On the Commutant of Multiplication Operators with Analytic Rational Symbol

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Abstract

Let $\mathcal{B}$ be a certain Banach space consisting of analytic functions defined on a bounded domain G in the complex plane. Let φ be an analytic multiplier of $\mathcal{B}$ we denote by M_φ and $\{M_\varphi\}'$ respectively, the operator of multiplication by φ and the commutant of M_φ. In this article under certain conditions on φ and G we characterize the commutant of M_φ. In particular, when φ is a rational function with poles off $\overline{G}$, under certain conditions on φ we show that $\{M_\varphi\}' = \{M_z\}'$. We extend some results obtained in [4] and [6] about the commutant of the operator M_φ.

1 Introduction

Let G be a bounded domain in the complex plane. Let $\mathcal{B}$ be a Banach space consisting of functions analytic on G such that $1 \in \mathcal{B}, z\mathcal{B} \subset \mathcal{B}$ and for every $\lambda \in G$ the linear functional e_λ of evaluation at λ is bounded on $\mathcal{B}$. Also assume that $\text{ran}(M_z - \lambda) = \ker(e_\lambda)$ for every $\lambda \in G$ and if $f \in \mathcal{B}$ and $|f(\lambda)| > c > 0$ for every $\lambda \in G$, then $\frac{1}{f}$ is a multiplier of $\mathcal{B}$.
In what follows G denotes a bounded domain in the complex plane, and by a Banach space of analytic functions $\mathcal{B}$ on G, we mean, one satisfying the above conditions.

Some examples of such spaces are as follows:

Received by the editors February 2011.

Communicated by F. Brakcx.

2000 *Mathematics Subject Classification :* Primary 47B35; Secondary 47B3.

Key words and phrases : commutant, multiplication operators, Banach space of analytic functions, rational function, only a simple zero.

1) The algebra $A(G)$ which is the algebra of all continuous functions on the closure of G that are analytic on G.

2) The Bergman space of analytic functions defined on G, $L_a^p(G)$ for $1 \leq p \leq \infty$.

3) The spaces D_α of all functions $f(z) = \sum \hat{f}(n)z^n$, holomorphic in the complex unit disc D, for which $\|f\|^2 = \sum (n+1)^\alpha \mid \hat{f}(n) \mid^2 < \infty$ for every $\alpha \geq 1$ or $\alpha \leq 0$.

4) The analytic Lipschitz spaces $Lip(\alpha, \overline{G})$ for $0 < \alpha < 1$, i.e., the space of all analytic functions defined on G that satisfy a Lipschitz condition of order α.

5) The subspace $lip(\alpha, \overline{G})$ of $Lip(\alpha, \overline{G})$, consisting of functions f in $Lip(\alpha, \overline{G})$ for which

$$lim_{z \to w} \frac{\mid f(z) - f(w) \mid}{\mid z - w \mid^\alpha} = 0.$$

6) The classical Hardy spaces H^p for $1 \leq p \leq \infty$.

Let E be a subset of $\mathbb{C}$. We say that f is in $H(E)$ if there is an open set U that contains E such that f is analytic in U. We denote by $B(a;r)$ the set $\{z \in \mathbb{C} : |z - a| < r\}$.

A complex valued function φ defined on G is called a multiplier of $\mathcal{B}$ if $\varphi\mathcal{B} \subset \mathcal{B}$ and the collection of all these multipliers is denoted by $\mathcal{M}(\mathcal{B})$. As it is shown in [11] each multiplier φ is bounded on G. Given a multiplier φ, we call M_φ, defined by $M_\varphi(f) = \phi f$ for every function $f \in \mathcal{B}$, the operator of multiplication by φ. The continuity of M_φ follows from the Closed Graph Theorem. We denote $\{M_\varphi\}'$ to be the set of all bounded linear operators X on $\mathcal{B}$ such that $M_\varphi X = X M_\varphi$, i.e., the commutant of M_φ. It is easy to see that $\{M_z\}' = \{M_\varphi : \varphi \in \mathcal{M}(\mathcal{B})\}$. Two good sources on this topics are [10] and [11].

