Substructures in algebras of associated homogeneous distributions on R

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Abstract

In previous work the author constructed a convolution algebra and an isomorphic multiplication algebra of one-dimensional associated homogeneous distributions with support in R. In this paper we investigate the various algebraic substructures that can be identified in these algebras. Besides identifying ideals and giving polynomial representations for six subalgebras, it is also shown that both algebras contain an interesting Abelian subgroup, which can be used to construct generalized integration/derivation operators of complex degree on the whole line R.

1 Introduction

In a series of preceding papers, [2]–[7], the author embarked on an in-depth study of the set $\mathcal{H}'(R)$ of Associated Homogeneous Distributions (AHDs) based on (i.e., with support in) the real line R, [9], [8]. The elements of $\mathcal{H}'(R)$ are the distributional analogues of power-log functions with domain in R and contain the majority of the distributions one encounters in (one-dimensional) physics applications (including the δ and $\eta \triangleq \frac{1}{\pi}x^{-1}$ distributions). For an introduction to AHDs, an overview of their properties and possible applications of this work, the reader is referred to [2] (or [1]).

The main result of this study was the construction of a convolution algebra and an isomorphic multiplication algebra of AHDs on R. The multiplication al-

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gebra provides a non-trivial example of how a distributional product can be defined, for an important subset of distributions containing a derivation and the delta distribution, and how this is influenced by Schwartz' impossibility theorem, [12]. Both constructed algebras are non-commutative and non-associative, but in a minimal and interesting way, see [6], [7].

In this paper, we examine the various abstract algebraic substructures that can be identified in these algebras. We consider their ideals and isolate the following substructures: (i) a special substructure, (ii) their general polynomial structure and (iii) polynomial representations of six subalgebras. The identified special substructure (i) further contains an interesting Abelian subgroup, which is useful for the construction of convolution or multiplication operators acting as generalized integration/derivation operators of complex degree on the whole line R.

We use the notation and definitions introduced in [2] (or [1]). For shorthand, we will write $\mathcal{H}'$ for $\mathcal{H}'(R)$ from now on. For a typical element $f_m^z \in \mathcal{H}'$, the superscript denotes its degree of homogeneity z and its subscript its order of association m.

2 Structures

From the construction of the convolution product ($*$), given in [4]–[5], and the multiplication product (.), defined in [2, eq. (82)] based on the generalized convolution theorem, together with the product formulas derived in [6, Theorem 6] and [7, Theorem 8], it follows that:

(i.1) $(\mathcal{H}', *)$ and $(\mathcal{H}', .)$ are non-commutative, non-associative unital derivation magmas. Both structures are isomorphic under Fourier transformation.

(i.2) If (the non-unique) extensions of partial distributions in $\mathcal{H}'$ (i.e., the "regularized" elements) are identified as equivalence sets, then $(\mathcal{H}', *)$ and $(\mathcal{H}', .)$ are derivation monoids for equivalence sets of AHDs on R. We adopt the latter identification throughout this paper.

(ii) Let $\mathcal{G}'$ denote the linear space of finite sums of elements of $\mathcal{H}'$ over $\mathbb{C}$. Then, $\mathcal{F}'_* \triangleq (\mathcal{G}', +, *)$ and $\mathcal{F}'_. \triangleq (\mathcal{G}', +, .)$ are non-associative unital derivation rings, which are in addition non-commutative, non-associative derivation algebras over $\mathbb{C}$.

2.1 Ideals

Definition 1. *Define the equivalence sets of extensions (indicated by the subscript e) in $\mathcal{H}'$,*

$$\begin{aligned} \left[f_e^k\right] &\triangleq \left\{f_e^k \in \mathcal{H}' : f_e^k \sim f_e^k + cx^k, \forall c \in \mathbb{C}\right\}, \forall k \in \mathbb{N}, \\ \left[f_e^{-l}\right] &\triangleq \left\{f_e^{-l} \in \mathcal{H}' : f_e^{-l} \sim f_e^{-l} + c\delta^{(l-1)}, \forall c \in \mathbb{C}\right\}, \forall l \in \mathbb{Z}_+. \end{aligned}$$

The zero distribution 0 is a regular Homogeneous Distribution (HD) of undefined degree. It is convenient to introduce the symbol 0^z for the zero distribution having degree of homogeneity $z \in \mathbb{C}$. The purpose of the symbol 0^z is only to be able to denote equivalence sets such as $\left[0^k\right]$ and $\left[0^{-l}\right]$.

Definition 2. *Define the sets,*

$$\begin{aligned}\mathcal{N}_0' &\triangleq \left\{\left[0^k\right], \forall k \in \mathbb{N}\right\},\\ \mathcal{E}_0' &\triangleq \left\{\left[0^{-l}\right], \forall l \in \mathbb{Z}_+\right\}.\end{aligned}$$

Theorem 3. *The subsemigroup $(\mathcal{N}_0', *)$ is a proper and principal ideal of $(\mathcal{H}', *)$.*

Proof. (i) It is clear from [6, eq. (C.3.3) and Theorem 3] that $(\mathcal{N}_0', *)$ is a (proper) subsemigroup of $(\mathcal{H}', *)$. Further, from the half-lines representation given in [3, Theorem 1] and [6, Theorem 1], it follows that $(\mathcal{N}_0', *)$ is an ideal of $(\mathcal{H}', *)$.

(ii) From the normalized parity representation [3, Theorem 3] together with [6, Theorem 2] follows that, for any $k \in \mathbb{N}$ and $\forall m \in \mathbb{N}$,

$$cx^k * \sum_{n=0}^{m} \left(q_{n,e}(z) D_z^n \Phi_e^z + q_{n,o}(z) D_z^n \Phi_o^z\right)$$

generates all non-zero elements of $\mathcal{N}_0'$ if z runs over $\mathbb{N}$ and the zero distribution if $z \in \mathbb{C}\backslash\mathbb{N}$. Hence, $\mathcal{N}_0'$ is a principal ideal with countable infinite generators. ∎

Theorem 4. *The subsemigroup $(\mathcal{E}_0', .)$ is a proper and principal ideal of $(\mathcal{H}', .)$.*

Proof. By Fourier transformation and Theorem 3. ∎

$\mathcal{N}_0'$ is not a prime ideal since it is not true that $\forall f, g \in \mathcal{H}'$, if $f * g \in \mathcal{N}_0'$ then $f \in \mathcal{N}_0'$ or $g \in \mathcal{N}_0'$, [10, p. 192]. A counter example is [6, eq. (48)]. We do have the following.

Theorem 5. *Let $f \in \mathcal{H}'$. If $f * f \in \mathcal{N}_0'$, then $f \in \mathcal{N}_0'$.*

Proof. A. Assume that f is an AHD of order of association $m > 0$.

