On the existence of infinitely many periodic solutions for second-order ordinary p-Laplacian system*

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Abstract

By using minimax methods in critical point theory, some new existence theorems of infinitely many periodic solutions are obtained for a second-order ordinary p-Laplacian system. The results obtained generalize many known works in the literature.

1. Introduction

Consider the periodic solutions of the following ordinary p-Laplacian system

$$\frac{d}{dt}(|\dot{u}(t)|^{p-2}\dot{u}(t)) - L(t)|u(t)|^{p-2}u(t) + \nabla F(t, u(t)) = 0, \quad a.e.\ t \in \mathbb{R}, \tag{1.1}$$

where $p > 1$, $T > 0$, $F : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}$ is T-periodic in t for all $u \in \mathbb{R}^n$, $\nabla F(t, u)$ is the gradient of $F(t, u)$ with respect to u. $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ is a positive definite symmetric matrix.

Throughout this paper, we always assume the following condition holds.

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(A) $F(t,x)$ is measurable in t for all $x \in \mathbb{R}^n$ and continuously differentiable in x for a.e. $t \in [0,T]$, and there exist $a \in C(\mathbb{R}^+, \mathbb{R}^+)$, $b \in L^1(0,T;\mathbb{R}^+)$ such that

$$|F(t,x)| \leq a(|x|)b(t), \quad |\nabla F(t,x)| \leq a(|x|)b(t)$$

for all $x \in \mathbb{R}^n$ and a.e. $t \in [0,T]$.

When $p = 2$, problem (1.1) becomes the following second-order Hamiltonian system

$$\ddot{u}(t) - L(t)u(t) + \nabla F(t,u(t)) = 0, \quad a.e.\ t \in \mathbb{R}. \tag{1.2}$$

When $L(t) = 0$, problem (1.2) reduces to the following Hamiltonian system

$$\ddot{u}(t) + \nabla F(t,u(t)) = 0, \quad a.e.\ t \in \mathbb{R}. \tag{1.3}$$

Taking $L(t) = 0$ in problem (1.1), then we have

$$\frac{d}{dt}(|\dot{u}(t)|^{p-2}\dot{u}(t)) + \nabla F(t,u(t)) = 0, \quad a.e.\ t \in \mathbb{R}. \tag{1.4}$$

Recently there are many papers concerning the existence of periodic solutions or homoclinic solutions for problems (1.2) and (1.3) via critical point theory. Here for identifying a few, we only mention [1,3,10,14-16,19,20,22]. However, there are not so many results about p-Laplacian systems. In [17], by using the dual least action principle in variational method, Tian and Ge obtained an existence result, which generalized Theorem 3.5 in [8]; in [4], Jebelean and Morosanu obtained two existence results by the least action principle and the Mountain Pass Lemma under nonlinear boundary conditions; Mawhin [6] got some existence results using the Schauder's fixed point theorem; the authors in [2,11] generalized problem (1.3) to differential inclusion systems, and got some existence results by the nonsmooth critical point theory; Paşca and Tang [12] obtained a result on the existence of infinite subharmonic solutions for sublinear differential inclusions systems with p-Laplacian by minimax methods in critical point theory; in [7], Manásevich and Mawhin discussed a general vector valued operator, and got some existence results by the topological methods; a multiplicity result was obtained in [5], where the nonlinearity $\nabla F(t,x)$ was assumed to be bounded; by using the Saddle Point Theorem in critical point theory, Xu and Tang [21] generalized the results of problem (1.3) of [19] and obtained some new results; Tang and Xiao [18] investigated homoclinic solutions of a more general ordinary p-Laplacian system and obtained a new result.

In [9], Ma and Zhang generalized the main result of [1] to p-Laplacian system (1.4) and established the existence of infinitely many periodic solutions for (1.4) by minimax methods in critical point theory. More precisely, they obtained the following main theorem.

Theorem A. *(See [9].) Suppose that F satisfies assumption* (A) *and the following conditions:*

(H1) $F(t,x) \geq 0$ *for all* $(t,x) \in \mathbb{R} \times \mathbb{R}^n$;

(H2) $\lim_{|x|\to 0} \frac{F(t,x)}{|x|^p} = 0$ *uniformly for a.e.* $t \in [0,T]$;

(H3) $\liminf_{|x|\to\infty} \frac{F(t,x)}{|x|^p} > 0$ *uniformly for a.e.* $t \in [0,T]$;

(H4) *There exists a positive constant* M *such that* $\limsup_{|x|\to\infty} \frac{F(t,x)}{|x|^r} \leq M$ *uniformly for a.e.* $t \in [0,T]$;

(H5) *There exists* $M_1 > 0$ *such that* $\liminf_{|x|\to\infty} \frac{(\nabla F(t,x),x) - pF(t,x)}{|x|^\mu} \geq M_1$ *uniformly for a.e.* $t \in [0,T]$;

where $r > p$ *and* $\mu > r - p$. *Then problem* (1.4) *has a sequence of distinct periodic solutions with period* k_jT *satisfying* $k_j \in \mathbb{N}$ *and* $k_j \to \infty$ *as* $j \to \infty$.

Motivated by the above papers, we consider the existence of periodic solutions for problem (1.1) and obtain the following theorem.

Theorem 1.1. *Suppose that F satisfies* (A), (H1), (H2), (H4), (H5). *Moreover, assume that the following conditions hold:*

(L) $L \in C(\mathbb{R}, \mathbb{R}^{n^2})$ *is positive definite symmetric T-periodic matrix for all* $t \in \mathbb{R}$ *and there exist constants* $c_2 \geq c_1 > 0$ *such that*

$$c_1|x|^p \leq (L(t)|x|^{p-2}x, x) \leq c_2|x|^p \text{ for all } t \in \mathbb{R} \text{ and } x \in \mathbb{R}^n;$$

(H3)′ $\liminf_{|x|\to\infty} \frac{F(t,x)}{|x|^p} > \frac{c_2}{p}$ *uniformly for a.e.* $t \in [0,T]$.