Let φ be an analytic function in a neighborhood of $\overline{G}$ and $\lambda \in \overline{G}$. If φ has a zero of order one at λ and $\varphi(z) \neq 0$ for all $z \neq \lambda$ in $\overline{G}$, we say that φ has only a simple zero in $\overline{G}$. Also for $\lambda \in G$ if $\varphi \in A(G)$ has a zero of order one at λ and $\varphi(z) \neq 0$ for all $z \neq \lambda$ in $\overline{G}$, then we say that φ has only a simple zero in $\overline{G}$.

Recall that a bounded linear operator T on a Banach space is called Fredholm, if it is invertible modulo of the compact operators. It is known that T is Fredholm if its range is closed and both ker$T$ and kerT^* are finite dimensional. If T is a Fredholm operator, we define the index of T as

$$\text{ind } (T) = \dim \ker T - \dim \ker T^*.$$

The commutant of multiplication operators on spaces of analytic functions on D, were investigated by many authors for certain multiplication operators. See for example, [1-8, 10-15]. only a few works has been done in studying commutants of multiplications operators on the spaces of analytic functions on bounded domains different from the unit disc. See for examples [4], [6], and [8].

The aim of this article is to investigate the commutant of the operator M_φ for certain function $\varphi \in \mathcal{M}(\mathcal{B})$. In particular, when φ is a polynomial or a rational function with poles off $\overline{G}$, under certain conditions on the its coefficients, we show that $\{M_\varphi\}' = \{M_z\}'$. In [4] Ž. Čučković and Dashan Fan have shown that if $G = \{z \in \mathbb{C} : r < |z| < 1\}$, $\mathcal{B} = L_a^2(G)$ and $p(z) = z + a_2 z^2 + \cdots + a_n z^n$, where

$a_i \geq 0$ and $p(z) - p(1)$ has n distinct zeros, then $\{M_p\}' = \{M_\Psi : \Psi \in \mathcal{M}(\mathcal{B})\}$. As a result, Theorem 2.4 and Corollary 2.6 extend the result obtained in [4] to Banach spaces of analytic functions on various domains G and certain polynomial or rational symbols. Also we extend the results obtained in [6]. Moreover in the both papers the authors used the condition that the zeros of the function p outside $\overline{G}$ are distinct which we omit this condition.

2 The main results

We begin this section with a theorem about the commutant of the multiplication operator M_φ. In fact we show that if for some $\lambda \in G$ the operator $M_{\varphi-\varphi(\lambda)}$is a Fredholm operator such that its index equal to -1, then $\{M_\varphi\}' = \{M_z\}'$.

Theorem 2.1 Let $\mathcal{B}$ be a Banach space of analytic functions on G, and let $\varphi \in \mathcal{M}(\mathcal{B})$. If there is a $\lambda \in G$ such that $M_{\varphi-\varphi(\lambda)}$ is a Fredholm operator with $\text{ind}(M_{\varphi-\varphi(\lambda)}) = -1$, then $\{M_\varphi\}' = \{M_z\}'$.

Proof. Let $T \in \{M_\varphi\}'$. It is easy to see that $T^*(e_\lambda)$ and e_λ are in $\ker(M_{\varphi-\varphi(\lambda)})^*$. Since $M_{\varphi-\varphi(\lambda)}$ is one to one and by assumption $\text{ind}(M_{\varphi-\varphi(\lambda)}) = -1$, we conclude that $\dim \ker(M_{\varphi-\varphi(\lambda)})^* = 1$. Therefore $T^*(e_\lambda)) = \psi(\lambda)e_\lambda$ for some constant $\psi(\lambda)$. Hence, we have

$$T(f)(\lambda) =< T(f), e_\lambda >=< f, T^*(e_\lambda) >= \psi(\lambda) < f, e_\lambda >= \psi(\lambda)f(\lambda),$$