Let $r \in \mathbb{C}$ and $k \in \mathbb{Z}_+$. Due to [6, Theorem 6], any square root in $(\mathcal{H}', *)$ of a distribution rx^{k-1} must have degree of homogeneity $k/2 - 1$. Let $f_m^{k/2-1} \in \mathcal{H}'$, of degree $k/2-1$ and order $m \in \mathbb{Z}_+$, represented by the normalized parity representation [6, eq. (51)],

$$f_m^{k/2-1} = \sum_{n=0}^{m} \alpha_n * \left(D_w^n \Phi_e^w\right)_{w=k/2},$$

wherein

$$\alpha_n = q_{n,e}\delta + q_{n,o}\eta \in \mathcal{H}_0'^{-1},$$

and $q_{n,e}, q_{n,o} \in \mathbb{C}$. The highest term of association of $f_m^{k/2-1} * f_m^{k/2-1}$ is

$$\left(\alpha_m * \left(D_w^m \Phi_e^w\right)_{w=k/2}\right) * \left(\alpha_m * \left(D_w^m \Phi_e^w\right)_{w=k/2}\right).$$

By [6, Theorems 3 and 4], this is equivalent to

$$(\alpha_m * \alpha_m) * \left(\left(D_k^{2m} \Phi_e^k\right)_0 + r_1 x^{k-1}\right) + r_2 x^{k-1},$$

with $r_1, r_2 \in \mathbb{C}$ arbitrary and $\left(D_k^{2m}\Phi_e^k\right)_0 = D_k^{2m}\Phi_e^k$ iff k is even. It is easily verified that $\forall \alpha_m \in \mathcal{H}_0^{\prime -1}$, $\alpha_m * \alpha_m = 0$ iff $\alpha_m = 0$. Hence, $f_m^{k/2-1} * f_m^{k/2-1}$ has order of association $2m$ (if k is even) or $2m+1$ (if k is odd). Consequently, the equation $f_m^{k/2-1} * f_m^{k/2-1} = rx^{k-1}$ can not have a solution in $\mathcal{H}'$ if $m \in \mathbb{Z}_+$.

B. Assume that f is a homogeneous distribution.

Any homogeneous distribution of degree $k/2-1$ has a complex representation of the form [3, eq. (25)], $c_\pm \in \mathbb{C}$,

$$f_0^{k/2-1} = c_+(x+i0)^{k/2-1} + c_-(x-i0)^{k/2-1}.$$

(i) For $k = 2p+2$, $\forall p \in \mathbb{N}$, we obtain, since $(x \pm i0)^p = x^p$ and by using [6, eq. (C.3.3)],

$$\begin{aligned} f_0^{k/2-1} * f_0^{k/2-1} &= \left(c_+(x+i0)^p + c_-(x-i0)^p\right) * \left(c_+(x+i0)^p + c_-(x-i0)^p\right), \\ &= \left(c_+ + c_-\right)^2 \left(x^p * x^p\right), \\ &= \left(c_+ + c_-\right)^2 x^{2p+1}. \end{aligned}$$

Hence, $f_0^p = cx^p \in \mathcal{N}_0'$, for some $c \in \mathbb{C}$.

(ii) For $k = 2p+1$, $\forall p \in \mathbb{N}$, we get from [2, eq. (345)] and [6, eqs. (47)–(49)],

$$\begin{aligned} & f_0^{k/2-1} * f_0^{k/2-1} \\ =\ & \left(c_+(x+i0)^{k/2-1} + c_-(x-i0)^{k/2-1}\right) * \left(c_+(x+i0)^{k/2-1} + c_-(x-i0)^{k/2-1}\right), \\ =\ & c_+^2 \frac{\Phi_{x+i0}^{k/2}}{\frac{1}{2\pi}\Gamma(1-k/2)e^{-i(\pi/2)(k/2-1)}} * \frac{\Phi_{x+i0}^{k/2}}{\frac{1}{2\pi}\Gamma(1-k/2)e^{-i(\pi/2)(k/2-1)}} + r'x^{k-1} \\ & + c_-^2 \frac{\Phi_{x-i0}^{k/2}}{\frac{1}{2\pi}\Gamma(1-k/2)e^{+i(\pi/2)(k/2-1)}} * \frac{\Phi_{x-i0}^{k/2}}{\frac{1}{2\pi}\Gamma(1-k/2)e^{+i(\pi/2)(k/2-1)}}, \end{aligned}$$

or

$$f_0^{k/2-1} * f_0^{k/2-1} = \\ r'x^{k-1} - \frac{(2\pi)^2}{\Gamma^2(1-k/2)}\left(c_+^2 e^{+ik\pi/2}\left(\Phi_{x+i0}^k\right)_0 + c_-^2 e^{-ik\pi/2}\left(\Phi_{x-i0}^k\right)_0\right).$$

Herein is, [2, eq. (349)],

$$e^{\pm ik\pi/2}\left(\Phi_{x\pm i0}^k\right)_0 = \frac{1}{2}(-1)^k\left(\frac{1}{2}\frac{x^{k-1}\operatorname{sgn}}{(k-1)!} + \frac{\pm i}{\pi}\frac{x^{k-1}}{(k-1)!}\left(\ln|x| - \psi(k)\right)\right).$$

For $f_0^{k/2-1}$ to be a convolutional square root of $r'x^{k-1}$ requires that simultaneously,

$$\begin{aligned} c_+^2 + c_-^2 &= 0, \\ c_+^2 - c_-^2 &= 0, \end{aligned}$$

so $c_+ = c_- = 0$. Hence, there are no non-zero homogeneous distributions of degree $k/2 - 1 = p - 1/2$ such that $f_0^{k/2-1} * f_0^{k/2-1} = rx^{k-1}$.

C. Collecting results, we showed that cx^{l-1}, $\forall l \in \mathbb{Z}_+$, are the only distributions in $\mathcal{H}'$ which square, with respect to the convolution product, to a distribution $c'x^{k-1}$ with $k \in \mathbb{Z}_+$. Hence, if $f * f \in \mathcal{N}'_0$ then is also $f \in \mathcal{N}'_0$. ∎

From Theorem 5 follows in particular that there does not exist in $(\mathcal{H}', *)$ a distribution $1^{-1/2}$, of degree $-1/2$, such that $1^{-1/2} * 1^{-1/2} = 1$, i.e., the one distribution 1 has no convolutional square root in $\mathcal{H}'$.

$\mathcal{E}'_0$ is not a prime ideal since it is not true that $\forall f, g \in \mathcal{H}'$, if $f.g \in \mathcal{E}'_0$ then $f \in \mathcal{E}'_0$ or $g \in \mathcal{E}'_0$. A counter example is the Fourier transform of [6, eq. (48)]. We also have the following.

Theorem 6. *Let $f \in \mathcal{H}'$. If $f.f \in \mathcal{E}'_0$, then $f \in \mathcal{E}'_0$.*

Proof. By Fourier transformation and Theorem 5. ∎

From Theorem 6 follows that there also does not exist in $(\mathcal{H}', .)$ a distribution $\delta^{1/2}$, of degree $-1/2$, such that $\delta^{1/2}.\delta^{1/2} = \delta$, i.e., the delta distribution δ has no multiplicative square root in $\mathcal{H}'$.

More generally, since $1 \in \mathcal{N}'_0$, $\delta \in \mathcal{E}'_0$ and $\mathcal{N}'_0 \cap \mathcal{E}'_0 = \{0\}$, we have that $\forall g \in \mathcal{N}'_0$, $\nexists f \in \mathcal{H}' : f * g = \delta$ and $\forall g \in \mathcal{E}'_0$, $\nexists f \in \mathcal{H}' : f.g = 1$.

2.2 Special substructure

The set of homogeneous distributions, $\mathcal{H}'_0$, is not closed under convolution, because $\forall f_0^{a-1}, g_0^{b-1} \in \mathcal{H}'_0 : a + b \in \mathbb{Z}_+, f_0^{a-1} * g_0^{b-1} \in \mathcal{H}'_1$, where $\mathcal{H}'_1$ denotes the set of AHDs on R of order of association 1. However, a special subset $\mathcal{SH}'_1$ is closed under convolution $\forall a, b \in \mathbb{C}$, as is shown in the proof of Theorem 8 below.

