F_x \in L^2_{\omega^*}(\mathcal{A},\mathbb{C})$ by $F_x(a,b) = \langle Da.b, x\rangle$, then the following relations hold.

$$\int \langle D(ca).b, x\rangle dM_\alpha(a,b) = \int F_x.c(a,b)dM_\alpha(a,b) = \langle c.M_\alpha, F_x\rangle \tag{3}$$

$$\int \langle Da.bc, x\rangle dM_\alpha(a,b) = \int c.F_x(a,b)dM_\alpha(a,b) = \langle M_\alpha.c, F_x\rangle \tag{4}$$

By (3) and (4) we have

$$|\langle \int D(ca).b\, dM_\alpha(a,b) - \int Da.bc\, dM_\alpha(a,b), x\rangle| \leq \|c.M_\alpha - M_\alpha.c\| \|F_x\|.$$

So

$$\| \int Dca.b\, dM_\alpha(a,b) - \int Da.bc\, dM_\alpha(a,b)\| \leq \|c.M_\alpha - M_\alpha.c\|\|D\|\|a\|\|b\|. \quad (5)$$

If we define $G : \mathcal{A} \times \mathcal{A} \longrightarrow L^2_{\omega^*}(\mathcal{A},\mathbb{C})^*$ by $G(a,b) = a \otimes b$, then for every F in $L^2_{\omega^*}(\mathcal{A},\mathbb{C})$ we have

$$\langle \int G dM_\alpha, F\rangle = \int \langle G(a,b), F\rangle dM_\alpha(a,b) = \int F(a,b) dM_\alpha(a,b) = \langle M_\alpha, F\rangle.$$

So $M_\alpha = \int (a \otimes b) dM_\alpha(a,b)$. Now for every $t \in \mathcal{A}_*$,

$$\langle \Delta_{\omega^*}(M_\alpha), t\rangle = \langle M_\alpha, \Delta^*_{\mathcal{A}}(t)\rangle = \int \langle a \otimes b, \Delta^*_{\mathcal{A}}(t)\rangle dM_\alpha(a,b) = \langle \int ab\, dM_\alpha(a,b), t\rangle.$$

Thus

$$\Delta_{\omega^*}(M_\alpha) = \int ab\, dM_\alpha(a,b). \quad (6)$$

Moreover we have

$$\begin{aligned}\langle Dc. \int ab\, dM_\alpha(a,b), x\rangle &= \int \langle ab, x.Dc\rangle dM_\alpha(a,b) \\ &= \int \langle Dc.ab, x\rangle dM_\alpha(a,b) = \langle \int Dc.ab\, dM_\alpha(a,b), x\rangle.\end{aligned}$$

Therefore

$$Dc. \int ab\, dM_\alpha(a,b) = \int Dc.ab\, dM_\alpha(a,b). \quad (7)$$

Now by (1), (2) and (7),

$$\begin{aligned} c.\phi_\alpha - \phi_\alpha.c &= \int D(ca).b\, dM_\alpha(a,b) - \int Dc.ab\, dM_\alpha(a,b) - \int Da.bc\, dM_\alpha(a,b) \\ &= \int D(ca).b\, dM_\alpha(a,b) - \int Da.bc\, dM_\alpha(a,b) - Dc. \int ab\, dM_\alpha(a,b).\end{aligned}$$

Applying our assumption and (5) to the above identity shows that

$$lim_\alpha(\phi_\alpha.c - c.\phi_\alpha) = Dc \quad (c \in \mathcal{A}).$$

Therefore $\mathcal{A}$ is approximately Connes-amenable. ■

We do not know whether the converse of Theorem 3.1 is true in general. However we show in Sections 4 and 5 that the converse is true for von Neumann algebras and measure algebras. For approximate strong Connes-amenability, the corresponding question is answered in the next theorem which is the approximate version of [26, Theorem 4.7]. First we need to give a precise definition of this new concept.

Definition. $\mathcal{A}$ is called approximately strongly Connes-amenable if for each unital Banach $\mathcal{A}$-bimodule $\mathcal{X}$, every $\omega^* - \omega^*$ continuous derivation $D \in Z^1(\mathcal{A}, \mathcal{X}^*)$ whose range consists of normal elements is approximately inner.

Theorem 3.2. The following conditions are equivalent.
(i) $\mathcal{A}$ has an approximate normal, virtual diagonal.
(ii) $\mathcal{A}$ is approximately strongly Connes-amenable.
Proof. $(i) \Longrightarrow (ii)$. This is similar to Theorem 3.1.
$(ii) \Longrightarrow (i)$ Since Δ_{ω^*} is $\omega^* - \omega^*$ continuous then $ker\Delta_{\omega^*}$ is ω^*-closed and

$$(L^2_{\omega^*}(\mathcal{A}, \mathbb{C})/^{\perp}ker\Delta_{\omega^*})^* = ker\Delta_{\omega^*}.$$

So $ker\Delta_{\omega^*}$ is a normal dual $\mathcal{A}$-module and $ad_{e\otimes e}$ attains its values in the normal elements of $ker\Delta_{\omega^*}$. By assumption there exists a net $(N_\alpha) \subset ker\Delta_{\omega^*}$ such that

$$ad_{e\otimes e}(a) = \lim_{\alpha} a.N_\alpha - N_\alpha.a \quad (a \in \mathcal{A}).$$

Let $M_\alpha = e \otimes e - N_\alpha$. It follows that

$$a.M_\alpha - M_\alpha.a \longrightarrow 0 \quad and \quad \Delta_{\omega^*}(M_\alpha) \longrightarrow e \quad (a \in \mathcal{A}).$$

Therefore (M_α) is an approximate normal virtual diagonal for $\mathcal{A}$. ∎

We saw that dual Banach algebras with an approximate normal, virtual diagonal are approximately Connes-amenable, but we conjecture that as its classical case [see 31], the converse is likely to be false in general. We now modify the definition of approximate normal, virtual diagonal and obtain the desired characterization of approximate Connes-amenability. Let $\mathcal{A}$ be a dual Banach algebra with predual $\mathcal{A}_*$ and let $\Delta : \mathcal{A}\widehat{\otimes}\mathcal{A} \longrightarrow \mathcal{A}$ be the multiplication map. From [30, Corollary 4.6], we conclude that Δ^* maps $\mathcal{A}_*$ into $\sigma WC((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)$. Consequently, Δ^{**} induces the homomorphism

$$\Delta_{\sigma WC} : \sigma WC((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^* \longrightarrow \mathcal{A}.$$

With these preparations made, we can now characterize approximately Connes-amenable, dual Banach algebras through the existence of certain approximate normal, virtual diagonals. This is indeed an approximate version of [30, Theorem 4.8].