for each $f \in \mathcal{B}$. In particular $\psi(\lambda) = T(1)(\lambda)$. Since $M_{\varphi-\varphi(\lambda)}$ is Fredholm, there is a positive number ϵ such that if U is a bounded linear operator on $\mathcal{B}$ and $||U|| < \epsilon$, then $M_{\varphi-\varphi(\lambda)} + U$ is Fredholm and $\text{ind}(M_{\varphi-\varphi(\lambda)}) = \text{ind}(M_{\varphi-\varphi(\lambda)} + U)$. Now by continuity of $\varphi - \varphi(\lambda)$ at λ, there is a positive number δ such that for each $t \in G$ with $|t - \lambda| < \delta$, we have $|\varphi(t) - \varphi(\lambda)| < \epsilon$. So the operator $M_{\varphi-\varphi(t)}$ is Fredholm and $\text{ind}(M_{\varphi-\varphi(t)}) = -1$. Hence $T(f)(t) = T(1)(t)f(t)$ for each $f \in \mathcal{B}$ and for every $t \in B(\lambda;\delta) \cap G$. Set $\psi = T(1)$. Since two analytic functions $T(f)$ and ψf are equal on $B(\lambda;\delta) \cap G$ and G is connected, we have $T(f) = \psi f$ for all $f \in \mathcal{B}$, which proves the theorem. ∎

Theorem 2.2 Let $\mathcal{B}$ be a Banach space of analytic functions on G, let $\varphi \in \mathcal{M}(\mathcal{B}) \cap A(G)$, and let $\lambda \in G$. If $\varphi(z) - \varphi(\lambda)$ has only a simple zero in $\overline{G}$, then $M_{\varphi-\varphi(\lambda)}$ is a Fredholm operator with $\text{ind}(M_{\varphi-\varphi(\lambda)}) = -1$, so by Theorem 2.1, $\{M_\varphi\}' = \{M_\Psi : \Psi \in \mathcal{M}(\mathcal{B})\}$.

Proof. First we show that $\text{ran}(M_\varphi - \varphi(\lambda)) = \ker e_\lambda$. It is easy to see that $\text{ran}(M_\varphi - \varphi(\lambda)) \subset \ker e_\lambda$.

To show the converse, since $\text{ran}(M_z - \lambda) = \ker e_\lambda$, we have $(\varphi - \varphi(\lambda))(z) = (z - \lambda)h(z)$ for some $h \in \mathcal{B}$. Because $\varphi \in A(G)$, h has a continuous extension on $\overline{G}$ which we denote it again with h. By assumption $h(z) \neq 0$ for every $z \in \overline{G}$. Therefore $\frac{1}{h}$ is in $\mathcal{M}(\mathcal{B})$ and we have $z - \lambda = \frac{\varphi(z)-\varphi(\lambda)}{h(z)}$. Now if $f \in \ker e_\lambda$, then

$f = (z-\lambda)g$ for some function $g \in \mathcal{B}$. Hence

$$f = \frac{\varphi - \varphi(\lambda)}{h} g = (\varphi - \varphi(\lambda))\frac{g}{h}.$$

Since $g \in \mathcal{B}$ and $\frac{1}{h} \in \mathcal{M}(\mathcal{B})$, we have $\frac{g}{h} \in \mathcal{B}$ and $\ker e_\lambda \subset \operatorname{ran}(M_\varphi - \varphi(\lambda))$.

From $\operatorname{ran}(M_\varphi - \varphi(\lambda)) = \ker e_\lambda$, we conclude that $\operatorname{ran}(M_\varphi - \varphi(\lambda))$ is closed and $\dim \ker(M_{\varphi-\varphi(\lambda)})^* = 1$. On the other hand $M_{\varphi-\varphi(\lambda)}$ is one to one , therefore $\dim \ker(M_{\varphi-\varphi(\lambda)}) = 0$. Hence $M_{\varphi-\varphi(\lambda)}$ is a Fredholm operator and its index equal to -1. ∎

From now on we assume that $r(z) = p(z)/q(z)$ is a rational function such that $p(z)$ and $q(z)$ are polynomials without common factors. Also the poles of $r(z)$ which are exactly the zeros of $q(z)$ are off $\overline{G}$.