Then problem (1.1) *has a sequence of distinct nonconstant periodic solutions with period* k_jT *satisfying* $k_j \in \mathbb{N}$ *and* $k_j \to \infty$ *as* $j \to \infty$.

Remark 1.1. The existence results of problem (1.3) have been generalized to p-Laplacian system (1.4) or differential inclusion system. However, similar generalization of problem (1.2) cannot be found in the literature due to the difficulty made by the matrix $L(t)$. In order to overcome this difficulty, we need other condition such as (L).

Remark 1.2. We point out that Theorem 1.1 generalizes Theorem A. From (H3), we know that $\liminf_{|x|\to\infty} \frac{F(t,x)}{|x|^p}$ is bounded from below uniformly for a.e. $[0,T]$, without loss of generality, we can choose a positive constant such as $\frac{c_2}{p}$ such that $\liminf_{|x|\to\infty} \frac{F(t,x)}{|x|^p} > \frac{c_2}{p}$ uniformly for a.e. $t \in [0,T]$, that is our condition (H3)′.

If we use other conditions to replace (H4) and (H5) in Theorem 1.1, then we obtain the following theorem.

Theorem 1.2. *Suppose that L satisfies* (L) *and F satisfies* (A), (H1), (H2), (H3)′ *and the following conditions:*

(H4)′ *There exists a positive constant* M_2 *such that* $\limsup_{|x|\to\infty} \frac{F(t,x)}{|x|^p} \leq M_2$ *uniformly for a.e.* $t \in [0,T]$;

(H6) *There exists* $f \in L^1(0,T;\mathbb{R}^+)$ *such that* $(\nabla F(t,x), x) - pF(t,x) \geq f(t)$ *for all* $x \in \mathbb{R}^n$ *and a.e.* $t \in [0,T]$;

(H7) $\lim_{|x|\to\infty}[(\nabla F(t,x), x) - pF(t,x)] = +\infty$ *for a.e.* $t \in [0,T]$.

Then problem (1.1) *has a sequence of distinct nonconstant periodic solutions with period* $k_j T$ *satisfying* $k_j \in \mathbb{N}$ *and* $k_j \to \infty$ *as* $j \to \infty$.

Theorem 1.3. *The conclusion in Theorem 1.2 is the same if conditions* (H6) *and* (H7) *are replaced by the following conditions, respectively:*

(H6)′ *There exists* $g \in L^1(0,T;\mathbb{R}^+)$ *such that* $(\nabla F(t,x),x) - pF(t,x) \le g(t)$ *for all* $x \in \mathbb{R}^n$ *and a.e.* $t \in [0,T]$;

(H7)′ $\lim_{|x|\to\infty}[(\nabla F(t,x),x) - pF(t,x)] = -\infty$ *for a.e.* $t \in [0,T]$.

Remark 1.3. Our results also hold true even if $L(t) = 0$ or $p = 2$, from this point, our results generalize many results in the literature. As far as we know, existence results of periodic solutions for problem (1.1) cannot be found in the literature. Besides, under the conditions of our theorems, all the periodic solutions we obtain in this paper are nonconstant.

2. Preliminaries

Let k be a positive integer and $W^{1,p}_{kT}$ be the Sobolev space defined by

$$W^{1,p}_{kT} = \{u : \mathbb{R} \to \mathbb{R}^n \mid u \text{ is absolutely continuous, } u(t+kT) = u(t) \text{ and } \dot{u} \in L^p(0,kT;\mathbb{R}^n)\}$$

with the norm

$$\|u\| = \left(\int_0^{kT} |u(t)|^p dt + \int_0^{kT} |\dot{u}(t)|^p dt\right)^{1/p}.$$

Define the functional φ_k on $W^{1,p}_{kT}$ by

$$\varphi_k(u) = \frac{1}{p}\int_0^{kT} [|\dot{u}(t)|^p + (L(t)|u(t)|^{p-2}u(t),u(t))]dt - \int_0^{kT} F(t,u(t))dt, \;\; u \in W^{1,p}_{kT}.$$

It follows from [8] and assumption (A) that the functional φ_k is continuously differentiable on $W^{1,p}_{kT}$ and

$$< \varphi_k'(u), v >= \int_0^{kT} [(|\dot{u}(t)|^{p-2}\dot{u}(t),\dot{v}(t)) + (L(t)|u(t)|^{p-2}u(t),v(t)) - (\nabla F(t,u(t)),v(t))]dt \quad (2.1)$$

for $u, v \in W^{1,p}_{kT}$. It is well known that the solutions of problem (1.1) correspond to the critical points of the functional φ_k.

For $u \in W^{1,p}_{kT}$, let $\bar{u} = \frac{1}{kT}\int_0^{kT} u(t)dt$ and $\tilde{u}(t) = u(t) - \bar{u}$, then it follows from the Proposition 1.1 in [8] that

$$\|u\|_\infty := \max_{t\in[0,kT]} |u(t)| \le ((kT)^{-1/p} + (kT)^{1/q})\|u\| = d_k\|u\|, \qquad (2.2)$$

where $d_k = (kT)^{-1/p} + (kT)^{1/q}$ and if $\frac{1}{kT}\int_0^{kT} u(t)dt = 0$, then

$$\|\tilde{u}\|_\infty := \max_{t\in[0,kT]} |\tilde{u}(t)| \le (kT)^{1/q}\|\dot{u}\|_{L^p}, \tag{2.3}$$

and

$$\|\tilde{u}\|_{L^p}^p \le (kT)^p \|\dot{u}\|_{L^p}^p, \tag{2.4}$$

where $\frac{1}{p}+\frac{1}{q} = 1$. Let $\tilde{W}_{kT}^{1,p} = \{u \in W_{kT}^{1,p} |\ \bar{u} = 0\}$, then $W_{kT}^{1,p} = \tilde{W}_{kT}^{1,p} \bigoplus \mathbb{R}^n$. We will use the following lemma to prove our main results.

