Non-archimedean function spaces and the Lebesgue dominated convergence theorem*

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Abstract

Let $M(X, \mathbb{K})$ be the non-archimedean Banach space of all additive and bounded $\mathbb{K}$-valued measures on the ring of all clopen subsets of a zero-dimensional compact space X, where $\mathbb{K}$ is a non-archimedean non-trivially valued complete field. It is known that $M(X, \mathbb{K})$ is isometrically isomorphic to the dual of the Banach space $C(X, \mathbb{K})$ of all continuous $\mathbb{K}$-valued maps on X with the sup-norm topology. Does the non-archimedean Lebesgue Dominated Convergence Theorem hold for the space $M(X, \mathbb{K})$? Only in the trivial case! We show (Theorem 2) that for every sequence $(f_n)_n$ in $C(X, \mathbb{K})$ such that $f_n(x) \to 0$ for all $x \in X$ and $\|f_n\| \leq 1$ for all $n \in \mathbb{N}$, one has $\int_X f_n d\mu \to 0$ for each $\mu \in M(X, \mathbb{K})$ iff X is finite. In the second part we characterize (Theorem 3) weakly Lindelöf non-archimedean Banach spaces E with a base as well as Corson $\sigma(E', E)$-compact unit balls in their duals E' (Theorem 17). We also look at the Kunen space from the non-archimedean point of view.

1 Introduction

Let X be a compact space and let $M(X, \mathbb{R})$ be the space of all regular Borel measures on X. The classical Riesz Representation Theorem and the Lebesgue Domi-

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nated Convergence Theorem imply that every continuous linear real-valued functional $C(X, \mathbb{R}) \to \mathbb{R}$ is represented by a unique regular Borel measure μ on X and, if $(f_n)_n$ is a sequence of real-valued continuous functions on X which converges pointwise to zero and is uniformly bounded, then $\int_X f_n d\mu \to 0$ for every $\mu \in M(X, \mathbb{R})$. What about this very popular theorem for the non-archimedean case?

Let $\mathbb{K}$ be a non-archimedean non-trivially valued complete field with valuation $|.|$. Let X be a compact zero-dimensional space and let $\Omega(X)$ be the ring of all clopen subsets of X. By a *measure* on X we mean a map $\mu : \Omega(X) \to \mathbb{K}$ which is additive and bounded, i.e. $\|\mu\| := \sup\{|\mu(U)| : U \in \Omega(X)\} < \infty$. The space $M(X, \mathbb{K})$ of measures on X, with the natural operations and the norm $\|.\|$, is a Banach space.

There is a simple way to integrate continuous $\mathbb{K}$-valued functions on X with respect to a measure μ, [15], [19].

Let $C(X, \mathbb{K})$ be the Banach space of all $\mathbb{K}$-valued continuous maps on X equipped with the usual supremum norm $\|.\|$. Let $C(X, \mathbb{K})'$ be the topological dual of the space $C(X, \mathbb{K})$. The following Riesz Representation Theorem can be found in [15, Theorem 2.5.30], see also [19, Theorem 7.18] for generalizations.

Theorem 1. *Let X be a compact zero-dimensional space. For every $\phi \in C(X, \mathbb{K})'$ there exists exactly one measure on X such that $\phi(f) = \int_X f d\mu$ for all $f \in C(X, \mathbb{K})$ and the map*

$$M(X, \mathbb{K}) \to C(X, \mathbb{K})', \mu \mapsto \phi_\mu,$$

is an isometrical isomorphism.

Let us identify $C(X, \mathbb{K})' = M(X, \mathbb{K})$. We say that X has the $\mathbb{K}$-*Lebesgue property* if for each sequence $(f_n)_n$ in $C(X, \mathbb{K})$ such that $f_n(x) \to 0$ for all $x \in X$ and $\|f_n\| \leq 1$ for all $n \in \mathbb{N}$, one has $\int_X f_n d\mu \to 0$ for each $\mu \in C(X, \mathbb{K})'$.

Since the space $C(X, \mathbb{K})$ has the Orlicz-Pettis property, i.e. every convergent sequence in the weak topology of $C(X, \mathbb{K})$ is norm-convergent, see [14, Corollary 2.5], we note that X has the $\mathbb{K}$-Lebesgue property iff for each sequence $(f_n)_n$ in $C(X, \mathbb{K})$ such that $f_n(x) \to 0$ for all $x \in X$ and $\|f_n\| \leq 1$ for all $n \in \mathbb{N}$, one has $\|f_n\| \to 0$.

Let $C_p(X, \mathbb{K})$ be the space of all $\mathbb{K}$-valued continuous maps on X endowed with the pointwise topology. By σ_X we denote the weak topology of $C(X, \mathbb{K})$.