Proposition 2.3 Let $\mathcal{B}$ be a Banach space of analytic functions on G, where G is the interior of $\overline{G}$, and let $r(z) = p(z)/q(z)$ be a rational function with poles off $\overline{G}$. If there are α and β in G such that $p(z) - p(\alpha)$ has only a simple zero in $\overline{G}$, $r(\beta) \neq 0$, and $|r(\beta)q(z) - p(\alpha)| < |p(z) - p(\alpha)|$ for each $z \in \partial G$, then $\{M_r\}' = \{M_z\}'$.

Proof. By assumptions, we have

$$r(z) - r(\beta) = \frac{p(z)q(\beta) - q(z)p(\beta)}{q(z)q(\beta)} = \frac{q(\beta)(p(z) - p(\alpha) - r(\beta)q(z) + p(\alpha))}{q(z)q(\beta)}.$$

Thus, $r(z) - r(\beta) = 0$ if and only if $(p(z) - p(\alpha) - r(\beta)q(z) + p(\alpha) = 0$. Using general form of Rouche's Theorem we conclude that $r(z) - r(\beta)$ has only a simple zero at β. So by Theorem 2.2, the proof is complete. ∎

Remark. The above proposition holds if there are α and β in G such that $q(z) - q(\alpha)$ has only a simple zero in $\overline{G}$ and $r(\beta) \neq 0$, moreover, $|p(z) - r(\beta)q(\alpha)| < |r(\beta)(q(z) - q(\alpha))|$ for each $z \in \partial G$.

In Theorem 2.2, λ is in G and $\varphi \in \mathcal{M}(\mathcal{B}) \cap A(G)$. The same proof does not work for $\lambda \in \overline{G}$. In the next theorem we obtain a similar result, whenever $\lambda \in \overline{G}$ and $\varphi \in H(\overline{G}) \cap \mathcal{M}(\mathcal{B})$.

Theorem 2.4 Let $\varphi \in H(\overline{G}) \cap \mathcal{M}(\mathcal{B})$, let $\lambda \in \overline{G}$ and let $\varphi(z) - \varphi(\lambda)$ has only a simple zero in $\overline{G}$. Then $\{M_\varphi\}' = \{M_\Psi : \ \Psi \in \mathcal{M}(\mathcal{B})\}$.

Proof. Let Ω be an open set that contains $\overline{G}$ such that $\varphi \in H(\Omega)$ and let g to be defined in $\Omega \times \Omega$ by

$$g(z,w) = \begin{cases} \frac{\varphi(z)-\varphi(w)}{z-w} & z \neq w, \\ \varphi'(z) & z = w. \end{cases}$$

It is obvious that g is continuous in $\Omega \times \Omega$ and so g is uniformly continuous in $\overline{G} \times \overline{G}$. Since by assumption $g(z,\lambda) \neq 0$ for each $z \in \overline{G}$ and $g(z,\lambda)$ is continuous as a function of z in $\overline{G}$, there is some $\varepsilon > 0$ such that $|g(z,\lambda)| > \varepsilon$ for each $z \in \overline{G}$. Now by uniform continuity of g in $\overline{G} \times \overline{G}$ there is an open set $U \subset G$

such that for each $w \in U$ and for all $z \in \overline{G}$, we have $|g(z,w)| > \frac{\varepsilon}{2}$. Therefore $\varphi(z) - \varphi(w)$ has only a simple zero in $\overline{G}$ for each $w \in U$. Now by Theorem 2.2, we have $\{M_\varphi\}' = \{M_\Psi : \ \Psi \in \mathcal{M}(\mathcal{B})\}$. ■

Corollary 2.5 Let $\mathcal{B}$ be a Banach space of analytic functions on G. Suppose that $\varphi \in H(\overline{G})$ and for some λ in $\overline{G}$ the function $\varphi(z) - \varphi(\lambda)$ has only a simple zero in $\overline{G}$. If $\psi \in H(\overline{G})$ is a univalent map from $\overline{G}$ onto $\overline{G}$, and $\varphi \circ \psi \in \mathcal{M}(\mathcal{B})$, then $\{M_{\varphi\circ\psi}\}' = \{M_z\}'$.