Definition. An approximate $\sigma WC-$virtual diagonal for $\mathcal{A}$ is a net (M_α) in $\sigma WC((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^*$ such that

$$a.M_\alpha - M_\alpha.a \longrightarrow 0 \quad and \quad \Delta_{\sigma WC}(M_\alpha) \longrightarrow e \quad (a \in \mathcal{A}),$$

the limits being taken in norm.

Theorem 3.3. The following conditions are equivalent.
(i) $\mathcal{A}$ is approximately Connes-amenable.
(ii) There is an approximate $\sigma WC-$virtual diagonal for $\mathcal{A}$.

Proof. $(i) \Longrightarrow (ii)$ The map

$$D : \mathcal{A} \longrightarrow \sigma WC((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^*, \quad a \longmapsto a \otimes e - e \otimes a$$

is a well defined bounded derivation, since $\mathcal{A}\widehat{\otimes}\mathcal{A}$ can be embedded canonically into $\sigma WC((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^*$. Since the dual module $\sigma WC((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^*$ is normal, then it follows that D is $\omega^* - \omega^*$-continuous. Clearly D attains its values in the ω^*-closed submodule $ker\Delta_{\sigma WC}$ which is a normal dual Banach $\mathcal{A}$-module. So there is a net $(N_\alpha) \subset ker\Delta_{\sigma WC}$ such that

$$Da = \lim_\alpha (a.N_\alpha - N_\alpha.a) \quad (a \in \mathcal{A}).$$

Letting $M_\alpha = e \otimes e - N_\alpha$, we see that it is an approximate $\sigma WC-$virtual diagonal for $\mathcal{A}$.
$(ii) \Longrightarrow (i)$ Let $\mathcal{X}$ be a normal dual Banach $\mathcal{A}$-bimodule. By Proposition 2.4 we may assume that $\mathcal{X}$ is unital. Let $D \in Z^1(\mathcal{A}, \mathcal{X})$ be a $\omega^* - \omega^*$-continuous derivation. Define

$$\theta_D : \mathcal{A}\widehat{\otimes}\mathcal{A} \longrightarrow \mathcal{X}, \quad a \otimes b \longmapsto a.Db.$$

By [28, Lemma 4.9], θ_D^* maps the predual $\mathcal{X}_*$ into $\sigma WC((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)$. Therefore $(\theta^*|_{\mathcal{X}_*})^*$ maps $\sigma WC((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^*$ into $\mathcal{X}$. Let $(M_\alpha) \subset \sigma WC((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^*$ be an approximate $\sigma WC-$virtual diagonal for $\mathcal{A}$ and let $x_\alpha = (\theta^*|_{\mathcal{X}_*})^*(M_\alpha)$. Observe that $\mathcal{A}\widehat{\otimes}\mathcal{A}$ is ω^*-dense in $\sigma WC((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)^*$. So for every α there is a net (u^α_β) in $\mathcal{A}\widehat{\otimes}\mathcal{A}$ such that $M_\alpha = \omega^* - \lim_\beta u^\beta_\alpha$. Suppose $c \in \mathcal{A}$ and $t \in \mathcal{X}_*$ are arbitrary. One can easily check that

$$\begin{aligned} x_\alpha.c &= \sigma(\mathcal{X}, \mathcal{X}_*) - \lim_\beta \theta_D(u^\beta_\alpha).c, \quad \text{and} \\ c.x_\alpha &= \sigma(\mathcal{X}, \mathcal{X}_*) - \lim_\beta c.\theta_D(u^\beta_\alpha) = (\theta^*|_{\mathcal{X}_*})^*(c.M_\alpha). \end{aligned} \tag{8}$$

On the other hand by [30, Lemma 4.6], $\Delta^*(\mathcal{A}_*) \subseteq \sigma WC((\mathcal{A}\widehat{\otimes}\mathcal{A})^*)$ and hence

$$\Delta_{\sigma WC}(M_\alpha) = \sigma(\mathcal{A}, \mathcal{A}_*) - lim_\beta \Delta(u^\beta_\alpha). \tag{9}$$

Suppose $u^\beta_\alpha = \Sigma_k a^{\alpha\beta}_k \otimes b^{\alpha\beta}_k$. Using identities (8) and (9), we obtain

$$\begin{aligned} c.x_\alpha - x_\alpha.c &= (\theta^*|_{\mathcal{X}_*})^*(c.M_\alpha) - \omega^* - \lim_\beta \Sigma_k a^{\alpha\beta}_k.Db^{\alpha\beta}_k.c \\ &= (\theta^*|_{\mathcal{X}_*})^*(c.M_\alpha) - \lim_\beta \Sigma_k a^{\alpha\beta}_k.D(b^{\alpha\beta}_k c) + \lim_\beta \Sigma_k a^{\alpha\beta}_k b^{\alpha\beta}_k.Dc \\ &= (\theta^*|_{\mathcal{X}_*})^*(c.M_\alpha - M_\alpha.c) + \Delta_{\sigma WC}(M_\alpha).Dc. \end{aligned} \tag{10}$$

By our assumption and (10), we have

$$Dc = \lim_\alpha (c.x_\alpha - x_\alpha.c) \quad (c \in \mathcal{A}).$$

This implies that $\mathcal{A}$ is approximately Connes-amenable. ∎

Remark 3.4. In the light of proposition 2.3 if we modify the definition of approximate normal virtual diagonal to the following one, then Theorem 3.1 holds also in the case that $\mathcal{A}$ does not have an identity.
"Let $\mathcal{A}$ be a dual Banach algebra (not necessarily unital). A net (M_α) in $L^2_{\omega^*}(\mathcal{A}^\#, \mathbb{C})^*$ is called an approximate normal, virtual diagonal for $\mathcal{A}$ if for every $a \in \mathcal{A}^\#$

$$a.M_\alpha - M_\alpha.a \longrightarrow 0 \quad and \quad \Delta_{\omega^*}(M_\alpha) \longrightarrow e.$$