The main result of Section 2 is the next one, which establishes that the non-archimedean Lebesgue Dominated Convergence Theorem holds only in the trivial case: when X is finite.

Theorem 2. *Let X be a compact zero-dimensional space. Let B be the closed unit ball in $C(X, \mathbb{K})$. The following assertions are equivalent.*

(i) X is finite.

(ii) B is a Fréchet-Urysohn space in the weak topology of $C(X, \mathbb{K})$.

(iii) For every decreasing sequence $(U_n)_n$ of clopen subsets of X there is an $m \in \mathbb{N}$ such that $U_n = U_m$ for all $n \geq m$.

(iv) X has the $\mathbb{K}$-Lebesgue property.

(v) Every uniformly bounded $C_p(X, \mathbb{K})$-compact ($C_p(X, \mathbb{K})$-metrizable) set is compact in the weak topology of $C(X, \mathbb{K})$.

On the other hand, in [11, Theorem 4.13] Katsaras showed a Lebesgue Dominated Convergence Theorem for a certain class of measures μ on X. Our Theorem 2 shows that such measures μ do not cover the whole dual $M(X, \mathbb{K})$.

Section 3 deals with compact zero-dimensional spaces X such that $C_p(X, \mathbb{K})$ is K-analytic (Lindelöf). It is known that if X is a compact scattered space (i.e., every closed subset L of X has an isolated point in L), then $C_p(X, \mathbb{R})$ is Lindelöf iff $C(X, \mathbb{R})$ is weakly Lindelöf. Also, it was proved in [17] that for a compact space X the real space $C_p(X, \mathbb{R})$ is K-analytic iff $C(X, \mathbb{R})$ is weakly K-analytic. The proofs of these classical results use the $\mathbb{R}$-Lebesgue property of X. The same argument cannot be used in the non-archimedean setting, since (as we show in Theorem 2) the Lebesgue theorem fails for this case. However, by using non-archimedean techniques we prove in Section 3 some p-adic versions of these classical results about $C_p(X, \mathbb{R})$. The key point to get these versions is the following

Theorem 3. *Let $\mathbb{K}$ be separable. Let E be a Banach space over $\mathbb{K}$ with a base. The following assertions are equivalent.*

(i) E is separable.

(ii) $(E, \sigma(E, E'))$ is separable.

(iii) E is analytic (K-analytic, Lindelöf).

(iv) $(E, \sigma(E, E'))$ is analytic (K-analytic, Lindelöf).

(v) E has a compact resolution.

(vi) $(E, \sigma(E, E'))$ has a compact resolution.

(vii) $B_{E'}$ is $\sigma(E', E)$-(ultra)metrizable (where $B_{E'}$ is the closed unit ball in E').

(viii) $(E', \sigma(E', E))$ is hereditarily separable, i.e., subsets of $(E', \sigma(E', E))$ are separable.

(ix) $(E', \sigma(E', E))$ is linear hereditarily separable, i.e., linear subspaces of $(E', \sigma(E', E))$ are separable.

(x) E is isomorphic to the Banach space $c_0(\mathbb{N}, \mathbb{K})$.

Let $\mathbb{K}$ be separable. It is known that $c_0(I, \mathbb{K})$ is separable iff I is countable. Applying Theorem 3 for $E := c_0(I, \mathbb{K})$ we obtain that $c_0(I, \mathbb{K})$ is weakly (K-)analytic iff I is countable. However, $c_0(I, \mathbb{R})$ is weakly K-analytic (but not K-analytic) for any set I (since $c_0(I, \mathbb{R})$ is a weakly compactly generated Banach space and Talagrand's [17] applies). Being motivated by remarkable Haydon-Kunen-Talagrand examples we characterize Corson $\sigma(E', E)$-compactness for the unit ball of the dual of any non-archimedean Banach space E over a locally compact $\mathbb{K}$, see Theorem 17.

For basics on non-archimedean normed and locally convex spaces we refer to [19] and [15], respectively.

2 Proof of Theorem 2

We start with the following example motivating also Theorem 2. Recall that, for a prime number p, $\mathbb{Q}_p$ is the field of the p-adic numbers equipped with its p-adic (non-archimedean) valuation, and $\mathbb{Z}_p$ is the corresponding closed unit ball in $\mathbb{Q}_p$.