Lemma 2.1. *(See [13].) Let E be a real Banach space with $E = X_1 \bigoplus X_2$, where X_1 is finite dimensional. Suppose that $\varphi \in C^1(E,\mathbb{R})$ satisfies the (PS) condition, and*

(a) *There exist constants $\rho, \alpha > 0$ such that $\varphi|_{\partial B_\rho \bigcap X_2} \ge \alpha$, where $B_\rho := \{u \in E |\ \|u\| \le \rho\}$, ∂B_ρ denotes the boundary of B_ρ;*

(b) *There exists an $e \in \partial B_1 \bigcap X_2$ and $L > \rho$ such that if $Q \equiv (\bar{B}_L \bigcap X_1) \bigoplus \{re |\ 0 \le r \le L\}$, then $\varphi|_{\partial Q} \le 0$.*

Then φ possesses a critical value $c \ge \alpha$ which can be characterized as

$$c = \inf_{h\in\Gamma} \max_{u\in Q} \varphi(h(u)),$$

where $\Gamma = \{h \in C(\bar{Q}, E) |\ h = id$ on $\partial Q\}$.

It is well known that a deformation lemma can be proved with the weaker condition (C) replacing the usual (PS) condition. So Lemma 2.1 holds true under condition (C).

3. Proofs of theorems

Proof of Theorem 1.1. The proof is divided into three steps. In the following, C_i $(i = 1, \cdots)$ denote different positive constants.

Step 1. The functional φ_k satisfies condition (C). Let $\{u_n\} \subset W_{kT}^{1,p}$ satisfying $(1 + \|u_n\|)\|\varphi_k'(u_n)\| \to 0$ as $n \to \infty$ and $\varphi_k(u_n)$ is bounded, then, there exists a constant C_1 such that

$$|\varphi_k(u_n)| \le C_1,\ \ (1 + \|u_n\|)\|\varphi_k'(u_n)\| \le C_1. \tag{3.1}$$

From (H4), there exists $M_3 > 0$ such that

$$F(t,x) \le M|x|^r \ \text{ for all } |x| \ge M_3 \ \text{ and a.e. } t \in [0,T]. \tag{3.2}$$

By assumption (A), for $|x| \le M_3$, there exists $C_2 = \max_{|x|\le M_3} a(|x|) > 0$ such that

$$|F(t,x)| \le C_2 b(t),$$

which together with (3.2) implies that

$$F(t,x) \le M|x|^r + C_2 b(t) \ \text{ for all } x \in \mathbb{R}^n \ \text{ and a.e. } t \in [0,T]. \tag{3.3}$$

By (3.1) and (3.3), we have

$$\begin{aligned}\varphi_k(u_n)+\int_0^{kT}F(t,u_n)dt &\le C_1+\int_0^{kT}(M|u_n(t)|^r+C_2b(t))dt\\ &= C_1+C_2k\|b\|_{L^1}+M\int_0^{kT}|u_n(t)|^r dt\\ &= C_3+M\int_0^{kT}|u_n(t)|^r dt. \qquad (3.4)\end{aligned}$$

On the other hand, from (L), we have

$$\begin{aligned}\varphi_k(u_n)+\int_0^{kT}F(t,u_n)dt &= \frac{1}{p}\int_0^{kT}[|\dot{u}_n(t)|^p+(L(t)|u_n(t)|^{p-2}u_n(t),u_n(t))]dt\\ &\ge \frac{1}{p}\int_0^{kT}[|\dot{u}_n(t)|^p+c_1|u_n(t)|^p]dt\\ &\ge \min\left\{\frac{1}{p},\frac{c_1}{p}\right\}\|u_n\|^p\\ &= C_4\|u_n\|^p. \qquad (3.5)\end{aligned}$$

By (3.4) and (3.5), we get

$$C_4\|u_n\|^p\le C_3+M\int_0^{kT}|u_n(t)|^r dt. \qquad (3.6)$$

From (H5), there exists $M_4>0$ such that

$$(\nabla F(t,x),x)-pF(t,x)\ge M_1|x|^\mu \text{ for } |x|\ge M_4 \text{ and a.e. } t\in[0,T]. \qquad (3.7)$$

By assumption (A), for $|x|\le M_4$, there exists $C_5=\max_{|x|\le M_4}a(|x|)>0$ such that

$$|(\nabla F(t,x),x)-pF(t,x)|\le C_5(p+M_4)b(t). \qquad (3.8)$$

Thus, from (3.7) and (3.8), we have

$$(\nabla F(t,x),x)-pF(t,x)\ge M_1|x|^\mu-M_1M_4^\mu-C_5(p+M_4)b(t) \text{ for } x\in\mathbb{R}^n \text{ and a.e. } t\in[0,T],$$

which together with (3.1) implies that

$$\begin{aligned}(p+1)C_1 &\ge p\varphi_k(u_n)-<\varphi_k'(u_n),u_n>\\ &= \int_0^{kT}[(\nabla F(t,u_n),u_n)-pF(t,u_n)]dt\\ &\ge M_1\int_0^{kT}|u_n(t)|^\mu dt-C_5(p+M_4)\int_0^{kT}b(t)dt-M_1M_4^\mu kT\\ &= M_1\int_0^{kT}|u_n(t)|^\mu dt-C_6.\end{aligned}$$