4 Approximate Connes-amenability of von Neumann algebras

In this section we prove the "approximate" analog of the main result of Effros in [10]. First recall some notations from [10]. Let $\mathcal{A}$ be a von Neumann algebra. We call a map $F \in L^2(\mathcal{A}, \mathbb{C})$ reduced if there exist states $p, q \in \mathcal{A}_*$ and a constant K such that for every $a, b \in \mathcal{A}$,

$$|F(a,b)| \leq Kp(aa^*)^{1/2}q(b^*b)^{1/2}.$$

The set $L^2_{\omega^*,0}(\mathcal{A}, \mathbb{C})$ of all such bilinear functionals is an $\mathcal{A}$- submodule of $L^2(\mathcal{A}, \mathbb{C})$ and $L^2_{\omega^*,0}(\mathcal{A}, \mathbb{C})^*$ is a normal dual Banach $\mathcal{A}-$bimodule [10, Lammas 2.1 and 2.2]. Also $\mathcal{A}_* \otimes \mathcal{A}_* \subseteq L^2_{\omega^*,0}(\mathcal{A}, \mathbb{C})$ and $\mathcal{A} \otimes \mathcal{A}$ is identified with an $\mathcal{A}-$submodule of $L^2_{\omega^*,0}(\mathcal{A}, \mathbb{C})^*$. If $\Delta : \mathcal{A} \otimes \mathcal{A} \longrightarrow \mathcal{A}$ is the multiplication map, then Δ^* maps $\mathcal{A}_*$ into $L^2_{\omega^*,0}(\mathcal{A}, \mathbb{C})$ and consequently Δ^{**} drops to an $\mathcal{A}$-bimodule homomorphism $\Delta_{\omega^*,0} : L^2_{\omega^*,0}(\mathcal{A}, \mathbb{C})^* \longrightarrow \mathcal{A}$.
We need the following Lemma in the proof of the next Theorem.

Lemma 4.1. [10, Lemma 2.3] Suppose $\mathcal{A}$ is a finite or properly infinite von Neumann algebra. Then there is a $\omega^* - \omega^*$ continuous linear $\mathcal{A}-$bimodule map

$$\Phi : L^2_{\omega^*,0}(\mathcal{A}, \mathbb{C})^* \longrightarrow L^2_{\omega^*}(\mathcal{A}, \mathbb{C})^*$$

such that $\Delta_{\omega^*} \circ \Phi = \Delta_{\omega^*,0}$.

Theorem 4.2. A von Neumann algebra $\mathcal{A}$ is approximately Connes-amenable if and only if it has an approximate normal virtual diagonal.
Proof. If $\mathcal{A}$ has an approximate normal virtual diagonal then by Theorem 3.1, $\mathcal{A}$ is approximately Connes-amenable.
Conversely, suppose $\mathcal{A}$ is approximately Connes-amenable. The dual Banach $\mathcal{A}-$bimodule
$L^2_{\omega^*,0}(\mathcal{A}, \mathbb{C})^*$ is normal and hence the bounded derivation D defined by

$$D : \mathcal{A} \longrightarrow L^2_{\omega^*,0}(\mathcal{A}, \mathbb{C})^* \quad , \quad a \longmapsto a \otimes e_{\mathcal{A}} - e_{\mathcal{A}} \otimes a$$

is $\omega^* - \omega^*$ continuous. Since $\Delta_{\omega^*,0}$ is $\omega^* - \omega^*$ continuous, then $ker\Delta_{\omega^*,0}$ is a ω^*-closed submodule of $L^2_{\omega^*,0}(\mathcal{A}, \mathbb{C})^*$ and we have a Banach $\mathcal{A}-$bimodule isomorphism

$$(L^2_{\omega^*,0}(\mathcal{A}, \mathbb{C})/^{\perp}ker\Delta_{\omega^*,0})^* \cong ker\Delta_{\omega^*,0},$$

As a result $ker\Delta_{\omega^*,0}$ is a normal dual Banach $\mathcal{A}-$bimodule and $D(\mathcal{A}) \subseteq \ker \Delta_{\omega^*,0}$. Since $\mathcal{A}$ is approximately Connes-amenable, then there exists a net $(N_\alpha) \subseteq \ker \Delta_{\omega^*,0}$ such that

$$Da = \lim_\alpha (a.N_\alpha - N_\alpha.a) \quad (a \in \mathcal{A}).$$

If we set $M_\alpha = e_\mathcal{A} \otimes e_\mathcal{A} - N_\alpha$, then

$$\lim_\alpha (a.M_\alpha - M_\alpha.a) = 0 \quad (a \in \mathcal{A}), \text{ and}$$
$$\lim_\alpha \Delta_{\omega^*,0}(M_\alpha) = \Delta_{\omega^*,0}(e_\mathcal{A} \otimes e_\mathcal{A}) = e_\mathcal{A}.$$

If $\mathcal{A}$ is finite or properly infinite then by Lemma 4.1, $\widetilde{M_\alpha} = \Phi(M_\alpha) \in L^2_{\omega^*}(\mathcal{A}, \mathbb{C})^*$ is an approximate normal virtual diagonal because of the following identities

$$\lim_\alpha (a.\widetilde{M_\alpha} - \widetilde{M_\alpha}.a) = \lim_\alpha \Phi(a.M_\alpha - M_\alpha.a) = 0 \quad (a \in \mathcal{A}),$$
$$\lim_\alpha \Delta_{\omega^*}(\widetilde{M_\alpha}) = \lim_\alpha \Delta_{\omega^*} \circ \Phi(M_\alpha) = \lim_\alpha \Delta_{\omega^*,0}(M_\alpha) = e_\mathcal{A}.$$

In the general case, there are central projections $p_1, p_2 \in \mathcal{A}$, such that $e_\mathcal{A} = p_1 + p_2$, $p_1\mathcal{A}$ is a finite von Neumann algebra and $p_2\mathcal{A}$ is a properly infinite von Neumann algebra. Since $\mathcal{A}$ is approximately Connes-amenable and $\mathcal{A} = p_1\mathcal{A} \oplus p_2\mathcal{A}$, then it is easy to see that the von Neumann algebras $p_1\mathcal{A}$ and $p_2\mathcal{A}$ are approximately Connes-amenable. Therefore there exist nets $(M_\alpha) \subseteq L^2_{\omega^*}(p_1\mathcal{A}, \mathbb{C})^*$ and $(M_\beta) \subseteq L^2_{\omega^*}(p_2\mathcal{A}, \mathbb{C})^*$ such that,

$$\lim_\alpha (a.M_\alpha - M_\alpha a) = 0 \quad (a \in p_1\mathcal{A}) \text{ and } \Delta_{\omega^*}(M_\alpha) \longrightarrow p_1, \quad (1)$$
$$\lim_\beta (a.M_\beta - M_\beta a) = 0 \quad (a \in p_2\mathcal{A}) \text{ and } \Delta_{\omega^*}(M_\beta) \longrightarrow p_2. \quad (2)$$