In the next corollary we extend the result obtained in Theorem 4 in [4] to Banach spaces of analytic functions, to more general domains, Also we show that it is not necessary that all of the n zeros of $p(z) - p(1)$ are distinct .

Corollary 2.6 Let $\mathcal{B}$ be a Banach space of analytic functions on G, let $G \subset D$ be such that $1 \in \overline{G}$ and let $p(z) = z + a_2z^2 + \cdots + a_nz^n$, where $a_i \geq 0$ for $i = 2, \cdots, n$. Then $\{M_p\}' = \{M_z\}'$.

Proof. It is easy to see that $p(1) > |p(z)|$ for all $z \in \overline{G} - \{1\}$, since $p'(1) \neq 0$ the function $p(z) - p(1)$ has only a simple zero in $\overline{G}$, and by Theorem 2.4 the proof is complete. ■

Let $r(z) = p(z)/q(z)$ be a rational function with poles off $\overline{G}$, if $n = \max\{\deg(p), \deg(q)\} = 1$, then r is univalent and it is well known that $\{M_r\}' = \{M_z\}'$. Therefore in the reminder of this section we assume that $n = \max\{\deg(p), \deg(q)\} \geq 2$. Let $\lambda \in \overline{G}$. If $r(z) - r(\lambda)$ has $n - 1$ zeros outside of $\overline{G}$, then $\{M_r\}' = \{M_z\}'$. In particular if $p(z)$ has only a simple zero at a point $\lambda \in \overline{G}$, then $r(z)$ has only a simple zero at λ and therefore, $\{M_r\}' = \{M_z\}'$.

From now on, we assume that $G \subset D$.

Corollary 2.7 Let $\mathcal{B}$ be a Banach space of analytic functions on G. Suppose that $n \geq 2$ is an integer, $a \neq 0$ is a complex number with $|a| > 1$ and $p(z) = z^n + az$. If $r(z) = \frac{p(z)}{q(z)}$ is a rational function with poles off $\overline{G}$ and $0 \in \overline{G}$, then $\{M_r\}' = \{M_z\}'$.

Proof. Let $\lambda = 0$ it is easy to see that $r(z) - r(\lambda) = r(z)$ has $n - 1$ distinct zeros outside of $\overline{D}$. Hence by Theorem 2.4, we have $\{M_r\}' = \{M_\Psi : \ \Psi \in \mathcal{M}(\mathcal{B})\}$.

In the next theorem we extend some results obtained in [6], in fact we omit the condition that the zeros of the polynomials outside $\overline{G}$ must be distinct.

Theorem 2.8 Let $\mathcal{B}$ be a Banach space of analytic functions on G and let $p = a_0 + a_1z + a_2z^2 + \cdots + a_nz^n$ be a polynomial of degree $n \geq 2$ such that $a_1 \neq 0$. If each of the following conditions holds, then $\{M_p\}' = \{M_\Psi : \ \Psi \in \mathcal{M}(\mathcal{B})\}$.

(a) For some real constant θ_0, we have $\text{Arg}a_i = \theta_0$ for $a_i \neq 0$ with $i \geq 1$ and $1 \in \partial G$.

(b) For each $a_i \neq 0$ with $i \geq 1$, $\text{Arg}a_i = \theta_0$ for i odd and $\text{Arg}a_i = \theta_0 + \pi$ or $\text{Arg}a_i = \theta_0 - \pi$ for i even, and $-1 \in \overline{G}$.

(c) There is a $z_0 \in \partial D \cap \partial G$ such that all nonzero terms $a_i{z_0}^i$ for $i \geq 1$ are positive or all are negative.