Hence, $\int_0^{kT}|u_n(t)|^\mu dt$ is bounded. If $\mu>r$, we have

$$\int_0^{kT}|u_n(t)|^r dt\le(kT)^{(\mu-r)/\mu}\left(\int_0^{kT}|u_n(t)|^\mu dt\right)^{r/\mu},$$

which together with (3.6) implies that $\|u_n\|$ is bounded. If $\mu \leq r$, then from (2.2), we get

$$\int_0^{kT} |u_n(t)|^r dt \leq \|u_n\|_\infty^{r-\mu} \left(\int_0^{kT} |u_n(t)|^\mu dt\right)^{r/\mu} \leq d_k^{r-\mu} \|u_n\|^{r-\mu} \left(\int_0^{kT} |u_n(t)|^\mu dt\right)^{r/\mu}.$$

Since $\mu > r - p$, it follows from (3.6) that $\|u_n\|$ is bounded too. Therefore $\|u_n\|$ is bounded in $W_{kT}^{1,p}$. Hence, there exists a subsequence, still denoted by $\{u_n\}$, such that

$$u_n \rightharpoonup u_0 \text{ weakly in } W_{kT}^{1,p}, \tag{3.9}$$

$$u_n \to u_0 \text{ strongly in } C(0, kT; \mathbb{R}^n). \tag{3.10}$$

$$u_n \to u_0 \text{ strongly in } L^p(0, kT; \mathbb{R}^n). \tag{3.11}$$

From (2.1), we have

$$\begin{aligned} & < \varphi_k'(u_n), u_n - u_0 > \\ = & \int_0^{kT} [(|\dot{u}_n(t)|^{p-2}\dot{u}_n(t), \dot{u}_n(t) - \dot{u}_0(t)) + (L(t)|u_n(t)|^{p-2}u_n(t), u_n(t) - u_0(t))]dt \\ & - \int_0^{kT} (\nabla F(t, u_n(t)), u_n(t) - u_0(t))dt. \end{aligned} \tag{3.12}$$

From (3.1) and (3.10), we have

$$| < \varphi_k'(u_n), u_n - u_0 > | \leq \|\varphi_k'(u_n)\| \|u_n - u_0\| \to 0 \text{ as } n \to \infty. \tag{3.13}$$

By (L), we know that $c_1 \leq \|L\| \leq c_2$, which together with the boundedness of $\{u_n\}$ and (3.11) implies that

$$\int_0^{kT} (L(t)|u_n(t)|^{p-2}u_n(t), u_n(t) - u_0(t))dt \leq \|L\| \|u_n\|_{L^p}^{p-1} \|u_n - u_0\|_{L^p} \to 0 \text{ as } n \to \infty. \tag{3.14}$$

It follows from (A), (3.10) and the boundedness of $\{u_n\}$ that

$$\int_0^{kT} (\nabla F(t, u_n(t)), u_n(t) - u_0(t))dt \to 0 \text{ as } n \to \infty,$$

which together with (3.12), (3.13) and (3.14) implies that

$$\int_0^{kT} (|\dot{u}_n(t)|^{p-2}\dot{u}_n(t), \dot{u}_n(t) - \dot{u}_0(t))dt \to 0 \text{ as } n \to \infty. \tag{3.15}$$

It is easy to see from the boundedness of $\{u_n\}$ and (3.10) that

$$\int_0^{kT} (|u_n(t)|^{p-2}u_n(t), u_n(t) - u_0(t))dt \to 0 \text{ as } n \to \infty. \tag{3.16}$$

Let $\phi(u) = \frac{1}{p}(\int_0^{kT} |u(t)|^p dt + \int_0^{kT} |\dot{u}(t)|^p dt)$. Then, we have

$$\begin{aligned} < \phi'(u_n), u_n - u_0 > \; = & \int_0^{kT} (|\dot{u}_n(t)|^{p-2}\dot{u}_n(t), \dot{u}_n(t) - \dot{u}_0(t))dt \\ & + \int_0^{kT} (|u_n(t)|^{p-2}u_n(t), u_n(t) - u_0(t))dt \end{aligned} \tag{3.17}$$

and

$$< \phi'(u_0), u_n - u_0 > \; = \int_0^{kT} (|\dot{u}_0(t)|^{p-2}\dot{u}_0(t), \dot{u}_n(t) - \dot{u}_0(t))dt + \int_0^{kT} (|u_0(t)|^{p-2}u_0(t), u_n(t) - u_0(t))dt. \quad (3.18)$$