Example 4. *Let $\mathbb{K}$ be locally compact (e.g. $\mathbb{K} = \mathbb{Q}_p$). Let X be the closed ball $B(a,r) := \{x \in \mathbb{K} : |x-a| \le r\}$ (e.g. $X = \mathbb{Z}_p$). Then X is a zero-dimensional compact space not having the $\mathbb{K}$-Lebesgue property.*

Proof. Since $\mathbb{K}$ is locally compact, the valuation of $\mathbb{K}$ is discrete. Let ρ be its uniformizing element, see [15]. We may assume that $a = 0$ and $r = \rho^s$ for some $s \in \mathbb{Z}$. For each $n \in \mathbb{N}$ set $A_n := \{x \in \mathbb{K} : |x| = \rho^{n-1}\rho^s\}$ and assume that f_n is the $\mathbb{K}$-valued characteristic function of A_n. Note that $(f_n)_n$ converges pointwise to zero on X and $\|f_n\| = 1$ for each $n \in \mathbb{N}$. Hence X does not have the $\mathbb{K}$-Lebesgue property. ∎

Now we prove Theorem 2.

Proof. (i) ⇒ (ii), (iv), (v) are obvious.

(iv) ⇒ (iii): Let $(U_n)_n$ be a decreasing sequence of clopen subsets of X. Let $x \in X$. If $x \in \bigcap_n U_n$, then $\chi_{U_n}(x) \to 1$, where χ_{U_n} is the $\mathbb{K}$-valued characteristic function of the set U_n. If $x \notin \bigcap_n U_n$, then $\chi_{U_n}(x) \to 0$. This implies that $(\chi_{U_n})_n$ is a Cauchy sequence in $C_p(X,\mathbb{K})$. Also, it is clear that $\|\chi_{U_n} - \chi_{U_m}\| \le 1$ for all $n, m \in \mathbb{N}$. Since X has the $\mathbb{K}$-Lebesgue property, $(\chi_{U_n})_n$ is Cauchy in the norm topology of $C(X,\mathbb{K})$, so there is an $m \in \mathbb{N}$ such that $\|\chi_{U_n} - \chi_{U_m}\| < 1$ for all $n \ge m$, i.e., $|\chi_{U_n}(x) - \chi_{U_m}(x)| < 1$ for all $x \in X$ and $n \ge m$. Hence $\chi_{U_n} = \chi_{U_m}$, and then $U_n = U_m$ for all $n \ge m$.

(iii) ⇒ (i): We prove that all the elements of X are isolated points. Then, by compactness of X we deduce that X is finite. Assume that there exists an element $x \in X$ which is not isolated; we derive a contradiction. Let $U_1 := U$ be a clopen neighbourhood of x. Since $U \neq \{x\}$, there are an $x_1 \in U \setminus \{x\}$ and a clopen neighbourhood U_2 of x such that $U_2 \subset U$ and $x_1 \in U \setminus U_2$. Again we have that $U_2 \neq \{x\}$ and with the same reasoning as before we find a clopen neighbourhood U_3 of x with $U_3 \subset U_2$ and an $x_2 \in U_2 \setminus U_3$. Continuing this procedure we construct a sequence $x_1, x_2, \ldots$ in X and a decreasing sequence $(U_n)_n$ of clopen subsets of X such that $x_n \in U_n \setminus U_{n+1}$ for all $n \in \mathbb{N}$. Thus, all the inclusions in that decreasing sequence are strict, a contradiction with (iii).

(ii) ⇒ (i): Let τ_B and σ_B be the restrictions to B of the norm topology and of the topology σ_X on $C(X,\mathbb{K})$ respectively. We prove that $\tau_B = \sigma_B$. For that, let $A \subset B$. Clearly $\overline{A}^{\tau_B} \subset \overline{A}^{\sigma_B}$. Now, let $f \in \overline{A}^{\sigma_B}$. By (ii) there is a sequence $(f_n)_n$ in A such that $f_n \to f$ in σ_X. Since $C(X,\mathbb{K})$ has the Orlicz-Pettis property we obtain that $f_n \to f$ in τ_B, hence $f \in \overline{A}^{\tau_B}$. Therefore, for any $A \subset B$ the closures of A in σ_B and τ_B coincide, i.e. $\tau_B = \sigma_B$.

Since B is compactoid in $(C(X,\mathbb{K}), \sigma_X)$ by [15, Theorem 5.4.1], and $\sigma_B = \tau_B$, we apply [15, Theorem 3.8.13] to deduce that B is a compactoid neighbourhood of zero in the Banach space $C(X,\mathbb{K})$. Then $C(X,\mathbb{K})$ is finite-dimensional by [15, Theorem 3.8.5], i.e. X is finite.