Proof. By assumption $\mid p(1) - a_0 \mid = \mid a_1 \mid + \mid a_2 \mid + \cdots + \mid a_n \mid$. Therefore $p(z) - p(1) = 0$ implies that

$$\mid a_1 z + a_2 z^2 + \cdots + a_n z^n \mid = \mid a_1 \mid + \mid a_2 \mid + \cdots + \mid a_n \mid .$$

For $z \in D$, we have

$$\mid a_1 z + a_2 z^2 + \cdots + a_n z^n \mid < \mid a_1 \mid + \mid a_2 \mid + \cdots + \mid a_n \mid,$$

so $p(z) - p(1)$ has no zero in D. On the other hand if $w \in \partial D$ is a zero of $p(z) - p(1)$, then

$$\begin{aligned} \mid a_1 \mid + \mid a_2 \mid + \cdots + \mid a_n \mid &= \mid a_1 w \mid + \mid a_2 w^2 \mid + \cdots + \mid a_n w^n \mid \\ &= \mid a_1 w + a_2 w^2 + \cdots + a_n w^n \mid . \end{aligned}$$

Hence $\mathrm{Arg}(a_1 w + a_2 w^2 + \cdots + a_n w^n) = \mathrm{Arg}\,(a_1 w)$. Since $p(w) - a_0 = p(1) - a_0 = e^{i\theta_0}(\mid a_1 \mid + \mid a_2 \mid + \cdots + \mid a_n \mid)$, we have $\mathrm{Arg}(a_1 w + a_2 w^2 + \cdots + a_n w^n) = \mathrm{Arg}(a_1 w) = \theta_0$, which implies that $w = 1$. It is easy to see that $p'(1) \neq 0$, so the polynomial $p(z) - p(1)$ has only a simple zero at 1, and by Theorem 2.4, (a) holds.

Using similar argument as used in the proof of part (a) we conclude (b) and (c). ■

Proposition 2.9 Let $\mathcal{B}$ be a Banach space of analytic functions on D, let p be a polynomial of degree $n \geq 2$ and let $r(z) = \frac{p(z)}{q(z)}$ be a rational function. If there is $z_0 \in \partial D$ such that $|r(z_0)| > |r(z)|$ for all $z \in \overline{D} - \{z_0\}$, then $\{M_r\}' = \{M_z\}'$. .

Proof. By assumptions $|r(z_0)| > |r(z)|$ for all $z \in \overline{D} - \{z_0\}$, which implies that $r(z) - r(z_0)$ has no zero in $\overline{D} - \{z_0\}$ and $r'(z_0) \neq 0$. So we conclude that $r(z) - r(z_0)$ has only a simple zero in $\overline{D}$, and by Theorem 2.4, the proof is complete. ■

Remark. Proposition 2.9 holds if there is $z_0 \in \partial D$ such that $|r(z_0)| \leq |r(z)|$ for all $z \in \overline{D} - \{z_0\}$ and $r'(z_0) \neq 0$.

Corollary 2.10 Suppose that $\mathcal{B}$ is a Banach space of analytic functions on G. Let $p(z) = a_n z^n + a_{n-1} z^{n-1} + \cdots + a_0$ be a polynomial of degree $n \geq 2$ with nonnegative real coefficients and let $1 \in \partial G$. If there is a positive integer $m \leq n$ such that a_m and a_{m-1} are not equal to zero, then $\{M_p\}' = \{M_\Psi : \ \Psi \in \mathcal{M}(\mathcal{B})\}$.

Proof. It is easy to see that $|p(1)| > |p(z)|$ for all $z \in \overline{D} - \{1\}$. In fact, if $z = e^{i\theta}$ for some θ, $-\pi < \theta \leq \pi$ and $|p(z)| = |p(1)|$, we have $\mid a_m e^{im\theta} + a_{m-1} e^{i(m-1)\theta} \mid = a_m + a_{m-1}$. Thus, $m\theta = (m-1)\theta + 2k\pi$ for some integer k. Hence $z = 1$, and so by Proposition 2.9, the proof is complete. ■

Lemma 2.11 Let functions $f(z)$ and $g(z)$ be analytic in the open unit disk D and continuous on ∂D. Suppose that there is a point $e^{i\theta_0} \in \partial D$ such that $|f(z)| > |g(z)|$ for all $z \in \partial D - \{e^{i\theta_0}\}$ and $f(e^{i\theta_0}) = -g(e^{i\theta_0}) \neq 0$. Let also the functions $f(z)$ and $g(z)$ have the derivatives at the point $z_0 = e^{i\theta_0}$ and the following inequality holds

$$\frac{e^{i\theta_0}(f'(e^{i\theta_0}) + g'(e^{i\theta_0}))}{f(e^{i\theta_0})} < 0.$$

Then N_{f+g} and N_f, the numbers of zeros of the functions $f+g$ and f according to multiplicity in D are equal.