For each $F \in L^2_{\omega^*}(\mathcal{A}, \mathbb{C})$ define

$$F_i(a,b) = F(a,b) \quad \textit{and} \quad a, b \in p_i\mathcal{A}. \quad (i = 1, 2)$$

Clearly $F_i \in L^2_{\omega^*}(p_i\mathcal{A}, \mathbb{C})$. Now define the net $(M_{(\alpha,\beta)}) \subseteq L^2_{\omega^*}(\mathcal{A}, \mathbb{C})^*$ by $M_{(\alpha,\beta)} = M_\alpha \oplus M_\beta$. Then

$$\langle M_{(\alpha,\beta)}, F \rangle = \langle M_\alpha \oplus M_\beta, F \rangle = \langle M_\alpha, F_1 \rangle + \langle M_\beta, F_2 \rangle.$$

For each $c \in \mathcal{A}$ and $a, b \in p_i\mathcal{A}$,

$$(Fc)_i(a,b) = (Fc)(a,b) = F(c_i a, b) = F_i c_i(a,b),$$
$$(cF)_i(a,b) = (cF)(a,b) = F(a, bc_i) = c_i F_i(a,b).$$

where $c = c_1 + c_2$, $c_i \in p_i\mathcal{A}$, $i = 1, 2$. Hence

$$\begin{aligned}
\langle c.M_{(\alpha,\beta)}, F \rangle &= \langle M_\alpha \oplus M_\beta, Fc \rangle \\
&= \langle M_\alpha, (Fc)_1 \rangle + \langle M_\beta, (Fc)_2 \rangle \\
&= \langle M_\alpha, F_1 c_1 \rangle + \langle M_\beta, F_2 c_2 \rangle \\
&= \langle c_1.M_\alpha, F_1 \rangle + \langle c_2.M_\beta, F_2 \rangle \\
&= \langle c_1.M_\alpha \oplus c_2.M_\beta, F \rangle.
\end{aligned}$$

Therefore $c.M_{(\alpha,\beta)} = c_1.M_\alpha \oplus c_2.M_\beta$. Similarly $M_{(\alpha,\beta)}.c = M_\alpha.c_1 \oplus M_\beta.c_2$. By (1) and (2) we have,

$$\begin{aligned}\lim_{(\alpha,\beta)}(c.M_{(\alpha,\beta)} - M_{(\alpha,\beta)}.c) &= \lim_{(\alpha,\beta)}(c_1.M_\alpha \oplus c_2.M_\beta - M_\alpha.c_1 \oplus M_\beta.c_2)\\ &= \lim_{(\alpha,\beta)}((c_1.M_\alpha - M_\alpha.c_1) \oplus (c_2.M_\beta - M_\beta.c_2)) = 0.\end{aligned}$$

For $a_* \in \mathcal{A}_*$ define $a_{*,i} \in (p_i\mathcal{A})_*$ by $a_{*,i}(b) = a_*(b)$ $(b \in p_i\mathcal{A})$. So for each $a, b \in p_iA$,

$$\Delta^*(a_{*,i})(a,b) = a_{*,i}(ab) = a_*(ab) = \Delta^*(a_*)(a,b) = (\Delta^*(a_*))_i(a,b).$$

Thus,

$$\begin{aligned}\langle \Delta_{\omega^*}(M_{(\alpha,\beta)}), a_*\rangle &= \langle M_\alpha \oplus M_\beta, \Delta^*(a_*)\rangle\\ &= \langle M_\alpha, \Delta^*(a_*)_1\rangle + \langle M_\beta, \Delta^*(a_*)_2\rangle\\ &= \langle M_\alpha, \Delta^*(a_{*,1})\rangle + \langle M_\beta, \Delta^*(a_{*,2})\rangle\\ &= \langle \Delta_{\omega^*}(M_\alpha), a_{*,1}\rangle + \langle \Delta_{\omega^*}(M_\beta), a_{*,2}\rangle\\ &= \langle \Delta_{\omega^*}(M_\alpha) \oplus \Delta_{\omega^*}(M_\beta), a_{*,1} \oplus a_{*,2}\rangle.\end{aligned}$$

As a result $\Delta_{\omega^*}(M_{(\alpha,\beta)}) = \Delta_{\omega^*}(M_\alpha) \oplus \Delta_{\omega^*}(M_\beta)$. Also

$$\Delta_{\omega^*}(M_{(\alpha,\beta)}) - e_{\mathcal{A}} = (\Delta_{\omega^*}(M_\alpha) - p_1) \oplus (\Delta_{\omega^*}(M_\beta) - p_2). \quad (3)$$

Finally based on (1),(2) and (3) we have

$$\lim_{(\alpha,\beta)} \Delta_{\omega^*}(M_{(\alpha,\beta)}) \longrightarrow e_{\mathcal{A}}.$$

It follows that net $(M_{(\alpha,\beta)})$ is an approximate normal virtual diagonal for $\mathcal{A}$. ∎

5 Approximate Connes-amenability of $M(G)$

In this section we characterize approximate Connes-amenable measure algebras on locally compact groups. Throughout this section G is a locally compact group, G^{op} denotes the same group, with reversed multiplication and $\Delta : M(G)\widehat{\otimes}M(G) \longrightarrow M(G)$ is the multiplication map. We recall some terminology from [28]. A bounded function $f : G \times G^{op} \longrightarrow \mathbb{C}$ is called separately C_0 if for each $x \in G$, the function $G^{op} \longrightarrow \mathbb{C}$, $y \mapsto f(x,y)$, belongs to $C_0(G^{op})$, and for each $y \in G^{op}$, the function $G \longrightarrow \mathbb{C}$, $x \mapsto f(x,y)$ belongs to $C_0(G)$. The collection of all separately C_0-functions is denoted by $SC_0(G \times G^{op})$. Let $LUC(G)$ be the commutative C^*-algebra of left uniformly continuous functions on G and G_{LUC} be its character space. The set

$$\{f \in LUC(G \times G^{op}) \ : \ \phi.f \in SC_0(G \times G^{op}) \text{ for all } \phi \in (G \times G^{op})_{LUC}\}$$

which is denoted by $LUCSC_0(G \times G^{op})$ is a closed $M(G)$-submodule of $SC_0(G \times G^{op})$ whose dual is a normal dual Banach $M(G)$-bimodule. Moreover