(v) ⇒ (iv): Let $(f_n)_n$ be a sequence such that $f_n \to 0$ in $C_p(X,\mathbb{K})$ and $\|f_n\| \le 1$ for all n. Clearly $L := \{f_n : n \in \mathbb{N}\} \bigcup \{0\}$ is $C_p(X,\mathbb{K})$-compact. Also, L is $C_p(X,\mathbb{K})$-metrizable, by [15, Theorem 3.8.24], hence σ_X-compact by (v). Therefore, the pointwise topology and σ_X coincide on L, so $f_n \to 0$ in σ_X. Thus, X has the $\mathbb{K}$-Lebesgue property. ∎

Remark 5. Implication (v) $\Rightarrow$ (i) shows that Grothendieck's Theorem, [6, Theorem 4.2], fails for the spaces $C(X,\mathbb{K})$.

3 Non-archimedean $C_p(X,\mathbb{K})$ spaces and the Lindelöf property

A topological space X has a *compact resolution* if X has a family $\{K_\alpha : \alpha \in \mathbb{N}^\mathbb{N}\}$ of compact subsets covering X such that $K_\alpha \subset K_\beta$ if $\alpha \leq \beta$ in $\mathbb{N}^\mathbb{N}$. X is called *analytic* if X is a continuous image of $\mathbb{N}^\mathbb{N}$; it is a *K-analytic space* if there is an upper semi-continuous compact-valued map from $\mathbb{N}^\mathbb{N}$ into X whose union is X, see [17], [16]. Note that separable complete metric spaces are analytic, analytic spaces are K-analytic, K-analytic spaces are Lindelöf and any K-analytic space admits a compact resolution. The converses of the above results fail, see [2], [16], [17]. Also, countable unions and products of K-analytic [analytic] spaces are K-analytic [analytic]; closed subspaces of a K-analytic [analytic] space are K-analytic [analytic], see [16], [18].

The following proposition is motivated by Theorem 2(v) and will be useful in the proof of Corollary 10.

Proposition 6. *Let X be a zero-dimensional space having a compact resolution (for example when X is σ-compact) and let $L \subset C_p(X,\mathbb{K})$ be a compact set. Then L is metrizable iff L is separable.*

Proof. It is well-known that every metrizable compact space is separable. Now, assume that L is separable. Let S be a countable dense subset of L. The space $C_p(S,\mathbb{K})$ is metrizable by [14, Theorem 3.7.2]. Clearly, the map

$$\varphi : C_p(L,\mathbb{K}) \to C_p(S,\mathbb{K}), f \mapsto f|S,$$

is a continuous injection onto its metrizable range. It follows that $C_p(L,\mathbb{K})$ admits a weaker metric topology. Let $\delta : X \to C_p(L,\mathbb{K})$ be the continuous map defined by $x \mapsto \delta_x$, $\delta_x(f) := f(x)$, $x \in X$, $f \in L$. $\delta(X)$ has a compact resolution and, as a subset of $C_p(L,\mathbb{K})$, $\delta(X)$ admits a weaker metric topology. Then by Talagrand's [17], see also [3, Corollary 4.3], the space $\delta(X)$ is analytic, hence separable. For every $f \in L$ let $f_0 : \delta(X) \to \mathbb{K}$ be the continuous function on $\delta(X)$ defined by $f_0(\delta_x) := f(x)$. Then $\{f_0 : f \in L\}$ is a compact subset of $C_p(\delta(X),\mathbb{K})$ homeomorphic to L. Since $\delta(X)$ is separable, $C_p(\delta(X),\mathbb{K})$ admits a weaker metric topology, so L is metrizable. ∎

Example 7. *Proposition 6 fails if "compactness" of L is replaced by "compactoidity".*

Indeed, let $\mathbb{K}$ be locally compact. Let $E := (c_0(\mathbb{N},\mathbb{K}), \sigma(c_0(\mathbb{N},\mathbb{K}), \ell^\infty(\mathbb{N},\mathbb{K})))$. Let $T := \{e_1, e_2, \ldots,\}$, where e_i are the unit vectors. Then T is bounded in E, hence compactoid by [15, Corollary 5.4.2], and T is metrizable and separable. By [15, Example 5.4.4] the closed absolutely convex hull L of T is nonmetrizable (although it is compactoid and separable). Finally, let X be the closed unit ball of E'. Observe that X is compact with respect to the weak*-topology on E' and that E is homeomorphically embedded in $C_p(X,\mathbb{K})$.

In [10] we showed a pure non-archimedean theorem stating that if E is a non-archimedean Banach space over a locally compact $\mathbb{K}$, then E endowed with the weak topology is a Lindelöf space iff E is separable. We extend this result (Theorem 3) for Banach spaces with a base over a separable $\mathbb{K}$ (if $\mathbb{K}$ is locally compact, then $\mathbb{K}$ is separable and its valuation is discrete, so every Banach space over $\mathbb{K}$ has a base, [15, Theorems 2.1.11, 2.5.4]). First we note the following

Lemma 8. *If E is a Banach space with a base, then every closed linear subspace of countable type of E is weakly closed.*

Proof. Let D be a closed linear subspace of countable type of E. By [19, Corollary 3.18], D is complemented in E, so there is a continuous linear projection $P : E \to E$ whose kernel is D. Then P is continuous if E is endowed with its weak topology (which is Hausdorff), so we obtain that $D = P^{-1}\{0\}$ is weakly closed. ∎

Note that, by separability of $\mathbb{K}$, a locally convex space E over $\mathbb{K}$ is separable iff there is a countable set whose linear hull is dense in E (iff the space is of countable type, in case E is metrizable).