2.2.1 Definition

Definition 7. *Let $q_e, q_o \in \mathcal{A}(\mathbb{C}, \mathbb{C})$. Define, using the normalized parity representation,*

$$\mathcal{SH}'_1 \triangleq \left\{ g^z \in \mathcal{H}'_1 : g^z = q_e(z)\, \Phi_e^{z+1} + q_o(z)\, \Phi_o^{z+1}, \forall z \in \mathbb{C} \right\} \tag{1}$$

and wherein any partial distribution Φ_e^{z+1} or Φ_o^{z+1} at $z = k \in \mathbb{N}$ is replaced by its equivalence set of extensions.

In (1) it is not required that the coefficient functions q_e, q_o satisfy conditions [3, eqs. (37)–(38)]. If these conditions were satisfied, then the normalized parity representation [3, eq. (36)] assures that $g^z \in \mathcal{H}'_0$ and that g^z is complex holomorphic in $\mathbb{C}$. Further, in a sufficiently small neighborhood of $z = 2p$ ($z = 2p + 1$), $p \in \mathbb{N}$, Φ_e^{z+1} (Φ_o^{z+1}) in (1) is to be replaced by its equivalence set of complex holomorphic extensions $\left(\Phi_e^{z+1}\right)_e$ ($\left(\Phi_o^{z+1}\right)_e$), which will be in $\mathcal{H}'_1$, since they are proper AHDs of first order of association. In particular at $z = 2p$ ($z = 2p + 1$), Φ_e^{2p+1} (Φ_o^{2p+2}) will contain a term proportional to $x^k \ln|x|$, $k = 2p$ ($2p + 1$). Finally, there are elements in $\mathcal{H}'_1$ which are not in $\mathcal{SH}'_1$ (e.g., $D_z\Phi_e^{z+1}$). Hence, $\mathcal{H}'_0 \subset \mathcal{SH}'_1 \subset \mathcal{H}'_1$.

2.2.2 General properties

Theorem 8. *The structure* $\left(\mathcal{SH}_1', *\right)$ *is the largest proper submonoid of* $(\mathcal{H}', *)$.

Proof. (i) From the convolution product formula, [6, Theorem 6], readily follows that $(\mathcal{H}_1', *)$ is not closed, due to the presence of terms of the form $D_z\Phi_e^{z+1}$ and $D_z\Phi_o^{z+1}$. For any subset of $\mathcal{H}_1'$ to be closed, it is necessary that it consists of AHDs for which the coefficient functions $q_{1,e}(z)$ and $q_{1,o}(z)$ in its normalized parity representation are zero. Such AHDs have a representation of the form (1).

Further, from the convolution product formula [6, Theorem 6], it readily follows that $\left(\mathcal{H}_p', *\right)$ for $p > 1$ is not closed, due to the presence of terms of the form $D_z^n\Phi_e^{z+1}$ and $D_z^n\Phi_o^{z+1}$ with $n \in \mathbb{Z}_{[1,p]}$. Consequently, no subset $\mathcal{SH}_p'$ can exist, with $\mathcal{H}_1' \subset \mathcal{SH}_p' \subset \mathcal{H}_p'$, that is closed under convolution.

(ii) From [6, Theorem 4] follows that the convolution product of two elements of $\mathcal{SH}_1'$ is again in $\mathcal{SH}_1'$. Hence $\left(\mathcal{SH}_1', *\right)$ is closed. Since the convolution product $*$ for equivalence sets of extensions is associative, [6, Theorem 3], and the $*$-identity $\delta \in \mathcal{SH}_1'$, it follows that $\left(\mathcal{SH}_1', *\right)$ is a submonoid of $(\mathcal{H}', *)$. Since $\mathcal{SH}_1'$ is the largest proper subset of $\mathcal{H}'$ that is closed, $\left(\mathcal{SH}_1', *\right)$ is the largest proper submonoid of $(\mathcal{H}', *)$. ■

The closure of $\left(\mathcal{SH}_1', *\right)$ is a consequence of the remarkable fact that the convolution products of extensions of the normalized parity basis AHDs at natural degrees, being of order of association 1, result in a homogeneous distribution and not an AHD of order 2 (as would normally be expected).

An equivalent form for the elements of $\mathcal{SH}_1'$ is the parity representation,

$$\mathcal{SH}_1' \triangleq \left\{g^z \in \mathcal{H}_1' : g^z = p_e(z)\,|x|^z + p_o(z)\left(|x|^z \operatorname{sgn}\right), \forall z \in \mathbb{C}\right\}, \tag{2}$$

with, [2, Appendix],

$$\begin{aligned} p_e(z) &= \frac{q_e(z)}{2\Gamma(z+1)\cos\left(\frac{\pi}{2}(z+1)\right)}, \\ p_o(z) &= \frac{q_o(z)\,1}{2\Gamma(z+1)\sin\left(\frac{\pi}{2}(z+1)\right)}, \end{aligned}$$

and wherein any partial distribution $|x|^z$ or $|x|^z \operatorname{sgn}$ at $z = -k \in \mathbb{Z}_-$ is replaced by its equivalence set of extensions.

Theorem 9. *The structure* $\left(\mathcal{SH}_1', .\right)$ *is the largest proper submonoid of* $(\mathcal{H}', .)$.

Proof. By Fourier transformation and Theorem 8. ■

Theorem 10. *The set* $\mathcal{SH}_1'$ *is invariant under Fourier transformation.*

Proof. Any element $g^z \in \mathcal{SH}_1'$ can be represented in terms of the normalized parity representation as in definition (1). Using the Fourier transforms [2, eqs. (321)–(322)],

$$\begin{aligned} \mathcal{F}\left[\Phi_e^z\right] &= |2\pi\chi|^{-z}, \forall z \in \mathbb{C}\backslash\mathbb{Z}_{o,+}, \\ \mathcal{F}\left[\Phi_o^z\right] &= -i\,|2\pi\chi|^{-z}\operatorname{sgn}, \forall z \in \mathbb{C}\backslash\mathbb{Z}_{e,+}, \end{aligned}$$

and [2, eqs. (325)–(326)] applied to general extensions,

$$\begin{aligned}\mathcal{F}\left[\left(\Phi_e^k\right)_e\right] &= (2\pi)^{-k}\left(\chi^{-k}\,\mathrm{sgn}\right)_e, \forall k \in \mathbb{Z}_{o,+},\\ \mathcal{F}\left[\left(\Phi_o^k\right)_e\right] &= -i\,(2\pi)^{-k}\left(\chi^{-k}\,\mathrm{sgn}\right)_e, \forall k \in \mathbb{Z}_{e,+},\end{aligned}$$

we see that the Fourier transform $\mathcal{F}[g^z]$ is of the form

$$g^z = p_e(z)\,|x|^z + p_o(z)\left(|x|^z\,\mathrm{sgn}\right)$$

and wherein any partial distribution $|x|^z$ or $|x|^z\,\mathrm{sgn}$ at $z = -k \in \mathbb{Z}_-$ is replaced by its equivalence set of extensions. The latter is the parity representation (2) for an element of $\mathcal{SH}_1'$. ■

Define $\mathcal{SH}_1'^{-1} \triangleq \{a(z)\,\delta + b(z)\,\eta, \forall a,b \in \mathcal{A}(\mathbb{C},\mathbb{C})\} \subset \mathcal{SH}_1'$. Any convolution operator $\omega *$, with kernel $\omega \in \mathcal{SH}_1'^{-1}$, is a degree of homogeneity and order of association preserving endomorphism of $\mathcal{SH}_1'$.