$\Delta_* = \Delta^*|_{C_0(G)}$ maps $C_0(G)$ into $LUCSC_0(G \times G^{op})$ [28, Theorem 4.4]. Therefore Δ^{**} turns into an $M(G)$-bimodule homomorphism $\widetilde{\Delta} : LUCSC_0(G \times G^{op})^* \longrightarrow M(G)$.

Proposition 5.1. If $M(G)$ is approximately Connes-amenable, then G is amenable.
Proof. First we show that here is a net $(M_\alpha) \subseteq LUCSC_0(G \times G^{op})^*$ such that

$$\mu.M_\alpha - M_\alpha.\mu \longrightarrow 0 \quad (\mu \in M(G)) \quad \text{and} \quad \widetilde{\Delta}(M_\alpha) \longrightarrow \delta_e.$$

It is easy to see that the map

$$D : M(G) \longrightarrow LUCSC_0(G \times G^{op})^*, \quad \mu \longmapsto \mu \otimes \delta_e - \delta_e \otimes \mu$$

is a bounded derivation. By [28, Proposition 3.2] the maps $\mu \longmapsto \mu \otimes \delta_e$ and $\mu \longmapsto \delta_e \otimes \mu$ form $M(G)$ into $M(G \times G^{op})$ are $\omega^* - \omega^*$continuous and hence so is D. Moreover $D(M(G)) \subseteq ker\widetilde{\Delta}$. Since $\widetilde{\Delta}$ is a $\omega^* - \omega^*$ continuous bimodule homomorphism, then $ker\widetilde{\Delta}$ is a ω^*- closed submodule and

$$(LUCSC_0(G \times G^{op})/{}^{\perp}ker\widetilde{\Delta})^* \cong ker\widetilde{\Delta}$$

as Banach $M(G)$-bimodules. By [28, Theorem 4.4(ii)], $LUCSC_0(G \times G^{op})^*$ is a normal dual Banach $M(G)$-bimodule, and so is $ker\widetilde{\Delta}$. Since $M(G)$ is approximately Connes-amenable, then there is a net $(N_\alpha) \subseteq ker\widetilde{\Delta}$ such that $D\mu = \lim_\alpha \mu.N_\alpha - N_\alpha.\mu \quad (\mu \in M(G))$. The identities $\mu.N_\alpha = (\mu \otimes \delta_e) * N_\alpha$ and $N_\alpha.\mu = N_\alpha * (\delta_e \otimes \mu)$ imply that if we set $M_\alpha = \delta_e \otimes \delta_e - N_\alpha$, then the net (M_α) has the required properties.
Since $\widetilde{\Delta}(M_\alpha) \longrightarrow \delta_e$, then we can suppose that $M_\alpha \neq 0$ for every α and if we consider M_α as a measure on the character space of the commutative C^*-algebra $LUCSC_0(G \times G^{op})$, then the total variation $|M_\alpha|$ is a non-zero element of $LUCSC_0(G \times G^{op})^*$. Observe that

$$|\delta_g.M_\alpha| = |(\delta_g \otimes \delta_e) * M_\alpha| = (\delta_g \otimes \delta_e) * |M_\alpha| = \delta_g.|M_\alpha|,$$

and similarly $|M_\alpha.\delta_g| = |M_\alpha|.\delta_g$. Thus

$$\|\delta_g.|M_\alpha| - |M_\alpha|.\delta_g\| = \||\delta_g.M_\alpha| - |M_\alpha.\delta_g|\| \leq \|M_\alpha.\delta_g - \delta_g.M_\alpha\| \longrightarrow 0.$$

On the other hand the convergence $\widetilde{\Delta}(M_\alpha) \longrightarrow \delta_e$ implies that the net $(1/\|M_\alpha\|)$ is bounded and so if we set $N_\alpha = |M_\alpha|/\|M_\alpha\|$, then $\delta_g.N_\alpha - N_\alpha.\delta_g \longrightarrow 0$. Let N be a ω^*- cluster point of (N_α). Then $\delta_g.N = N.\delta_g$ for every $g \in G$. As in the proof of [28, Theorem 5.3], $LUC(G \times G^{op})$ can be considered as a C^*-subalgebra of $LUCSC_0(G \times G^{op})^{**}$; So in particular $\langle f, N\rangle$ is well defined for each $f \in LUC(G \times G^{op})$. Note that the embedding of $LUC(G \times G^{op})$ into $LUCSC_0(G \times G^{op})^{**}$ is an $M(G)-$bimodule homomorphism. Define

$$m : LUC(G) \longrightarrow C \quad , \quad f \longmapsto \langle N, f \otimes 1\rangle.$$

Since $f \otimes 1 \in LUC(G \times G^{op})$, then m is a well-defined, positive, linear functional whose normalization is a left invariant mean on $LUC(G)$ as in the proof of [28, Theorem 5.3]. Therefore G is amenable. ■

Combination of the preceding proposition, Theorem 3.1 and the main result of [29] leads to the following result.

Theorem 5.2. The following conditions are equivalent.
(i) G is amenable.
(i) $M(G)$ is approximately Connes-amenable.
(iii) $M(G)$ is Connes-amenable.
(iv) $M(G)$ has an approximate normal, virtual diagonal.
The algebra $WAP(G)$ of weakly almost periodic functions on G is a commutative C^*-algebra which is a left introverted subspace of $L^\infty(G)$ [34, Lemma 6.3]. Thus $WAP(G)^*$ is a dual Banach algebra which is identified with $WAP(L^\infty(G))^*$. In the next proposition we identify the relationship between approximate amenability of $\mathcal{A}$ and approximate Connes-amenability of $WAP(\mathcal{A}^*)^*$ in the special case of group algebras.