Now we prove Theorem 3.

Proof. Clearly (i) $\Rightarrow$ (ii) and (viii) $\Rightarrow$ (ix). (ii) $\Rightarrow$ (i) follows from Lemma 8 and (v) $\Rightarrow$ (i) from [15, Theorem 11.5.3].

Recall again that every separable complete metric space is analytic, that analytic $\Rightarrow$ K-analytic $\Rightarrow$ existence of a compact resolution; K-analytic $\Rightarrow$ Lindelöf, and that these last four properties are preserved by passing from the norm to the weak topology. Hence we have (i) $\Rightarrow$ (iii) $\Rightarrow$ (iv), (v) $\Rightarrow$ (vi). Next we prove the remaining implications.

$(E, \sigma(E, E'))$ is Lindelöf $\Rightarrow$ (i): Let $(e_i)_{i\in I}$ be a base of E. Then E is linearly homeomorphic to $c_0(I, \mathbb{K})$, [19, Corollary 3.8], so it is enough to prove this implication for $E = c_0(I, \mathbb{K})$.

We follow the argument from [10, Theorem 7, (9) $\Rightarrow$ (4)]: Assume I is uncountable; we derive a contradiction. Let $x \in E$. Put $I_x = \{i \in I : x_i = 0\}$. The functional $f_x : E \to \mathbb{K}, \quad y \mapsto \sum_{i\in I_x} y_i$ is well defined, linear, and continuous. Set $\mathcal{W} := \{W_x : x \in E\}$, where $W_x = \{y \in E : |f_x(y)| < 1\}$. Since $x \in W_x$ for all $x \in E$, the family $\mathcal{W}$ covers E. By assumption $\mathcal{W}$ contains a countable subfamily $\{W_x : x \in X\}$ covering E, where X is a countable subset of E. There exists $j \in I$ such that $x_j = 0$ for each $x \in X$. Then $j \in \bigcap_{x\in X} I_x$. Consequently, $f_x(e_j) = 1$ for each $x \in X$, where $e_j \in E$ such that $(e_j)_i := \delta_{ji}$ for all $i \in I$. We proved that $e_j \notin W_x$ for all $x \in X$, a contradiction. Hence I is countable, so E is separable.

(vi) $\Rightarrow$ (v): Every separable closed linear subspace of E is complemented in E, [19, Corollary 3.18]. This implies that every continuous linear functional defined on a separable linear subspace of E admits a continuous linear extension to the whole space. Then every weakly compact subset of E is norm-compact by [15, Theorem 5.8.5], and we are done.

(i) $\Leftrightarrow$ (vii): It follows from [15, Theorem 7.6.10] and its proof.

(i) $\Rightarrow$ (viii): Since every separable Banach space is linearly homeomorphic to $c_0 := c_0(\mathbb{N}, \mathbb{K})$, it suffices to prove this implication for $E := c_0$. Let $e_1, e_2, \ldots$ be the unit vectors of E. It is easily seen that every $x = (x_n)_n \in \ell^\infty$ $(= E')$ can be written as $x = \sum_n x_n e_n$ in the weak*-topology $\sigma^* := \sigma(\ell^\infty, c_0)$ and that $e_n \to 0$ in this topology. In particular, $B_{E'}$ is the σ^*-closed absolutely convex hull of $\{e_1, e_2, \ldots\}$, and by separability of $\mathbb{K}$ we obtain that $B_{E'}$ is σ^*-separable. Also, by [15, Theorem 3.8.24], $B_{E'}$ is σ^*-metrizable. Then $B_{E'}$ and all of its subsets are σ^*-separable.

Now, let $F \subset E'$ and let $\lambda \in \mathbb{K}$ with $|\lambda| > 1$. By the above, $F \cap \lambda^n B_{E'}$ is σ^*-separable for all n. Then, $F = \bigcup_n (F \cap \lambda^n B_{E'})$, being a countable union of σ^*-separable sets, is also σ^*-separable. Therefore, E' is σ^*-hereditarily separable.