It follows from (3.15) and (3.16) that

$$< \phi'(u_n), u_n - u_0 > \to 0 \text{ as } n \to \infty. \quad (3.19)$$

From (3.9), we get

$$< \phi'(u_0), u_n - u_0 > \to 0 \text{ as } n \to \infty. \quad (3.20)$$

By (3.17), (3.18) and Hölder's inequality, we have

$$\begin{aligned}
& < \phi'(u_n) - \phi'(u_0), u_n - u_0 > \\
= & \int_0^{kT} (|\dot{u}_n(t)|^{p-2}\dot{u}_n(t), \dot{u}_n(t) - \dot{u}_0(t))dt + \int_0^{kT} (|u_n(t)|^{p-2}u_n(t), u_n(t) - u_0(t))dt \\
& - \int_0^{kT} (|\dot{u}_0(t)|^{p-2}\dot{u}_0(t), \dot{u}_n(t) - \dot{u}_0(t))dt - \int_0^{kT} (|u_0(t)|^{p-2}u_0(t), u_n(t) - u_0(t))dt \\
= & \|u_n\|^p + \|u_0\|^p - \int_0^{kT} (|\dot{u}_n(t)|^{p-2}\dot{u}_n(t), \dot{u}_0(t))dt - \int_0^{kT} (|u_n(t)|^{p-2}u_n(t), u_0(t))dt \\
& - \int_0^{kT} (|\dot{u}_0(t)|^{p-2}\dot{u}_0(t), \dot{u}_n(t))dt - \int_0^{kT} (|u_0(t)|^{p-2}u_0(t), u_n(t))dt \\
\geq & \|u_n\|^p + \|u_0\|^p - (\|u_n\|_{L^p}^{p-1}\|u_0\|_{L^p} + \|\dot{u}_n\|_{L^p}^{p-1}\|\dot{u}_0\|_{L^p}) \\
& -(\|u_0\|_{L^p}^{p-1}\|u_n\|_{L^p} + \|\dot{u}_0\|_{L^p}^{p-1}\|\dot{u}_n\|_{L^p}) \\
\geq & \|u_n\|^p + \|u_0\|^p - (\|u_n\|_{L^p}^p + \|\dot{u}_n\|_{L^p}^p)^{(p-1)/p}(\|u_0\|_{L^p}^p + \|\dot{u}_0\|_{L^p}^p)^{1/p} \\
& -(\|u_0\|_{L^p}^p + \|\dot{u}_0\|_{L^p}^p)^{(p-1)/p}(\|u_n\|_{L^p}^p + \|\dot{u}_n\|_{L^p}^p)^{1/p} \\
= & \|u_n\|^p + \|u_0\|^p - (\|u_n\|^{p-1}\|u_0\| + \|u_0\|^{p-1}\|u_n\|) \\
= & (\|u_n\|^{p-1} - \|u_0\|^{p-1})(\|u_n\| - \|u_0\|).
\end{aligned}$$

Hence, from (3.19) and (3.20), we obtain

$$0 \leq (\|u_n\|^{p-1} - \|u_0\|^{p-1})(\|u_n\| - \|u_0\|) \leq < \phi'(u_n) - \phi'(u_0), u_n - u_0 > \to 0 \quad \text{as } n \to \infty.$$

That is $\|u_n\| \to \|u_0\|$ as $n \to \infty$. Since $W_{kT}^{1,p}$ has the Kadec-Klee property, we have $u_n \to u_0$ in $W_{kT}^{1,p}$. Therefore, the functional φ_k satisfies condition (C).

Step 2. From (H2), for any $\varepsilon = \varepsilon(k) > 0$, there exists $\delta > 0$ such that

$$F(t, u) \leq \varepsilon |u|^p \text{ for } |u| \leq \delta \text{ and a.e. } t \in [0, kT]. \quad (3.21)$$

For $u \in \tilde{W}_{kT}^{1,p}$ and $\|u\|^p = \rho_k^p = \frac{\delta^p}{(kT)^{\frac{p}{q}}}$, then it follows from (2.3) that

$$\|u\|_\infty^p \leq (kT)^{\frac{p}{q}}\|\dot{u}\|_{L^p}^p \leq (kT)^{\frac{p}{q}}\|u\|^p = \delta^p,$$

which implies that $|u(t)| \leq \delta$. Then from (L) and (3.21), we have

$$\begin{aligned}
\varphi_k(u) &= \frac{1}{p}\int_0^{kT} |\dot{u}(t)|^p dt + \frac{1}{p}\int_0^{kT} (L(t)|u(t)|^{p-2}u(t), u(t))dt - \int_0^{kT} F(t,u)dt \\
&\geq \frac{1}{p}\int_0^{kT} |\dot{u}(t)|^p dt + \frac{1}{p}\int_0^{kT} c_1|u(t)|^p dt - \int_0^{kT} \varepsilon|u(t)|^p dt \\
&\geq \min\left\{\frac{1}{p}, \frac{c_1}{p}\right\} \|u\|^p - kT\varepsilon\delta^p \\
&= C_4\|u\|^p - kT\varepsilon\delta^p. \qquad (3.22)
\end{aligned}$$

Let $\varepsilon = \varepsilon(k) \in (0, \frac{C_4}{2(kT)^p})$, then from (3.22), we have

$$\varphi_k(u) \geq C_4\rho_k^p - kT\varepsilon\delta^p \geq \frac{C_4}{2}\rho_k^p \equiv \alpha > 0$$

for all $u \in \tilde{W}_T^{1,p}$ and $\|u\| = \rho_k$. This implies that condition (a) of Lemma 2.1 holds.

Step 3. From (H1) and (H3)′, there exists $C_7 > \frac{c_2}{p}$ such that

$$F(t,u) \geq C_7|u|^p \text{ for all } u \in \mathbb{R}^n \text{ and a.e. } t \in [0,T], \qquad (3.23)$$

Thus, from (L) and (3.23), we have

$$\begin{aligned}
\varphi_k(u) &= \frac{1}{p}\int_0^{kT} (L(t)|u|^{p-2}u, u)dt - \int_0^{kT} F(t,u)dt \\
&= \frac{k}{p}\int_0^{T} (L(t)|u|^{p-2}u, u)dt - k\int_0^{T} F(t,u)dt \\
&\leq \frac{c_2 k}{p}\int_0^{T} |u|^p dt - k\int_0^{T} C_7|u|^p dt
\end{aligned}$$

for all $u \in \mathbb{R}^n$. Since $C_7 > \frac{c_2}{p}$, we obtain

$$\varphi_k(u) \leq 0 \text{ for all } u \in \mathbb{R}^n. \qquad (3.24)$$

Let $\overline{W}_{kT}^{1,p} = \text{span}\{e_k\} + \mathbb{R}^n$, where $e_k = (k^{-1}\sin(k^{-1}\omega t))$, $\omega = 2\pi/T$. Since $\overline{W}_T^{1,p}$ is finite dimensional, there exists a constant $d > 0$ such that

$$\left(\int_0^T |x|^p dt\right)^{1/p} \geq d\left(\int_0^T |x|^2 dt\right)^{1/2}, \quad \forall\, x \in \overline{W}_T^{1,p}. \qquad (3.25)$$