Proposition 5.3. G is amenable if and only if $WAP(L^\infty(G))^*$ is approximately Connes-amenable.
Proof. Suppose G is amenable. By [21, Theorem 2.5] and [13, Theorem 3.2], G is amenable if and only if $L^1(G)$ is approximately amenable. Since the image of $L^1(G)$ is ω^*-dense in WAP$(L^\infty(G))^*$, then WAP$(L^1(G)^*)^*$ is Connes-amenable [26, Proposition 4.2(i)].
Conversely suppose $WAP(L^\infty(G))^*$ is approximately Connes-amenable. Since $C_0(G) \subseteq WAP(G)$, the restriction map from WAP$(G)^*$ onto $M(G)$ is a $\omega^* - \omega^*$ continuous algebra homomorphism. Consequently $M(G)$ is approximately Connes-amenable and by Theorem 5.2, G is amenable. ■

6 Approximate Connes-amenability of $PM_p(G)$

In this section we study approximate Connes-amenability of the algebra of p-pseudomeasures. First we need a more general result in the abstract setting. Let $\mathcal{A}$ be a Banach algebra, $\mathcal{B}$ be a dual Banach algebra and $\Theta : \mathcal{A} \longrightarrow \mathcal{B}$ be a homomorphism. We can consider $\mathcal{B}$ as a Banach $\mathcal{A}$-bimodule in a natural way and then we can equip $\mathcal{X} := \mathcal{B}\hat{\otimes}\mathcal{B}_*$ with the $\mathcal{A}$-bimodule operation

$$a\,.\,(b\otimes\phi) := b\otimes\Theta(a)\,.\,\phi \;\text{ and }\; (b\otimes\phi)\,.\,a := b\otimes\phi\,.\,\Theta(a) \;\; (a\in\mathcal{A},\, \phi\in\mathcal{B}_*,\, b\in\mathcal{B}).$$

Identifying $\mathcal{X}^*$ with $\mathcal{L}(\mathcal{B})$ via the identity

$$\langle b\otimes\phi, T\rangle = \langle Tb, \phi\rangle \quad (b\in\mathcal{B},\, \phi\in\mathcal{B}_*,\, T\in\mathcal{L}(\mathcal{B})),$$

we obtain the corresponding dual $\mathcal{A}$-bimodule operation on $\mathcal{L}(\mathcal{B})$,

$$(a\,.\,T)(b) = \Theta(a)\;(Tb) \quad (T\,.\,a)(b) = (Tb)\;\Theta(a) \quad (a\in\mathcal{A},\, b\in\mathcal{B},\, T\in\mathcal{L}(\mathcal{B})).$$

Note that the left action of $\mathcal{A}$ on $\mathcal{L}(\mathcal{B})$ coincides with the natural one, but the right action is different.

Proposition 6.1. Let $\mathcal{A}$ be a Banach algebra, $\mathcal{B}$ be a dual Banach algebra and let $\Theta : \mathcal{A} \longrightarrow \mathcal{B}$ be a homomorphism. Suppose that one of the following holds.
(i) $\mathcal{A}$ is approximately amenable.
(ii) $\mathcal{A}$ is an approximately Connes-amenable dual Banach algebra, and Θ is $\omega^* - \omega^*$-continuous.
Then there is a net $\{Q_i\}$ in $\mathcal{L}(\mathcal{B})$ such that each Q_i is the identity map on $\Theta(\mathcal{A})'$,

$$a \,.\, Q_i - Q_i \,.\, a \longrightarrow 0 \quad (a \in \mathcal{A}), \quad \text{and}$$

$$Q_i(z_1 b z_2) = z_1 Q_i(b) z_2 \quad (z_1, z_2 \in \Theta(\mathcal{A})',\ b \in \mathcal{B}).$$

Proof. Suppose $\mathcal{X} := \mathcal{B} \hat{\otimes} \mathcal{B}_*$ and $\mathcal{X}^*$ equipped with the above mentioned $\mathcal{A}$-module operation. Let F be the subspace of $\mathcal{X}^*$ consisting of those $T \in \mathcal{X}^*$ such that

$$\begin{aligned}
&\langle zb \otimes \phi - b \otimes \phi \,.\, z, T\rangle = 0, \\
&\langle bz \otimes \phi - b \otimes z \,.\, \phi, T\rangle = 0, \\
&\langle z \otimes \phi, T\rangle = 0, \quad (b \in \mathcal{B},\ \phi \in \mathcal{B}_*,\ z \in \Theta(\mathcal{A})').
\end{aligned}$$

Then F is a ω^*-closed $\mathcal{A}$-bimodule of $\mathcal{X}^*$ and thus a dual Banach $\mathcal{A}$-bimodule. Consider the derivation $D = ad_{id_{\mathcal{B}}}$ from $\mathcal{A}$ into $\mathcal{L}(\mathcal{B})$. As it was shown in the proof [27, Theorem 4.4.11], $D(\mathcal{A}) \subseteq F$. Thus if (i) holds, D is approximately inner. If (ii) holds, then $\mathcal{X}^*$ is a normal dual Banach $\mathcal{A}$-bimodule and D is $\omega^* - \omega^*$-continuous. Thus again D is approximately inner.
Therefore in any case there is a net $(T_i) \subseteq F$ such that $D = \lim_i ad_{T_i}$. Setting $Q_i := id_{\mathcal{B}} - T_i$, for each i, it is immediate that $a \,.\, Q_i - Q_i \,.\, a \longrightarrow 0$. Since $\langle z \otimes \phi, T_i\rangle = 0$, for $z \in \Theta(\mathcal{A})'$ and $\phi \in \mathcal{B}_*$, each Q_i is the identity map on $\Theta(\mathcal{A})'$.
For $b \in \mathcal{B}$, $z \in \Theta(\mathcal{A})'$, and $\phi \in \mathcal{B}_*$, we have $T_i(zb) = zT_i(b)$, and $T_i(bz) = T_i(b)(z)$. Therefore $Q_i(z_1 b z_2) = z_1 Q_i(b) z_2$, for $z_1, z_2 \in \Theta(\mathcal{A})'$, and $b \in \mathcal{B}$. ∎