(ix) $\Rightarrow$ (i): It is enough to prove this implication for $E = c_0(I, \mathbb{K})$, where I is a set with the same cardinality as a base of E. Then clearly we have $(E', \sigma(E', E)) = (\ell^\infty(I, \mathbb{K}), \sigma(\ell^\infty(I, \mathbb{K}), c_0(I, \mathbb{K})))$.

Assume (ix) holds for $(\ell^\infty(I, \mathbb{K}), \sigma(\ell^\infty(I, \mathbb{K})), c_0(I, \mathbb{K}))$. Let $\{e_i : i \in I\}$ be the set formed by the canonical unit vectors of E and let D be the linear hull of this set. By assumption D is separable in $\sigma := \sigma(\ell^\infty(I, \mathbb{K}), c_0(I, \mathbb{K}))$, hence there exist $y_1, y_2, \ldots \in D$ such that $D \subset \overline{\{y_1, y_2, \ldots\}}^{\sigma}$. Then each element of D has null coordinates off of the countable set $\bigcup_n \{i \in I : (y_n)_i \neq 0\}$. Therefore I is countable, so $c_0(I, \mathbb{K})$ is separable.

$(i) \Leftrightarrow (x)$: It follows from [15, Corollary 2.3.9]. ∎

We often use the terms "weakly separable", "weakly analytic", ... instead of "$\sigma(E, E')$-separable", "$\sigma(E, E')$-analytic", ...

It was proved in [17] that for a compact space X the real space $C_p(X, \mathbb{R})$ is K-analytic iff $C(X, \mathbb{R})$ is weakly K-analytic. By using the previous results of this section, we will give in Corollary 11 a non-archimedean counterpart of the above classical result, when X is a zero-dimensional compact abelian group, see also Corollary 12.

We will use the following additional fact

Theorem 9. *Let X be a zero-dimensional compact space. The following are equivalent.*

(i) X is metrizable.

(ii) X is ultrametrizable.

If, in addition, X is a zero-dimensional compact abelian group, then (i), (ii) are equivalent to:

(iii) X has countable tightness (i.e., if $A \subset X$ and $x \in \overline{A}$, then there exists a countable subset $B \subset A$ such that $x \in \overline{B}$).

Proof. (ii) $\Rightarrow$ (i) $\Rightarrow$ (iii) is obvious. (i) $\Rightarrow$ (ii): Since X is compact and metrizable, X is second-countable. Hence X is ultrametrizable by [19, p. 39]. If X is a zero-dimensional compact abelian group, (iii) $\Rightarrow$ (i) follows from [8, Theorem 2]. ∎

For any non-empty set Γ put $\Sigma(\Gamma) := \{x \in \mathbb{R}^\Gamma : \{x(\gamma) \neq 0\} \text{ is countable}\}$ endowed with the product topology. It is known that each space $\Sigma(\Gamma)$ is Fréchet-Urysohn, see [13].

A compact space X is called *Corson-compact* if X is homeomorphic to a compact subset of some $\Sigma(\Gamma)$. We refer to [5] and [12] for the properties of Corson-compact spaces used in the sequel.

Corollary 10. *Let X be a zero-dimensional compact space and let $\mathbb{K}$ be separable. The following assertions are equivalent.*

(i) The Banach space $C(X, \mathbb{K})$ is separable (analytic, K-analytic, Lindelöf, has a compact resolution).

(ii) $(C(X, \mathbb{K}), \sigma_X)$ is separable (analytic, K-analytic, Lindelöf, has a compact resolution).

(iii) $C_p(X, \mathbb{K})$ is separable (analytic).

(iv) X is (ultra)metrizable.

(v) X is Corson-compact and separable.

(vi) X is separable and $C_p(X, \mathbb{K})$ is K-analytic.

Proof. (i) $\Leftrightarrow$ (ii) is a direct consequence of Theorem 3, as $C(X, \mathbb{K})$ has an orthonormal base [15, Theorem 2.5.22]. (iv) $\Rightarrow$ (i) follows from Theorem 9 and [15, Theorem 2.5.24]. (ii) $\Rightarrow$ (iii) is obvious since, on $C(X, \mathbb{K})$, the pointwise topology is weaker than the weak topology. (iii) $\Rightarrow$ (iv): If $C_p(X, \mathbb{K})$ is separable, then (iv) follows from [15, Theorem 4.3.4].

(iv) $\Leftrightarrow$ (v): Every metric compact space is separable and Corson-compact, and every separable Corson-compact space is metrizable (note that $\Sigma(\Gamma)$ is dense $\mathbb{R}^\Gamma$), so ultrametrizable by Theorem 9.

From what we have already proved, it is clear that any of the properties (i)$-$(v) implies (vi).