From (3.23) and (3.25), we have

$$\begin{aligned}
&\varphi_k(u+re_k)\\
&= \frac{1}{p}\int_0^{kT}|r\dot{e}_k(t)|^p dt-\int_0^{kT}F(t,u+re_k(t))dt\\
&\quad +\frac{1}{p}\int_0^{kT}(L(t)|u+re_k(t)|^{p-2}(u+re_k(t)),u+re_k(t))dt\\
&\le \frac{1}{p}k^{-2p}r^p\omega^p\int_0^{kT}|\cos(k^{-1}\omega t)|^p dt+\frac{c_2}{p}\int_0^{kT}|u+re_k(t)|^p dt\\
&\qquad -\int_0^{kT}C_7|u+re_k(t)|^p dt\\
&\le \frac{1}{p}k^{-2p+1}r^p\omega^p\int_0^{T}|\cos(\omega t)|^p dt-k\int_0^{T}\left(C_7-\frac{c_2}{p}\right)|u+re_1(t)|^p dt\\
&\le \frac{T}{p}k^{-2p+1}r^p\omega^p-kd^p\left(C_7-\frac{c_2}{p}\right)\left(\int_0^{T}|u+re_1(t)|^2 dt\right)^{p/2}\\
&\le \frac{T}{p}k^{-2p+1}r^p\omega^p-kd^p\left(C_7-\frac{c_2}{p}\right)\left(\int_0^{T}(|u|^2+r^2|e_1(t)|^2 dt\right)^{p/2}\\
&\le \frac{T}{p}k^{-2p+1}r^p\omega^p-kd^p\left(C_7-\frac{c_2}{p}\right)\left(T|u|^2+\frac{Tr^2}{2}\right)^{p/2},\ \ \forall\, r\ge 0,\ u\in\mathbb{R}^n.
\end{aligned}$$

If $k\ge \frac{2^{5/4}T^{(2-p)/(4p)}\omega^{1/2}}{(C_7-\frac{c_2}{p})^{1/(2p)}d^{1/2}}$, then we have

$$\varphi_k(u+re_k)\le 0,\ \ \text{for all } r\ge 0 \text{ and } u\in\mathbb{R}^n. \tag{3.26}$$

From (3.26), we can choose two positive constants $r_1>\rho_k$ and $r_2>\rho_k$ such that

$$\varphi_k(u+re_k)\le 0,\ \ \text{for all } r\ge r_1 \text{ and } \|u\|\ge r_2. \tag{3.27}$$

Set

$$Q_k=\{re_k|\ 0\le r\le r_1, e_k\in \tilde{W}^{1,p}_{kT}\}\bigoplus\{u\in\mathbb{R}^n|\ \|u\|\le r_2\},$$

then we have $\partial Q_k=Q_{1k}\bigcup Q_{2k}\bigcup Q_{3k}$, where

$$Q_{1k}=\{u\in\mathbb{R}^n|\ \|u\|\le r_2\},$$

$$Q_{2k}=\{u+re_k|\ \|u\|=r_2, r\in[0,r_1], e_k\in\tilde{W}^{1,p}_{kT}\},$$

$$Q_{3k}=\{u+re_k|\ \|u\|\le r_2, r=r_1, e_k\in\tilde{W}^{1,p}_{kT}\}.$$

By (3.24) and (3.26), we get

$$\varphi(u)\le 0,\ \ u\in\partial Q_k=Q_{1k}\bigcup Q_{2k}\bigcup Q_{3k}. \tag{3.28}$$

Furthermore, for all $u + re_k \in Q_k$, from (H1) and (L), we have

$$\begin{aligned}
& \varphi_k(u+re_k) \\
= & \frac{1}{p}\int_0^{kT}|r\dot{e}_k(t)|^p dt - \int_0^{kT} F(t,u+re_k(t))dt \\
& +\frac{1}{p}\int_0^{kT}(L(t)|u+re_k(t)|^{p-2}(u+re_k(t)),u+re_k(t))dt \\
\leq & \frac{1}{p}r^p\int_0^{kT}|\dot{e}_k(t)|^p dt + \frac{c_2}{p}\int_0^{kT}|u+re_k(t)|^p dt \\
\leq & \frac{1}{p}k^{-2p}r^p\omega^p\int_0^{kT}|\cos(k^{-1}\omega t)|^p dt + \frac{2^{p-1}c_2}{p}\int_0^{kT}(|u|^p + r^p k^{-p}|\sin(k^{-1}\omega t)|^p)dt \\
\leq & \frac{1}{p}k^{-2p+1}r^p\omega^p\int_0^{T}|\cos(\omega t)|^p dt + \frac{2^{p-1}c_2}{p}\left(\|u\|^p + r^p k^{-p+1}\int_0^T|\sin(\omega t)|^p dt\right) \\
\leq & \frac{T}{p}k^{-2p+1}r^p\omega^p + \frac{2^{p-1}c_2}{p}(\|u\|^p + r^p k^{-p+1}T) \\
\leq & \frac{T}{p}r_1^p\omega^p + \frac{2^{p-1}c_2}{p}(r_2^p + r_1^p T).
\end{aligned}$$

Then by Lemma 2.1, for any positive integer $k \geq \frac{2^{5/4}T^{(2-p)/(4p)}\omega^{1/2}}{(C_7-\frac{c_2}{p})^{1/(2p)}d^{1/2}}$, φ_k has at least one critical point u_k in $W^{1,p}_{kT}$, and the corresponding critical value c_k satisfies

$$0 < \alpha \leq c_k = \varphi_k(u_k) \leq \frac{1}{p}r_1^p + \frac{2^{p-1}c_2}{p}(r_2^p + r_1^p). \tag{3.29}$$