Proof. Set $F(z) = f(e^{i\theta_0}z)$ and $G(z) = g(e^{i\theta_0}z)$. Then $F(1) = -G(1) \neq 0$, for all $z \in \partial D - \{1\}$ we have $|F(z)| > |G(z)|$ and $\frac{F'(1)+G'(1)}{F(1)} < 0$. Now by Corollary 2 in [9], the lemma follows. ∎

Proposition 2.12 Let $\mathcal{B}$ be a Banach space of analytic functions on D, let f and g belong to $H(\overline{D})$ and let f, g and $e^{i\theta_0}$ satisfy in the conditions of Lemma 2.11. If N_f, the number of zeros of f according to multiplicity in D is equal to zero, then $\{M_{f+g}\}' = \{M_z\}'$.

Proof. By Lemma 2.11, we have $N_{f+g} = 0$. Hence by assumption $f+g$ has only a simple zero at $e^{i\theta_0}$, and by Theorem 2.4, the proof is complete. ∎

In the next example we present some applications of the above theorems.

Example 2.13

a) If $q(z)$ is a polynomial which has no zero in $\overline{D}$, then there is a point $\lambda = e^{i\theta_0}$ such that $|q(\lambda)| \leq |q(z)|$ for all $z \in \overline{D}$. Now let $a = |a|e^{i\theta_0}$ be a nonzero constant, $p(z) = z^n + az^{n-1}$, and $\lambda \in \overline{G}$. It is not hard to see that $|p(z)| < |p(\lambda)|$ for every $z \in \overline{D} - \{\lambda\}$. Hence by the proof of Proposition 2.9, $r(z) - r(z_0)$ has only a simple zero in $\overline{D}$, and therefore in $\overline{G}$. Now by Theorem 2.4, we have $\{M_r\}' = \{M_z\}'$, where $r(z) = \frac{p(z)}{q(z)}$. For example $r(z) = \frac{z^7+iz^6}{(z-2i)^4(z-5i)^2}$ when $G = \{z \in \mathbf{C} : \mathbf{c} < |\mathbf{z}| < \mathbf{1}\}$ for some nonnegative constant $0 \leq c < 1$, or $G = D$ is such a rational function.

b) Let $r(z) = \frac{z^2+z+4}{z^3+2z^2+6z+4}$ be a rational function, if in the remark after Proposition 2.3 we set $\alpha = \beta = 0$, then $r(z) - r(0)$ has only a simple zero in $\overline{D}$, so $\{M_r\}' = \{M_z\}'$.

c) Let G be an open set such that $i \in \partial G$ (recall that after Proposition 2.5, we assume that $G \subset D$). Let $p(z) = z^8 - z^6 + 2iz^3 - 4$ and let $q(z)$ be a polynomial with zeros off $\overline{G}$ without common factor with $p(z)$. If in Proposition 2.12 we set $f(z) = 2iz^3 - 4$, $g(z) = z^8 - z^6$ and $\theta_0 = \frac{\pi}{2}$ we have

$$\frac{e^{i\theta_0}(f'(e^{i\theta_0}) + g'(e^{i\theta_0}))}{f(e^{i\theta_0})} = \frac{-17}{2}.$$

Moreover $|g(z)| \leq 2 \leq |f(z)|$. In the other hand $|g(z)| = |z^2 - 1| = 2$ if and only if $z = i$ or $z = -i$. But $|f(-i)| = 6$, so we have $|f(z)| > |g(z)|$ for all $z \in \partial D - \{e^{i\theta_0}\}$ and $f(e^{i\theta_0}) = -g(e^{i\theta_0)} \neq 0$. Therefore p has only a simple zero at i on $\overline{D}$. Now if $r(z) = \frac{p(z)}{q(z)}$, then $r(z)$ has only a simple zero at i in $\overline{G}$, and we have $\{M_r\}' = \{M_z\}'$.

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