(vi) $\Rightarrow$ (iv): $C_p(X, \mathbb{K})$ is K-analytic, so it has a compact resolution. Also, X is homeomorphically embedded in $C_p(C_p(X, \mathbb{K}), \mathbb{K})$. Then applying Proposition 6 we get that X is metrizable and so ultrametrizable by Theorem 9. ∎

Corollary 11. *Let X be a zero-dimensional compact abelian group. Then (i)$-$(vi) are equivalent to*

(vii) X is Corson-compact.

(viii) X has countable tightness.

(ix) $C_p(X, \mathbb{K})$ is K-analytic (Lindelöf).

If, in addition, $\mathbb{K}$ is locally compact, then (i)$-$(vi) are equivalent to

(x) $C_p(X, \mathbb{K})$ has a compact resolution.

Proof. Clearly (v) $\Rightarrow$ (vii). (iv) $\Leftrightarrow$ (viii) follows from Theorem 9. Also, (ii) $\Rightarrow$ (ix), (x) are obvious, since on $C(X, \mathbb{K})$, the pointwise topology is weaker than the weak one.

(vii) $\Rightarrow$ (viii): Every Corson-compact space is Fréchet-Urysohn, hence has countable tightness.

(ix) $\Rightarrow$ (viii): Assume that $C_p(X, \mathbb{K})$ is Lindelöf. Let $A \subset X$ and let $x \in \overline{A}$. Set $\mathcal{F} := \{f \in C(X, \mathbb{K}) : f(x) = 1\}$. Since $\mathcal{F}$ is a closed subspace of $C_p(X, \mathbb{K})$, the space $\mathcal{F}$ is Lindelöf. For each $y \in A$ set

$$V_y := \{g \in C(X, \mathbb{K}) : g(y) \neq 0\}.$$

Clearly each V_y is open in $C_p(X, \mathbb{K})$. Fix $f \in \mathcal{F}$. Since $f(\overline{A}) \subset \overline{f(A)}$, there exists $y \in A$ such that $f(y) \neq 0$. This implies that $\mathcal{F} \subset \bigcup\{V_y : y \in A\}$. Hence there exists a countable set $B \subset A$ such that $\mathcal{F} \subset \bigcup\{V_y : y \in B\}$. We show that $x \in \overline{B}$. Assume $x \notin \overline{B}$. By zero-dimensionality of X there is a clopen set U in X such that $x \in U$ and $\overline{B} \subset X \setminus U$. Then the characteristic function on U, $\chi_U : X \to \mathbb{K}$, is continuous and satisfies

$$\chi_U(x) = 1, \ \ \chi_U(X \setminus U) = \{0\}.$$

Since $\chi_U \in \mathcal{F}$, there exists $y \in B$ such that $\chi_U \in V_y$. As $y \in U$ we conclude that $y \in U \cap B$, a contradiction. Therefore, X has countable tightness.

Now assume that $\mathbb{K}$ is locally compact. By [9, Theorem 14] $C_p(X, \mathbb{K})$ is K-analytic if it has a compact resolution, which proves (x) $\Rightarrow$ (ix). ∎

Corollary 11 is not true for zero-dimensional compact spaces in general, as we will see in Remark 16.

As a direct consequence of Corollary 10 we note

Corollary 12. *Let X be a zero-dimensional compact space and let $\mathbb{K}$ be separable. Then $C_p(X, \mathbb{K})$ is separable (analytic) iff $C(X, \mathbb{K})$ is weakly separable (weakly analytic).*

Alster and Pol [1, Theorem] proved

Proposition 13. *If X is Corson-compact and M is a separable metric space, then $C(X, M)$ is Lindelöf in the pointwise topology.*

The remarkable Haydon-Kunen-Talagrand example (under the continuum hypothesis) of a non-separable Corson-compact space X such that the real space $C_p(X, \mathbb{R})$ is Lindelöf but $C(X, \mathbb{R})$ is not weakly Lindelöf can be found in [12, Theorem 5.9].

The following corollary provides a large class of compact spaces related with a non-archimedean version of the mentioned Haydon-Kunen-Talagrand example.

Corollary 14. *Let $\mathbb{K}$ be separable. If X is a non-separable zero-dimensional Corson-compact space, then $C_p(X, \mathbb{K})$ is Lindelöf and $C(X, \mathbb{K})$ is not weakly Lindelöf.*

Proof. $C_p(X, \mathbb{K})$ is Lindelöf by Proposition 13. $C(X, \mathbb{K})$ is not weakly Lindelöf by Corollary 10. ∎

In [1, Example 7] Alster and Pol constructed a non-separable Corson-compact space $X_0 \subset \{0, 1\}^T$ with $|T| = \aleph_1$ such that $X_0 \subset \Sigma(\aleph_1)$ and $C_p(X_0, \mathbb{R})$ is not K-analytic. Note the following non-archimedean version of the Alster-Pol's example (which supplements also Proposition 13).