Let G be a locally compact group, $p \geq 1$ and λ_p and ρ_p be the left and right regular representations of G on $L^p(G)$ respectively. We can extend λ_p to the measure algebra $M(G)$ and thus on $L^1(G)$. The Banach algebra $PM_p(G)$ of p-pseudomeasures on G is the closure of $\lambda_p(L^1(G))$ in $\mathcal{L}(L^p(G))$, with respect to the weak operator topology.
Recall that G is called inner amenable if there is a mean m on $L^\infty(G)$ such that

$$\langle \delta_g \star \phi \star \delta_{g^{-1}}, m\rangle = \langle \phi, m\rangle \quad (g \in G, \phi \in L^\infty(G))\,.$$

For a function f on a locally compact group G, define $\check{f}$ by $\check{f}(g) = f(g^{-1})$. Let $p, q \in (1, \infty)$ such that $1/p + 1/q = 1$. The space $A_p(G)$ consists of those $f \in C_0(G)$ such that there are sequences $(\xi_n) \subseteq L^p(G)$ and $(\eta_n) \subseteq L^q(G)$ with

$$f = \sum_{n=1}^{\infty} \xi_n \star \check{\eta}_n\,,\ \sum_{n=1}^{\infty} ||\xi_n||_p\, ||\eta_n||_q < \infty\,.$$

$A_p(G)$, which is called a Figa-Talamanca-Herz algebra, is a Banach algebra with respect to pointwise operations [20], and the dual space $A_p(G)^*$ can be identified with the algebra $PM_q(G)$ [25], where the duality is given by

$$\langle \xi \star \check{\eta}, T \rangle := \langle T\eta, \xi \rangle \quad (T \in PM_q(G),\ \xi \in L^p(G),\ \eta \in L^q(G))\ .$$

Theorem 6.2. Let G be an inner amenable locally compact group, $p \in (1, \infty)$, and let $PM_p(G)$ be approximately Connes-amenable. Then G is amenable.
Proof. Since G is inner amenable, then there is a net $(f_i)_i \subseteq P(G)$ such that

$$||\ \lambda_1(g^{-1})f_i - \rho_1(g)f_i\ || \longrightarrow 0 \quad (g \in G)\ .$$

Let $q \in (1, \infty)$ be such that $\dfrac{1}{p} + \dfrac{1}{q} = 1$. Let $\xi_i := f_i^{1/p}$ and $\eta_i := f_i^{1/q}$, so that $\xi_i \in L^p(G)$ and $\eta_i \in L^q(G)$. It follows that

$$||\ \lambda_p(g^{-1})\xi_i - \rho_p(g)\xi_i\ ||_p \longrightarrow 0\ ,\ \ ||\ \lambda_q(g^{-1})\eta_i - \rho_q(g)\eta_i\ ||_q \longrightarrow 0\ \ (g \in G)\ .$$

For $\phi \in L^\infty(G)$ let $\mathcal{M}_\phi \in \mathcal{L}(L^p(G))$ be the multiplication operator with symbole ϕ.
By Proposition 6.1, with $\mathcal{A} = PM_p(G)$, $\mathcal{B} = \mathcal{L}(L^p(G))$, and Θ as the inclusion, there exists a net $Q_\alpha : \mathcal{L}(L^p(G)) \longrightarrow \mathcal{L}(L^p(G))$ such that each Q_α is the identity on $PM_p(G)^{'}$,

$$Q_\alpha(T_1 S T_2) = T_1\ Q_\alpha(S)\ T_2 \quad (T_1, T_2 \in PM_p(G)^{'},\ S \in \mathcal{L}(L^p(G)))\ ,$$

and

$$T\ Q_\alpha(S) - Q_\alpha(S)\ T \longrightarrow 0 \quad (T \in PM_p(G),\ S \in \mathcal{L}(L^p(G)))\ .$$

In particular

$$\|\lambda_p(g)\ Q_\alpha(S) - Q_\alpha(S)\ \lambda_p(g)\| \longrightarrow 0 \quad (g \in G,\ S \in \mathcal{L}(L^p(G))).$$

Define $m_{i,\alpha} \in L^\infty(G)^*$ by

$$\langle \phi, m_{i,\alpha} \rangle := \langle Q_\alpha(\mathcal{M}_\phi)\xi_i, \eta_i \rangle \quad (\phi \in L^\infty(G))\ .$$

For each α, let m_α be a ω^*-cluster point of $(m_{i,\alpha})_i$. Passing to a subnet, we may suppose that $m_\alpha = \omega^* - \lim_i m_{i,\alpha}$, so that

$$\langle \phi, m_\alpha \rangle = \lim_i \langle \phi, m_{i,\alpha} \rangle \quad (\phi \in L^\infty(G))\ .$$

An argument similar to [26, Theorem 5.3], shows that

$$\langle \phi \star \delta_g, m_\alpha \rangle - \langle \phi, m_\alpha \rangle = \lim_i (\langle \lambda_p(g)\ Q_\alpha(\mathcal{M}_\phi)\ \lambda_p(g^{-1})\xi_i, \eta_i \rangle - \langle Q_\alpha(\mathcal{M}_\phi)\xi_i, \eta_i \rangle)\ ,$$

for $g \in G$ and $\phi \in L^\infty(G)$.
Thus

$$\begin{aligned} |\langle \phi \star \delta_g - \phi, m_\alpha \rangle| &\leq \lim_i \int_G ||\lambda_p(g)\ Q_\alpha(\mathcal{M}_\phi)\ \lambda_p(g^{-1}) - Q_\alpha(\mathcal{M}_\phi)||\ |\xi_i(h)|\ |\eta_i(h)|\ dh \\ &= ||\lambda_p(g)\ Q_\alpha(\mathcal{M}_\phi)\ \lambda_p(g^{-1}) - Q_\alpha(\mathcal{M}_\phi)||. \end{aligned}$$

Therefore
$$\lim_{\alpha} |\langle \phi \star \delta_g - \phi, m_\alpha \rangle| = 0.$$
Normalizing n, where n is a ω^*-cluster point of $(m_\alpha)_\alpha$, we obtain a right invariant mean on $L^\infty(G)$, so that G is amenable. ■

Corollary 6.3. For a locally compact group G, consider the following.
(i) G is amenable.
(ii) $PM_p(G)$ is approximately Connes-amenable, for every $p \in (1, \infty)$.
(iii) $VN(G)$ is approximately Connes-amenable.
(iv) $PM_p(G)$ is approximately Connes-amenable, for one $p \in (1, \infty)$.
Then we have $(i) \Longrightarrow (ii) \Longrightarrow (iii) \Longrightarrow (iv)$,
and if G is inner amenable, $(iv) \Longrightarrow (i)$ holds.
Proof. $(i) \Longrightarrow (ii)$ If G is amenable, then for every $p \in (1, \infty)$ $PM_p(G)$ is Connes-amenable [26, Theorem 5.3] and hence is approximately Connes-amenable.
$(ii) \Longrightarrow (iii)$ and $(iii) \Longrightarrow (iv)$ are trivial, because $VN(G) = PM_2(G)$. If G is inner amenable, then $(iv) \Longrightarrow (i)$ follows from Theorem 6.2. ■

7 Approximate Connes amenability of $\mathcal{A}^{**}$

If $\mathcal{A}$ is a dual Banach algebra such that $\mathcal{A}^{**}$ is Connes-amenable, then so is $\mathcal{A}$ [8]. In the following theorem we extend this result to approximate Connes-amenability.