Define $\mathcal{SH}_1'^{0} \triangleq \{a(z)\,1 + b(z)\,\mathrm{sgn}, \forall a,b \in \mathcal{A}(\mathbb{C},\mathbb{C})\} \subset \mathcal{SH}_1'$. Any multiplication operator $\omega.$, with kernel $\omega \in \mathcal{SH}_1'^{0}$, is a degree of homogeneity and order of association preserving endomorphism of $\mathcal{SH}_1'$.

2.2.3 Structure theorem

The only idempotents in $\left(\mathcal{SH}_1', *\right)$ are $\frac{1}{2}(\delta \pm i\eta)$ (the Heisenberg distributions) and the only idempotents in $\left(\mathcal{SH}_1', .\right)$ are $1_\pm$ (the Heaviside distributions).

Definition 11. *Define*

$$\mathcal{I}'_{*,\pm} \triangleq \left\{h^z \in \mathcal{SH}_1' : h^z = \frac{1}{2}(\delta \pm i\eta) * f^z, \forall f^z \in \mathcal{SH}_1'\right\}, \tag{3}$$

$$\mathcal{I}'_{.,\pm} \triangleq \left\{h^z \in \mathcal{SH}_1' : h^z = 1_\pm . f^z, \forall f^z \in \mathcal{SH}_1'\right\}, \tag{4}$$

and

$$\mathcal{I}'_{*} \triangleq \left\{g^z \in \mathcal{SH}_1' : q_e^2(z) + q_o^2(z) \neq 0, \forall z \in \mathbb{C}\right\}, \tag{5}$$

$$\mathcal{I}'_{.} \triangleq \left\{g^z \in \mathcal{SH}_1' : p_e^2(z) - p_o^2(z) \neq 0, \forall z \in \mathbb{C}\right\}. \tag{6}$$

The sets $\mathcal{I}'_{*,\pm}$ are proper and principal ideals of $\left(\mathcal{SH}_1', *\right)$. Consequently there are zero divisors in $\left(\mathcal{SH}_1', *\right)$, since $\forall f_-^{a-1} \in \mathcal{I}'_{*,-}$ and $\forall f_+^{b-1} \in \mathcal{I}'_{*,+}$ holds that $f_-^{a-1} * f_+^{b-1} = 0, \forall (a+b-1) \in \mathbb{C}\backslash\mathbb{N}$. Further, it is easy to show, based on the representation (1) and the linearly independence of Φ_e^{z+1} and Φ_o^{z+1}, $\forall z \in \mathbb{C}$, that $\mathcal{I}'_{*,-} \cap \mathcal{I}'_{*,+} = \{0\}$.

Similarly, $\mathcal{I}'_{.,\pm}$ are proper and principal ideals of $\left(\mathcal{SH}_1', .\right)$. Consequently, $\forall f_-^a \in \mathcal{I}'_{.,-}$ and $\forall f_+^b \in \mathcal{I}'_{.,+}$ holds that $f_-^a . f_+^b = 0, \forall (a+b) \in \mathbb{C}\backslash\mathbb{Z}_-$. Further, $\mathcal{I}'_{.,-} \cap \mathcal{I}'_{.,+} = \{0\}$.

From (3), (4) and Theorem 10 follows that $\mathcal{I}'_{.,\pm} = \mathcal{F}\left[\mathcal{I}'_{*,\pm}\right]$, respectively. Also, $\mathcal{I}'_{.} = \mathcal{F}\left[\mathcal{I}'_{*}\right]$.

Definition 12. *The operators $A_\pm : \mathcal{H}' \to \mathcal{H}'$ such that $f \mapsto \widehat{f}_\pm \triangleq A_\pm f$, with*

$$\begin{aligned} A_\pm &\triangleq \mp \frac{1}{2\pi i} (x \pm i0)^{-1} *, && (7)\\ &= \Phi^0_{x\pm i0} *, && (8)\\ &= \frac{1}{2} (\delta \pm i\eta) *, && (9) \end{aligned}$$

are generalized analyticity operators *on R. The resulting distributions $\widehat{f}_\pm$ are said to be* complex analytic on R *and are called the* generalized analytic extensions *of f on R.*

Eq. (8) shows that the kernels of the analyticity operators $A_\pm$ are the Heisenberg distributions $\Phi^0_{x\pm i0}$. A distribution complex analytic on R, can be regarded as a generalization of what in physics is called an analytic signal. Hence, the ideals $\mathcal{I}'_{*,\pm}$ consist of the complex analytic extensions of the elements in $\mathcal{SH}'_1$.

It follows from [6, Theorem 4] that the set $\mathcal{I}'_*$ consists of all the (uniquely) invertible AHDs on R under convolution. From the associativity of equivalence sets, [6, Theorem 3], and since $\delta \in \mathcal{I}'_*$ it follows that $(\mathcal{I}'_*, *)$ is an Abelian group. The convolutional inverse, $\left(g^{z-1}\right)^{-1}_*$, of

$$g^{z-1} = q_e (z-1) \Phi^z_e + q_o (z-1) \Phi^z_o \in \mathcal{SH}'_1,$$

exists provided $q^2_e (z-1) + q^2_o (z-1) \neq 0$ and is given by

$$\left(g^{z-1}\right)^{-1}_* = \frac{q_e (z-1)}{q^2_e (z-1) + q^2_o (z-1)} \Phi^{-z}_e - \frac{q_o (z-1)}{q^2_e (z-1) + q^2_o (z-1)} \Phi^{-z}_o.$$

Similarly, from [7, Theorem 6] follows that the set $\mathcal{I}'_.$ consists of all the (uniquely) invertible AHDs on R under multiplication. From the associativity of equivalence sets, [7, Theorem 3], and since $1 \in \mathcal{I}'_.$ it follows that $(\mathcal{I}'_., .)$ is an Abelian group. The multiplicative inverse, $(g^z)^{-1}_.$, of

$$g^z = p_e (z) |x|^z + p_o (z) \left(|x|^z \operatorname{sgn}\right) \in \mathcal{SH}'_1,$$

exists provided $p^2_e (z) - p^2_o (z) \neq 0$ and is given by

$$(g^z)^{-1}_. = \frac{p_e (z)}{p^2_e (z) - p^2_o (z)} |x|^{-z} - \frac{p_o (z)}{p^2_e (z) - p^2_o (z)} \left(|x|^{-z} \operatorname{sgn}\right).$$

We have the following structure theorem for $\mathcal{SH}'_1$.

Theorem 13. *There holds, $\mathcal{I}'_{*,-} \cup \mathcal{I}'_* \cup \mathcal{I}'_{*,+} = \mathcal{SH}'_1 = \mathcal{I}'_{.,-} \cup \mathcal{I}'_. \cup \mathcal{I}'_{.,+}$.*

Proof. A. $\mathcal{SH}'_1 = \mathcal{I}'_{*,-} \cup \mathcal{I}'_* \cup \mathcal{I}'_{*,+}$.