Similar to the proof of [9], let u_{k_1} be a k_1T-periodic solution, we can prove that there exists a positive integer $k_2 > k_1$ such that $u_{kk_1} \neq u_{k_1}$ for all $kk_1 \geq k_2$. Otherwise, $\varphi_k(u_{kk_1}) = k\varphi_k(u_{k_1}) \to \infty$ as $k \to \infty$, which contradicts to (3.29). Repeating this process, we can obtain a sequence $\{u_{k_j}\}$ of distinct periodic solutions of problem (1.1). From (3.24), we know that u_{k_j} is nonconstant. The proof is complete. ∎

Proof of Theorem 1.2. The proof of Theorem 1.2 is the same as that of Theorem 1.1 except for the proof of the boundedness of $\{u_n\}$. So, here we only prove that $\{u_n\}$ is bounded in $W^{1,p}_{kT}$. Otherwise, going to a subsequence if necessary, we can assume that $\|u_n\| \to \infty$ as $n \to \infty$. Set $z_n = \frac{u_n}{\|u_n\|}$, $z_n = \tilde{z}_n + \bar{z}_n$, then $\|z_n\| = 1$. Hence, there exists a subsequence, still denoted by $\{z_n\}$, such that

$$z_n \rightharpoonup z_0 \quad \text{weakly in} \quad W^{1,p}_{kT},$$
$$z_n \to z_0 \quad \text{strongly in} \quad C(0,kT;\mathbb{R}^n).$$

Then, we have

$$\bar{z}_n \to \bar{z}_0. \tag{3.30}$$

From (3.1), we have

$$\lim_{n\to\infty}[(\varphi_k'(u_n),u_n) - p\varphi_k(u_n)] = -pC_1,$$

which implies that

$$\lim_{n\to\infty}\int_0^{kT}[(\nabla F(t,u_n),u_n)-pF(t,u_n)]=pC_1. \tag{3.31}$$

From (H4)′, there exists $M_5>0$ such that

$$F(t,x)\le M_2|x|^p \text{ for all } |x|\ge M_5 \text{ and a.e. } t\in[0,T]. \tag{3.32}$$

From (A), for $|u|\le M_5$, there exists $C_8=\max_{|u|\le M_5}a(|u|)$ such that

$$|F(t,x)|\le C_8 b(t). \tag{3.33}$$

It follows from (3.32) and (3.33) that

$$F(t,x)\le M_2|x|^p+C_8 b(t) \text{ for all } x\in\mathbb{R}^n \text{ and a.e. } t\in[0,T]. \tag{3.34}$$

Hence, from (L) and (3.34), we obtain

$$\begin{aligned}
\varphi_k(u_n) &= \frac{1}{p}\int_0^{kT}|\dot u_n(t)|^p dt+\frac{1}{p}\int_0^{kT}(L(t)|u_n(t)|^{p-2}u_n(t),u_n(t))dt\\
&\quad -\int_0^{kT}F(t,u_n(t))dt\\
&\ge \frac{1}{p}\int_0^{kT}|\dot u_n(t)|^p dt+\frac{c_1}{p}\int_0^{kT}|u_n(t)|^p dt-M_2\int_0^{kT}|u_n(t)|^p dt\\
&\quad -C_8\int_0^{kT}b(t)dt\\
&= \frac{1}{p}\int_0^{kT}|\dot u_n(t)|^p dt-\left(M_2-\frac{c_1}{p}\right)\int_0^{kT}|u_n(t)|^p dt-C_9,
\end{aligned}$$

thus, for $n\to\infty$,

$$0\leftarrow\frac{\varphi_k(u_n)}{\|u_n\|^p}\ \ge\ \frac{1}{p}\|\dot z_n\|_{L^p}^p-\left(M_2-\frac{c_1}{p}\right)\int_0^{kT}|z_n(t)|^p dt-\frac{C_9}{\|u_n\|^p}.$$

Hence, $z_0\ne 0$. Let $\Omega\subset[0,kT]$ be the set on which $z_0\ne 0$. The measure of Ω is positive. Moreover, $|u_n|\to\infty$ as $n\to\infty$ for $t\in\Omega$. Thus, from (H6), we have

$$\begin{aligned}
&\int_0^{kT}[(\nabla F(t,u_n),u_n)-pF(t,u_n)]\\
=\ &\int_\Omega[(\nabla F(t,u_n),u_n)-pF(t,u_n)]dt+\int_{[0,kT]\setminus\Omega}[(\nabla F(t,u_n),u_n)-pF(t,u_n)]dt\\
\ge\ &\int_\Omega[(\nabla F(t,u_n),u_n)-pF(t,u_n)]dt+\int_{[0,kT]\setminus\Omega}f(t)dt.
\end{aligned}$$

It follows from Fatou's lemma and (H7) that

$$\lim_{n\to\infty}\int_0^{kT}[(\nabla F(t,u_n),u_n)-pF(t,u_n)]=+\infty,$$

which contradicts to (3.31). If $\|\dot{z}_0\|_{L^p} = 0$, hence from (2.4), $\tilde{z}_0 \to 0$ uniformly for a.e. $t \in [0, kT]$, then together with (3.30), we have $z_0 = \bar{z}_0$ and $kT|\bar{z}_0|^p = \|\bar{z}_0\|^p \to 1$. Consequently, $|u_n| \to \infty$ as $n \to \infty$ uniformly for a.e. $t \in [0, kT]$. From (H1) and (H3)′, we have

$$\begin{aligned}
\liminf_{|u_n|\to\infty} \frac{\int_0^{kT} F(t,u_n(t))dt}{\|u_n\|^p} &\geq \frac{\int_0^{kT} [\liminf_{|u_n|\to\infty} F(t,u_n(t))]dt}{\|u_n\|^p} \\
&= \int_0^{kT} [\liminf_{|u_n|\to\infty} \frac{F(t,u_n(t))}{|u_n(t)|^p} |z_n(t)|^p]dt \\
&= \int_0^{kT} [\liminf_{n\to\infty} \frac{F(t,u_n(t))}{|u_n(t)|^p} |z_0|^p]dt \\
&> \frac{c_2}{p}. \qquad (3.35)
\end{aligned}$$