Theorem 7.1. Let $\mathcal{A}$ be an Arens regular Banach algebra such that $\mathcal{A}^{**}$ is approximately Connes-amenable.
(i) If $\mathcal{A}$ is a dual Banach algebra, then $\mathcal{A}$ is approximately Connes-amenable.
(ii) If $\mathcal{A}$ is an ideal in $\mathcal{A}^{**}$ and $\mathcal{A}^{**}$ has an identity then $\mathcal{A}$ is approximately amenable.
Proof. (i) Suppose $\mathcal{X}$ is a normal dual Banach $\mathcal{A}$-bimodule, and $\pi : \mathcal{A}^{**} \longrightarrow \mathcal{A}$ is the restriction map to $\mathcal{A}_*$. Then π is a $\omega^* - \omega^*$ continuous homomorphism. Therefore $\mathcal{X}$ is a normal dual Banach $\mathcal{A}^{**}$-bimodule with the following actions
$$a^{**}.x = \pi(a^{**})x \quad , \quad x.a^{**} = x\pi(a^{**}) \quad (x \in \mathcal{X}, a^{**} \in \mathcal{A}^{**}).$$
Let $D : \mathcal{A} \longrightarrow \mathcal{X}$ be a $\omega^* - \omega^*$ continuous derivation. It is easy to see that $Do\pi : \mathcal{A}^{**} \longrightarrow \mathcal{X}$ is a $\omega^* - \omega^*$ continuous derivation. Since $\mathcal{A}^{**}$ is approximately Connes-amenable, than there exists a net $(x_\alpha) \subseteq \mathcal{X}$ such that
$$Do\pi(a^{**}) = \lim_{\alpha} a^{**}.x_\alpha - x_\alpha.a^{**} \quad (a^{**} \in \mathcal{A}^{**}).$$
So
$$D(a) = \lim_{\alpha} a.x_\alpha - x_\alpha.a \quad (a \in \mathcal{A}).$$
(ii) By [13, Proposition 2.5] in order to show that $\mathcal{A}$ is approximately amenable it is sufficient to show that every $D \in Z^1(\mathcal{A}, \mathcal{X}^*)$ is approximately inner for each neo-unital Banach $\mathcal{A}$-module.

Let $\mathcal{X}$ be a neo-unital Banach $\mathcal{A}$-bimodule, and let $D \in Z^1(\mathcal{A}, \mathcal{X}^*)$. As in the proof of [27, Theorem 4.4.8] one can show that $\mathcal{X}^*$ is a normal dual Banach $\mathcal{A}^{**}$-bimodule and D has a unique extension $\widetilde{D} \in Z^1(\mathcal{A}^{**}, \mathcal{X}^*)$. From the approximate Connes-amenability of $\mathcal{A}^{**}$ we conclude that $\widetilde{D}$, and hence D is inner. It follows that $\mathcal{A}$ is approximately amenable. ∎

Theorem 7.2. Suppose $\mathcal{A}$ is a Banach algebra with a bounded approximate identity (e_β) and $B(\mathcal{A}, \mathcal{A}^*) = W(\mathcal{A}, \mathcal{A}^*)$. If $\mathcal{A}^{**}$ is approximately strongly Connes-amenable, then $\mathcal{A}$ is approximately amenable.

Proof. Following the argument of [26, Theorem 4.8] we see that $\mathcal{A}$ is Arens regular and hence $\mathcal{A}^{**}$ is a dual Banach algebra. Moreover

$$(\mathcal{A}\widehat{\otimes}\mathcal{A})^{**} \cong L^2_{\omega^*}(\mathcal{A}^{**}, \mathbb{C})^*$$

as Banach $\mathcal{A}$-bimodules. Since $\mathcal{A}$ is Arens regular and has a bounded approximate identity, then $\mathcal{A}^{**}$ has an identity e. Thus by Theorem 3.2, $\mathcal{A}^{**}$ has an approximate normal virtual diagonal $(M_\alpha) \subset L^2_{\omega^*}(\mathcal{A}^{**}, \mathbb{C})^*$. Now set $M''_{(\alpha,\beta)} = M_\alpha + e_\beta \otimes e_\beta$ and $F_{(\alpha,\beta)} = G_{(\alpha,\beta)} = e_\beta$. Then for every $a \in \mathcal{A}$ we have

$$\begin{aligned} aM''_{(\alpha,\beta)} - M''_{(\alpha,\beta)}a + F_{(\alpha,\beta)} \otimes a - a \otimes G_{(\alpha,\beta)} & \\ &= aM_\alpha - M_\alpha a + ae_\beta \otimes e_\beta - e_\beta \otimes e_\beta a + e_\beta \otimes a - a \otimes e_\beta \\ &= (aM_\alpha - M_\alpha a) + (ae_\beta - a) \otimes e_\beta + e_\beta \otimes (a - e_\beta a) \longrightarrow 0. \end{aligned}$$

Moreover $aF_{(\alpha,\beta)} \longrightarrow a, \quad G_{(\alpha,\beta)}a \longrightarrow a$ and

$$\Delta^{**}(M''_{(\alpha,\beta)})a - F_{(\alpha,\beta)}a - G_{(\alpha,\beta)}a = \Delta^{**}(M_\alpha)a + e_\beta^2 a - e_\beta a - e_\beta a \longrightarrow 0.$$

Therefore by [13, Corollary 2.2] $\mathcal{A}$ is approximately amenable. ∎

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