Example 15. *Let $\mathbb{K}$ be separable. Then $C_p(X_0, \mathbb{K})$ is not K-analytic although it is Lindelöf.*

Indeed, $C_p(X_0, \mathbb{K})$ is Lindelöf by Proposition 13. To prove the first claim, suppose that $C_p(X_0, \mathbb{K})$ is K-analytic; we derive a contradiction. Then $C_p(X_0, \mathbb{K}^{\mathbb{N}})$ is K-analytic, since it is homeomorphic to $C_p(X_0, \mathbb{K})^{\mathbb{N}}$, a countable product of K-analytic spaces. Also, by [9, Proposition 19] there exists a continuous surjection $S_\varphi : C_p(X_0, \mathbb{K}^{\mathbb{N}}) \to C_p(X_0, \mathbb{R})$. So $C_p(X_0, \mathbb{R})$ is K-analytic, a contradiction.

Remark 16. Corollary 14 and Example 15 show that Corollary 11 is not true for general zero-dimensional compact spaces X.

Gul'ko proved that in the classical case the closed unit ball of the dual of a weakly K-analytic Banach space is Corson-compact in the weak* topology. Also, Kunen constructed (under the continuum hypothesis) an uncountable separable compact scattered (hence zero-dimensional) space Z such that $C(Z, \mathbb{R})$ is weakly

Lindelöf and the weak* dual is hereditarily separable, see for example [7]. Next we provide non-archimedean variants of the above classical facts.

If $\mathbb{K}$ is locally compact, the closed unit ball $B_{E'}$ in the dual of a non-archimedean Banach space E is a weak*-compact abelian group. Also, $\mathbb{K}$ is separable and its valuation is discrete, so every Banach space over $\mathbb{K}$ has a base, [15, Theorems 2.1.11, 2.5.4]. Hence the previous results of this section apply to get the following

Theorem 17. *Let E be a Banach space over a locally compact $\mathbb{K}$. Then (i)$-$(ix) of Theorem 3 are equivalent to (i)$-(x)$ of Corollaries 10 and 11, by taking $X := B_{E'}$ equipped with the restriction to X of the weak* topology $\sigma(E', E)$.*

Proof. It follows immediately from Theorem 3 and Corollaries 10, 11 (one has just to look at Theorem 3(vii) and Corollary 10(iv)). ∎

When $\mathbb{K}$ is locally compact, the weak* dual F of the non-separable Banach space $\ell^\infty(\mathbb{N}, \mathbb{K})$ is separable ([4, Proposition 4]), but the space F is not hereditarily separable by Theorem 17. Also, the weak* dual F of the real Banach space $\ell^\infty(\mathbb{N}, \mathbb{R})$ is separable but not hereditarily separable. Indeed, otherwise it is easily seen that F has countable tightness. Hence $\ell^\infty(\mathbb{N}, \mathbb{R})$ has the Corson's property (C) by [5, Theorem 12.41], a contradiction, [5, Excercise 12.44] (recall that a real or complex Banach space F has the *Corson's property (C)* if for every family of closed convex subsets of F with empty intersection there is a countable subfamily with empty intersection).

On the other hand, there are situations in sharp contrast with the classical case, as we show in the next examples.

Example 18. *Let $\mathbb{K}$ be locally compact and let I be an uncountable set. Let $B_{\mathbb{K}}$ and $B_{\mathbb{R}}$ be the closed unit ball in the dual of the Banach space $c_0(I, \mathbb{K})$ and $c_0(I, \mathbb{R})$, respectively. Then $B_{\mathbb{R}}$ is Corson-compact in the weak* topology and $B_{\mathbb{K}}$ is not.*

Indeed, the claim about $B_{\mathbb{K}}$ follows from Theorem 17, as $c_0(I, \mathbb{K})$ is not separable. $c_0(I, \mathbb{R})$ is weakly K-analytic, since it is a WCG Banach space, [5]; so Gul'ko result (mentioned above) applies.

Example 19. *Let $\mathbb{K}$ be locally compact and let Z be the Kunen compact space. Then the weak* dual of $C(Z, \mathbb{R})$ is hereditarily separable and the weak* dual of $C(Z, \mathbb{K})$ is not.*

Indeed, the claim about $C(Z, \mathbb{R})$ was already mentioned above (see the comments after Remark 16). Now, suppose that the weak* dual of $C(Z, \mathbb{K})$ is hereditarily separable. Then $C(Z, \mathbb{K})$ is separable by Theorem 17 and so Z must be ultrametrizable by Corollary 10. On the other hand, every metrizable scattered compact space is countable, a contradiction (since Z by assumption is uncountable).

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