By the boundedness of $\varphi_k(u_n)$ and (L), we have

$$\begin{aligned}
\frac{\varphi_k(u_n)}{\|u_n\|^p} &= \frac{\frac{1}{p}\int_0^{kT} |\dot{u}_n(t)|^p dt}{\|u_n\|^p} + \frac{\frac{1}{p}\int_0^{kT} (L(t)|u_n(t)|^{p-2}u_n(t), u_n(t))dt}{\|u_n\|^p} - \\
&\quad \frac{\int_0^{kT} F(t,u_n(t))dt}{\|u_n\|^p} \\
&\leq \frac{1}{p}\|\dot{z}_n\|_{L^p}^p + \frac{\frac{c_2}{p}\int_0^{kT} |u_n(t)|^p dt}{\|u_n\|^p} - \frac{\int_0^{kT} F(t,u_n(t))dt}{\|u_n\|^p} \\
&= \frac{1}{p}\|\dot{z}_n\|_{L^p}^p + \frac{c_2}{p}\|z_n\|_{L^p}^p - \frac{\int_0^{kT} F(t,u_n(t))dt}{\|u_n\|^p},
\end{aligned}$$

which together with $\|\dot{z}_0\|_{L^p} = 0$ and $\|z_0\| = 1$ implies that

$$\liminf_{n\to\infty} \frac{\int_0^{kT} F(t,u_n(t))dt}{\|u_n\|^p} \leq \frac{c_2}{p}.$$

But this contradicts to (3.35). Thus, $\{u_n\}$ is bounded in $W_{kT}^{1,p}$. ■

Proof of Theorem 1.3. The proof of Theorem 1.3 is similar to that of Theorem 1.2, we omit the detail here. ■

4. Examples

In this section, we give some examples to illustrate our results.

Example 4.1. In problem (1.1), let $p = 3, r = 5, \mu = 4, \omega = \frac{2\pi}{T}$,

$$L(t) = \text{diag}\,(1 + \exp(1 - \sin(k^{-1}\omega t)), \cdots, 1 + \exp(1 - \sin(k^{-1}\omega t))),$$

and

$$F(t,x) = \begin{cases} \frac{1+e}{3}(2 + \sin(k^{-1}\omega t))|x|^5, & |x| > 1, \\ (2 + \sin(k^{-1}\omega t)) \ln^3(1 + |x|^2), & |x| \leq 1. \end{cases}$$

It is easy to check that $L(t)$ satisfies (L) and F satisfies (A), (H1) and (H2). By a direct computation, we have

$$\liminf_{|x|\to\infty} \frac{F(t,x)}{|x|^3} > \frac{1+e}{3}, \quad \limsup_{|x|\to\infty} \frac{F(t,x)}{|x|^5} \leq 1+e$$

and

$$\liminf_{|x|\to\infty} \frac{(\nabla F(t,x),x)) - 3F(t,x)}{|x|^4} \geq \frac{2(1+e)}{3},$$

which show that (H3)′, (H4) and (H5) hold. Hence, from Theorem 1.1, problem (1.1) has a sequence of distinct nonconstant periodic solutions with period k_jT satisfying $k_j \in \mathbb{N}$ and $k_j \to \infty$ as $j \to \infty$.

Example 4.2. In problem (1.1), let $p = 4$ and $L(t)$ be the same as in Example 4.1. Let

$$F(t,x) = \frac{1+e}{4\pi}(5+\sin(k^{-1}\omega t))[|x|^4 - \ln(1+|x|^4)] \arctan|x|^4,$$

It is easy to check that $L(t)$ satisfies (L) and F satisfies (A), (H1) and (H2). By an easy calculation, we get

$$\liminf_{|x|\to\infty} \frac{F(t,x)}{|x|^4} > \frac{1+e}{4}, \quad \limsup_{|x|\to\infty} \frac{F(t,x)}{|x|^4} \leq \frac{3(1+e)}{4}$$

which imply that (H3)′ and (H4) hold. Moreover, there exists $f \in L^1(0,T;\mathbb{R}^+)$ such that

$$\begin{aligned}
&(\nabla F(t,x),x) - 4F(t,x) \\
&= \frac{1+e}{\pi}(5+\sin(k^{-1}\omega t))\left[\ln(1+|x|^4) - \frac{|x|^4}{1+|x|^4}\right]\arctan|x|^4 \\
&\quad + \frac{(1+e)|x|^4}{\pi(1+|x|^8)}(5+\sin(k^{-1}\omega t))[|x|^4 - \ln(1+|x|^4)] \\
&\geq f(t),
\end{aligned}$$

and

$$\lim_{|x|\to\infty} [(\nabla F(t,x),x) - 4F(t,x)] = +\infty.$$

Then, conditions (H6) and (H7) hold. Hence, it follows from Theorem 1.2 that problem (1.1) has a sequence of distinct nonconstant periodic solutions with period k_jT satisfying $k_j \in \mathbb{N}$ and $k_j \to \infty$ as $j \to \infty$.

If we let $p = 4$ and $L(t)$ be the same as in Example 4.1. And let

$$F(t,x) = \frac{1+e}{4\pi}(5+\sin(k^{-1}\omega t))[|x|^4 + \ln(1+|x|^4)] \arctan|x|^4.$$

Similarly, we can check that $F(t,x)$ satisfies all the conditions of Theorem 1.3, then problem (1.1) has a sequence of distinct nonconstant periodic solutions with period k_jT satisfying $k_j \in \mathbb{N}$ and $k_j \to \infty$ as $j \to \infty$.

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