Any element $f^z \in \mathcal{SH}'_1$ can be represented as in definition (1). If $q^2_e (z) + q^2_o (z) \neq 0, \forall z \in \mathbb{C}$, then $f^z \in \mathcal{I}'_*$. Else, if $q^2_e (z) + q^2_o (z) = 0$ for some z, then $q_o (z) = \pm i q_e (z)$ and $f^z \in \mathcal{I}'_{*,-}$ or $f^z \in \mathcal{I}'_{*,+}$.

B. Similarly, using (2). ∎

It will be shown elsewhere that among the elements of $\mathcal{I}'_*$ are kernels of convolution operators that act as generalized complex order integration/derivation operators for the whole line R. Similarly, among the elements of $\mathcal{I}'_.$ are kernels of multiplication operators which also act as a second type of (homomorphic) generalized complex order integration/derivation operators for the whole line R.

Both types of generalized integration over R are distinguished as follows. Convolution operators, having kernels of negative integer degree of homogeneity $z = -k \in \mathbb{Z}_-$ from $\mathcal{I}'_*$, are generalized 'multiplication' derivations of the form $\delta^{(k)}*$ (since they satisfy Leibniz' rule with respect to the multiplication product). Multiplication operators, having kernels of non-negative integer degree of homogeneity $z = k \in \mathbb{N}$ from $\mathcal{I}'_.$, are generalized 'convolution' derivations of the form x^k. (since they satisfy Leibniz' rule with respect to the convolution product). Also see [2, Section 3.2].

2.3 General polynomial structure

Theorem 14. *Any element of $\mathcal{H}'$ is a polynomial in the variable $(D_w \Phi_e^w)_{w=0}$, of degree equal to the order of association, with coefficients in $(\mathcal{SH}'_1, *)$ and for which the external product is convolution.*

Proof. Using [6, eq. (38)], we can rewrite the representation [6, eq. (51)] for an element of $\mathcal{H}'$ of order of association m in the following form,

$$f_m^z = \sum_{n=0}^{m} \left(\alpha_n(z) * \Phi_e^{z+1} \right) * \left(D_w^n \Phi_e^w \right)_{w=0}. \tag{10}$$

The coefficients in (10),

$$\begin{aligned} g^z &\triangleq \alpha_n(z) * \Phi_e^{z+1}, \\ &= \left(q_{n,e}(z)\,\delta + q_{n,o}(z)\,\eta \right) * \Phi_e^{z+1}, \\ &= q_{n,e}(z)\,\Phi_e^{z+1} + q_{n,o}(z)\,\Phi_o^{z+1}, \end{aligned}$$

are elements of $\mathcal{SH}'_1$, because (i) if $z \in \mathbb{C}\backslash\mathbb{Z}_+$, then $g^z \in \mathcal{H}'_0 \subset \mathcal{SH}'_1$ and (ii) if $z = k \in \mathbb{Z}_+$, either Φ_e^{z+1} or Φ_o^{z+1} is to be replaced by any extension $\left(\Phi_e^{2p+1}\right)_e$ or $\left(\Phi_o^{2p+2}\right)_e$, respectively, and then $g^z \in \mathcal{SH}'_1$.

Further, by [6, eq. (38)] again, we have that

$$\left(D_w^p \Phi_e^w\right)_{w=0} * \left(D_w^q \Phi_e^w\right)_{w=0} = \left(D_w^{p+q} \Phi_e^w\right)_{w=0}. \tag{11}$$

Eqs. (10) and (11) shows that any element in $\mathcal{H}'$ is a polynomial, in the variable $(D_w \Phi_e^w)_{w=0}$, with coefficients taken from $\mathcal{SH}'_1$. ■

The normalized parity basis HD Φ_e^z is a generating distribution for the monomial sequence $\{(D_w^n \Phi_e^w)_{w=0}, \forall n \in \mathbb{N}\}$ through its Maclaurin series about $z = 0$.

Using [2, eqs. (303), (10)–(11), (253), (173) and (259)–(260)], we find that (with γ the Euler-Mascheroni constant),

$$\zeta_{e,*} \triangleq (D_w \Phi_e^w)_{w=0} = \gamma\delta + \pi \left(\frac{1}{2} (\eta \operatorname{sgn})_0 \right). \tag{12}$$

The elements of the monomial sequence, $\zeta_{e,*}^n = (D_w^n \Phi_e^w)_{w=0}$, $\forall n \in \mathbb{Z}_+$, will be called *even associated* delta *distributions* (because $\Phi_e^0 = \delta$). We will refer to expression (10) as the *polynomial convolution representation* of an AHD on R.

Theorem 15. *Any element of $\mathcal{H}'$ is a polynomial in the variable $\left(D_w |x|^w\right)_{w=0}$, of degree equal to the order of association, with coefficients in $\left(\mathcal{SH}_1', .\right)$ and for which the external product is multiplication.*

Proof. By Fourier transformation and Theorem 14. ∎

The polynomial representation now takes the form

$$\begin{aligned} f_m^z &= \sum_{n=0}^{m} \left(\alpha_n (z) \,.\, |x|^z\right) . \left(D_w^n |x|^w\right)_{w=0}, && (13) \\ \alpha_n (z) &= p_{n,e} (z)\, 1 + i p_{n,o} (z) \left(-i \operatorname{sgn}\right). && (14) \end{aligned}$$

The polynomial variable is in this case

$$\zeta_{e,\cdot} \triangleq \left(D_w |x|^w\right)_{w=0} = \ln |x| \,. \tag{15}$$

The elements of the monomial sequence, $\zeta_{e,\cdot}^n = \left(D_w^n |x|^w\right)_{w=0} = \ln^n |x|$, $\forall n \in \mathbb{Z}_+$, will be called *even associated one distributions* (because $|x|^0 = 1$). We will refer to expression (13) as the *polynomial multiplication representation* of an AHD on R. This is the form that is classically used to represent AHDs.

2.4 Polynomial substructures

It follows from the product formula for $(\mathcal{H}', *)$, [6, Theorem 6], and for $(\mathcal{H}', .)$, [7, Theorem 8], that all the following substructures are closed.

2.4.1 Half-line convolution subalgebras

Definition 16. *For all $a_{0,\pm} \in \mathcal{A}(\mathbb{C}, \mathbb{C}) : a_{0,\pm}(z) \neq 0$ in $\mathbb{C}$ and $\forall a_{n,\pm} \in \mathcal{A}(\Omega \subseteq \mathbb{C}, \mathbb{C})$, $\forall n \in \mathbb{Z}_+$, define*

$$\begin{aligned} \mathcal{H}_{0,\pm,*}' &\triangleq \left\{ f_{0,\pm}^z \in \mathcal{H}_0' : f_{0,\pm}^z = a_{0,\pm}(z) \Phi_{\pm}^{z+1}, \forall z \in \mathbb{C} \right\}, && (16) \\ \mathcal{H}_{\pm,*}' &\triangleq \left\{ f_{m,\pm}^z \in \mathcal{H}' : f_{m,\pm}^z = \sum_{n=0}^{m} a_{n,\pm}(z) D_z^n \Phi_{\pm}^{z+1}, \forall z \in \Omega, \forall m \in \mathbb{N} \right\}. && (17) \end{aligned}$$

Obviously, $\mathcal{H}_{+,*}' \subset \mathcal{D}_R'$ and $\mathcal{H}_{-,*}' \subset \mathcal{D}_L'$.

Theorem 17. *Any element of $\mathcal{H}'_{\pm,*}$ is a polynomial with coefficients from the Abelian group $\left(\mathcal{H}'_{0,\pm,*}, *\right)$ and in the variable $(D_w \Phi^w_\pm)_{w=0}$, for the respective sign.*

Proof. (i) Due to the fact that $\Phi^0_\pm = \delta$ and in view of the restrictions placed on the coefficient functions $a_{0,\pm}$, the properties [4, eqs. (22)–(23)] of the distributions $\Phi^{z+1}_\pm$ and the commutativity of the convolution product in $\left(\mathcal{H}'_{0,\pm,*}, *\right)$, we have that $\left(\mathcal{H}'_{0,\pm,*}, *\right)$ are Abelian groups.

(ii) Due to [4, eqs. (22)–(23)], any $f^z_{m,\pm} \in \mathcal{H}'_{\pm,*}$ can be written as

$$f^z_{m,\pm} = \sum_{n=0}^{m} \left(a_{n,\pm}(z)\Phi^{z+1}_\pm\right) * (D^n_w \Phi^w_\pm)_{w=0}\,, \tag{18}$$

showing that any element of $\mathcal{H}'_{\pm,*}$ is a polynomial with coefficients from $\mathcal{H}'_{0,\pm,*}$. ■

The normalized half-line basis AHDs $\Phi^z_\pm$ are generating distributions for the respective polynomial sequences $\left\{(D^n_w \Phi^w_\pm)_{w=0}\,, \forall n \in \mathbb{N}\right\}$, through their Maclaurin series about $z = 0$. Using [2, eqs. (253), (10), (171) and (259)–(260)], we find the polynomial variable in this case to be

$$\zeta_{\pm,*} \triangleq (D_w \Phi^w_\pm)_{w=0} = \gamma\delta + \pi\eta_{\pm,0}. \tag{19}$$

The elements $\zeta^n_{\pm,*} = (D^n_w \Phi^w_\pm)_{w=0}$, $\forall n \in \mathbb{Z}_+$, will be called *half-line associated delta distributions* (because $\Phi^0_\pm = \delta$).

The sets $\mathcal{H}'_{0,\pm,*}$ contain the convolution kernels for complex degree integration/derivation over half-lines, so $\left(\mathcal{H}'_{0,\pm,*}, *\right)$ is the structure which serves as justification for the distributional generalization of the classical fractional calculus on half-lines.

Definition 18. *Denote by $\mathcal{F}'_{0,\pm,*}$ and $\mathcal{F}'_{\pm,*}$ the set of all finite sums of elements of $\mathcal{H}'_{0,\pm,*}$ and $\mathcal{H}'_{\pm,*}$ over $\mathbb{C}$, respectively.*

The structures $\left(\mathcal{F}'_{\pm,*}, +, *\right)$ are convolution algebras over $\mathbb{C}$, called the *half-line convolution subalgebras*.

2.4.2 Complex multiplication subalgebras

Definition 19. *For all $c_{0,\pm} \in \mathcal{A}(\mathbb{C},\mathbb{C}) : c_{0,\pm}(z) \neq 0$ in $\mathbb{C}$ and $\forall c_{n,\pm} \in \mathcal{A}(\Omega \subseteq \mathbb{C},\mathbb{C})$, $\forall n \in \mathbb{Z}_+$, define*

$$\mathcal{H}'_{0,x\pm i0,\cdot} \triangleq \left\{f^z_{0,\pm} \in \mathcal{H}'_0 : f^z_{0,\pm} = c_{0,\pm}(z)\,(x \pm i0)^z\,, \forall z \in \mathbb{C}\right\}, \tag{20}$$

$$\mathcal{H}'_{x\pm i0,\cdot} \triangleq \left\{f^z_{m,\pm} \in \mathcal{H}' : f^z_{m,\pm} = \sum_{n=0}^{m} c_{n,\pm}(z) D^n_z\,(x \pm i0)^z\,, \forall z \in \Omega, \forall m \in \mathbb{N}\right\}. \tag{21}$$

Obviously, $\mathcal{H}'_{x\pm i0,\cdot} \subset \mathcal{Z}'_\pm$ (see [2]).

Theorem 20. *Any element of $\mathcal{H}'_{x\pm i0,\cdot}$ is a polynomial with coefficients from the Abelian group $\left(\mathcal{H}'_{0,x\pm i0,\cdot},.\right)$ and in the variable $\left(D_w\left(x\pm i0\right)^w\right)_{w=0}$, respectively.*

Proof. By Fourier transformation and Theorem 17. ∎

The polynomial representation (18) now takes the form

$$f^z_{m,\pm}=\sum_{n=0}^{m}\left(c_{n,\pm}(z)\left(x\pm i0\right)^z\right).\left(D^n_w\left(x\pm i0\right)^w\right)_{w=0}, \tag{22}$$

The complex basis AHDs $(x\pm i0)^z$ are generating distributions for the respective polynomial sequences $\left\{\left(D^n_w\left(x\pm i0\right)^w\right)_{w=0},\forall n\in\mathbb{N}\right\}$, through their Maclaurin series about $z=0$. Using [2, eq. (189)], we find that the polynomial variable in this case is

$$\zeta_{x\pm i0,\cdot}\triangleq\left(D_w\left(x\pm i0\right)^w\right)_{w=0}=\mp i\pi 1-\ln\left(x\mp i0\right)=-\left(\ln|x|\pm i\pi\frac{1}{2}\operatorname{sgn}\right). \tag{23}$$

The elements $\zeta^n_{x\pm i0,\cdot}=\left(D^n_w\left(x\pm i0\right)^w\right)_{w=0}$, $\forall n\in\mathbb{Z}_+$, will be called *complex associated one distributions* (because $(x\pm i0)^0=1$).

Definition 21. *Denote by $\mathcal{F}'_{0,x\pm i0,\cdot}$ and $\mathcal{F}'_{x\pm i0,\cdot}$ the set of all finite sums of elements of $\mathcal{H}'_{0,x\pm i0,\cdot}$ and $\mathcal{H}'_{x\pm i0,\cdot}$ over $\mathbb{C}$, respectively.*

The structures $\left(\mathcal{F}'_{x\pm i0,\cdot},+,.\right)$ are multiplication algebras over $\mathbb{C}$, called the *complex multiplication subalgebras*.

2.4.3 Even convolution subalgebra

Definition 22. *For all $q_{0,e}\in\mathcal{A}(\mathbb{C},\mathbb{C}):q_{0,e}(z)\neq 0$ in $\mathbb{C}$ and $\forall q_{n,e}\in\mathcal{A}(\Omega\subseteq\mathbb{C},\mathbb{C})$, $\forall n\in\mathbb{Z}_+$, define*

$$\mathcal{SH}'_{1,e,*}\triangleq\left\{f^z_{1,e}\in\mathcal{SH}'_1:f^z_{1,e}=q_{0,e}(z)\Phi^{z+1}_e,\forall z\in\mathbb{C}\right\}, \tag{24}$$

$$\mathcal{H}'_{e,*}\triangleq\left\{f^z_{m,e}\in\mathcal{H}':f^z_{m,e}=\sum_{n=0}^{m}q_{n,e}(z)D^n_z\Phi^{z+1}_e,\forall z\in\Omega,\forall m\in\mathbb{N}\right\}. \tag{25}$$

At $z=k\in\mathbb{Z}_{o,+}$, the distributions $D^n_z\Phi^z_e$ in (24)–(25) are to be replaced by their equivalence set of extensions.

Theorem 23. *Any element of $\mathcal{H}'_{e,*}$ is a polynomial with coefficients from the Abelian group $\left(\mathcal{SH}'_{1,e,*},*\right)$ and in the variable $\left(D_w\Phi^w_e\right)_{w=0}$.*

Proof. (i) Due to the fact that $\Phi^0_e=\delta$ and the restriction placed on the coefficient function $q_{0,e}$, the convolution property [6, eq. (38) with $m=n=0$] of the distribution Φ^{z+1}_e and the commutativity of the convolution product of equivalence sets in $\left(\mathcal{SH}'_{1,e,*},*\right)$, we have that $\left(\mathcal{SH}'_{1,e,*},*\right)$ is an Abelian group.

(ii) Due to [6, eq. (38)], any $f^z_{m,e} \in \mathcal{H}'_{e,*}$ can be written as

$$f^z_{m,e} = \sum_{n=0}^{m} \left(q_{n,e}(z)\Phi_e^{z+1} \right) * \left(D^n_w \Phi^w_e \right)_{w=0}, \tag{26}$$

showing that any element of $\mathcal{H}'_{e,*}$ is a polynomial with coefficients from $\mathcal{SH}'_{1,e,*}$ and in the variable $\zeta_{e,*}$ given by (12). ∎

Definition 24. *Denote by $\mathcal{SF}'_{1,e,*}$ and $\mathcal{F}'_{e,*}$ the set of all finite sums of elements of $\mathcal{SH}'_{1,e,*}$ and $\mathcal{H}'_{e,*}$ over $\mathbb{C}$, respectively.*

The structure $\left(\mathcal{SF}'_{1,e,*}, +, *\right)$ is a non-associative ring with identity. The structure $\left(\mathcal{F}'_{e,*}, +, *\right)$ is a convolution algebra over $\mathbb{C}$, called the *even convolution subalgebra*.

2.4.4 Even multiplication subalgebra

Definition 25. *For all $p_{0,e} \in \mathcal{A}(\mathbb{C}, \mathbb{C}) : p_{0,e}(z) \neq 0$ in $\mathbb{C}$ and $\forall p_{n,e} \in \mathcal{A}(\Omega \subseteq \mathbb{C}, \mathbb{C})$, $\forall n \in \mathbb{Z}_+$, define*

$$\mathcal{SH}'_{1,e,\cdot} \triangleq \left\{ f^z_{1,e} \in \mathcal{SH}'_1 : f^z_{1,e} = p_{0,e}(z) \left|x\right|^z, \forall z \in \mathbb{C} \right\}, \tag{27}$$

$$\mathcal{H}'_{e,\cdot} \triangleq \left\{ f^z_{m,e} \in \mathcal{H}' : f^z_{m,e} = \sum_{n=0}^{m} p_{n,e}(z) D^n_z \left|x\right|^z, \forall z \in \Omega, \forall m \in \mathbb{N} \right\}. \tag{28}$$

At $z = -k \in \mathbb{Z}_{0,-}$, the distributions $D^n_z \left|x\right|^z$ in (27)–(28) are to be replaced by their equivalence set of extensions.

Theorem 26. *Any element of $\mathcal{H}'_{e,\cdot}$ is a polynomial with coefficients from the Abelian group $\left(\mathcal{SH}'_{1,e,\cdot}, +, .\right)$ and in the variable $\left(D_w \left|x\right|^w\right)_{w=0}$.*

Proof. By the Fourier transformation and Theorem 23. ∎

The polynomial representation (26) now takes the form

$$f^z_{m,e} = \sum_{n=0}^{m} \left(p_{n,e}(z) \left|x\right|^z \right) . \left(D^n_w \left|x\right|^w \right)_{w=0}, \tag{29}$$

with the polynomial variable $\zeta_{e,\cdot}$ given by (15).

Definition 27. *Denote by $\mathcal{SF}'_{1,e,\cdot}$ and $\mathcal{F}'_{e,\cdot}$ the set of all finite sums of elements of $\mathcal{SH}'_{1,e,\cdot}$ and $\mathcal{H}'_{e,\cdot}$ over $\mathbb{C}$, respectively.*

The structure $\left(\mathcal{F}'_{e,\cdot}, +, .\right)$ is a convolution algebra over $\mathbb{C}$, called the *even multiplication subalgebra*.

2.4.5 Complex convolution subalgebras

Definition 28. *For all $c_{0,\pm} \in \mathcal{A}(\mathbb{C},\mathbb{C}) : c_{0,\pm}(z) \neq 0$ in $\mathbb{C}$ and $\forall c_{n,\pm} \in \mathcal{A}(\Omega \subseteq \mathbb{C},\mathbb{C})$, $\forall n \in \mathbb{Z}_+$, define*

$$\mathcal{SH}'_{1,x\pm i0,*} \triangleq \left\{ f^z_{1,x\pm i0} \in \mathcal{H}'_1 : f^z_{1,x\pm i0} = c_{0,\pm}(z)\Phi^{z+1}_{x\pm i0}, \forall z \in \mathbb{C} \right\}, \tag{30}$$

$$\mathcal{H}'_{x\pm i0,*} \triangleq \left\{ f^z_{m,x\pm i0} \in \mathcal{H}' : f^z_{m,x\pm i0} = \sum_{n=0}^{m} c_{n,\pm}(z) D^n_z \Phi^{z+1}_{x\pm i0}, \forall z \in \Omega, \forall m \in \mathbb{N} \right\}. \tag{31}$$

At $z = k \in \mathbb{Z}_+$, the distributions $D^n_z \Phi^z_{x\pm i0}$ in (30)–(31) are to be replaced by their equivalence set of extensions.

Theorem 29. *Any element of $\mathcal{H}'_{x\pm i0,*}$ is a polynomial with coefficients from the Abelian group $(\mathcal{SH}'_{1,x\pm i0,*}, *)$ and in the variable $(D_w \Phi^w_{x\pm i0})_{w=0}$, respectively.*

Proof. (i) Due to the fact that the Heisenberg distributions $\Phi^0_{x\pm i0}$ are identity elements in $(\mathcal{SH}'_{1,x\pm i0,*}, *)$ and the restrictions placed on the coefficient functions $c_{0,\pm}$, the convolution properties [6, eqs. (47) and (49) with $m = n = 0$] of the distributions $\Phi^z_{x\pm i0}$ and the commutativity of the convolution product of equivalence sets in $(\mathcal{SH}'_{1,x\pm i0,*}, *)$, we have that $(\mathcal{SH}'_{1,x\pm i0,*}, *)$ are Abelian groups.

(ii) Due to [6, Corollary 5], any $f^z_{m,x\pm i0} \in \mathcal{H}'_{x\pm i0,*}$ can be written as

$$f^z_{m,x\pm i0} = \sum_{n=0}^{m} \left(c_{n,\pm}(z)\Phi^z_{x\pm i0}\right) * \left(D^n_w \Phi^w_{x\pm i0}\right)_{w=0}, \tag{32}$$

showing that any element of $\mathcal{H}'_{x\pm i0,*}$ is a polynomial with coefficients from $\mathcal{SH}'_{1,x\pm i0,*}$ in the variables $(D_w \Phi^w_{x\pm i0})_{w=0}$, respectively. ∎

The normalized complex basis AHDs $\Phi^z_{x\pm i0}$ are generating distributions for the respective polynomial sequences $\left\{ \left(D^n_w \Phi^w_{x\pm i0}\right)_{w=0}, \forall n \in \mathbb{N} \right\}$, through their Maclaurin series about $z = 0$. Using [2, eqs. (354), (10)–(11), (133), (152), (172), (173) and (259)–(260)], we find that the polynomial variable in this case is

$$\zeta_{x\pm i0,*} \triangleq \left(D_w \Phi^w_{x\pm i0}\right)_{w=0} = \gamma\frac{1}{2}(\delta \pm i\eta) + \pi\frac{1}{2}\left(\frac{1}{2}(\eta\,\mathrm{sgn})_0 \pm i\frac{1}{\pi}\eta \ln|x|\right). \tag{33}$$

The elements $\zeta^n_{x\pm i0,*} = \left(D^n_w \Phi^w_{x\pm i0}\right)_{w=0}$ will be called *associated normalized complex distributions*. Due to [2, eq. (346)], $\zeta_{x+i0,*} + \zeta_{x-i0,*} = \zeta_{e,*}$.

Definition 30. *Denote by $\mathcal{SF}'_{1,x\pm i0,*}$ and $\mathcal{F}'_{x\pm i0,*}$ the set of all finite sums of elements of $\mathcal{SH}'_{1,x\pm i0,*}$ and $\mathcal{H}'_{x\pm i0,*}$ over $\mathbb{C}$, respectively.*

The structures $\left(\mathcal{F}'_{x\pm i0,*}, +, *\right)$ are convolution algebras over $\mathbb{C}$, called the *complex convolution subalgebras*.

2.4.6 Half-line multiplication subalgebras

Definition 31. *For all $a_{0,\pm} \in \mathcal{A}(\mathbb{C},\mathbb{C}) : a_{0,\pm}(z) \neq 0$ in $\mathbb{C}$ and $\forall a_{n,\pm} \in \mathcal{A}(\Omega \subseteq \mathbb{C},\mathbb{C})$, $\forall n \in \mathbb{Z}$, define*

$$\mathcal{SH}'_{1,\pm,\cdot} \triangleq \left\{ f^z_{1,\pm} \in \mathcal{H}'_1 : f^z_{1,\pm} = a_{0,\pm}(z) x^z_\pm, \forall z \in \mathbb{C} \right\}, \tag{34}$$

$$\mathcal{H}'_{\pm,\cdot} \triangleq \left\{ f^z_{m,\pm} \in \mathcal{H}' : f^z_{m,\pm} = \sum_{n=0}^{m} a_{n,\pm}(z) D^n_z x^z_\pm, \forall z \in \Omega, \forall m \in \mathbb{N} \right\}. \tag{35}$$

At $z = -k \in \mathbb{Z}_-$, the distributions $D^n_z x^z_\pm$ in (34)–(35) are to be replaced by their equivalence set of extensions.

Theorem 32. *Any element of $\mathcal{H}'_{\pm,\cdot}$ is a polynomial with coefficients from the Abelian group $\left(\mathcal{SH}'_{1,\pm,\cdot}, .\right)$ and in the variable $\left(D_w x^w_\pm\right)_{w=0}$, respectively.*

Proof. By Fourier transformation and Theorem 29. ∎

The polynomial representation (32) now takes the form

$$f^z_{m,\pm} = \sum_{n=0}^{m} \left(a_{n,\pm}(z) x^z_\pm\right) . \left(D^n_w x^w_\pm\right)_{w=0}, \tag{36}$$

The half-line basis AHDs $x^z_\pm$ are generating distributions for the respective polynomial sequences $\left\{ \left(D^n_w x^w_\pm\right)_{w=0}, \forall n \in \mathbb{N} \right\}$, through their Maclaurin series about $z = 0$. Using [2, eq. (111)], we find for the polynomial variable in this case

$$\zeta_{\pm,\cdot} \triangleq \left(D_w x^w_\pm\right)_{w=0} = 1_\pm \ln|x|. \tag{37}$$

The elements $\zeta^n_{\pm,\cdot} = \left(D^n_w x^w_\pm\right)_{w=0} = 1_\pm \ln^n|x|$ will be called *half-line associated step distributions*. Due to [2, eq. (130)], $\zeta_{+,\cdot} + \zeta_{-,\cdot} = \zeta_{e,\cdot}$.

Definition 33. *Denote by $\mathcal{SF}'_{1,\pm,\cdot}$ and $\mathcal{F}'_{\pm,\cdot}$ the set of all finite sums of elements of $\mathcal{SH}'_{1,\pm,\cdot}$ and $\mathcal{H}'_{\pm,\cdot}$ over $\mathbb{C}$, respectively.*

The structures $\left(\mathcal{F}'_{\pm,\cdot}, +, .\right)$ are multiplication algebras over $\mathbb{C}$, called the *half-line multiplication subalgebras*.

2.5 Factor ring structures

(i) In the algebra $(\mathcal{F}'_*/\mathcal{N}'_0, +, *)$ we have commutativity and associativity for the convolution product, single-valuedness of critical products with resulting degree $k \in \mathbb{N}$, homogeneous and unique extensions at $\mathbb{Z}_+$ (e.g., $\left(\Phi^k_{e,o}\right)_0$), equivalence of x^k_+ with x^k_-, etc., and this structure becomes a commutative and associative ring with $*$-identity. This is a consequence of a forthcoming impossibility theorem, stating that enforcing a prolongation of the convolution product from a subset of distributions (for which the commutative and associative convolution product is defined) to a superset of generalized functions, while retaining all product properties, makes us loose the 1 ideal $\mathcal{N}'_0$.

(ii) In the algebra $(\mathcal{F}'_{\cdot}/\mathcal{E}'_0, +, .)$ we have commutativity and associativity for the multiplication product, single-valuedness of critical products with resulting degree $k \in \mathbb{Z}_-$, homogeneous and unique extensions at $\mathbb{Z}_-$ (e.g., $x_{\pm,0}^{-k}$), equivalence of $\eta_{+,0}^{(k)}$ with $\eta_{-,0}^{(k)}$, etc., and this structure becomes a commutative and associative ring with .-identity. This is a consequence of Schwartz' impossibility theorem, [12], stating that enforcing a prolongation of the multiplication product from the set of continuous functions (for which the commutative and associative multiplication product is defined) to a superset of generalized functions, while retaining all product properties, makes us loose the δ ideal $\mathcal{E}'_0$.

(iii) In the algebras $(\mathcal{F}'_*/(\mathcal{N}'_0 \cup \mathcal{E}'_0), +, *)$ and $(\mathcal{F}'_{\cdot}/(\mathcal{N}'_0 \cup \mathcal{E}'_0), +, .)$ we restore associativity, single-valuedness and get homogeneous and unique extensions at integer degrees of homogeneity. Both structures are homomorphic under the Fourier transformation. Notice that $\mathcal{N}'_0 \cup \mathcal{E}'_0$ is not an ideal, not with respect to the convolution product nor with respect to the multiplication product.

(iv) When $\mathcal{F}'_*$ or $\mathcal{F}'_{\cdot}$ is considered as an algebra of partial distributions, defined only on $\mathcal{D}_{\mathbb{Z}}$, all these partial distributions are regular. Then, the resulting distributional convolution algebra is isomorphic to a convolution algebra of integrable functions and the resulting distributional multiplication algebra is isomorphic to a multiplication algebra of integrable functions.

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