Half–linear Sturm–Liouville problem with weights

Pavel Drábek Komil Kuliev

Abstract

We prove a necessary and sufficient conditions for discreteness of the set of all eigenvalues (with the usual Sturm–Liouville properties) of half–linear eigenvalue problem with locally integrable weights. Our conditions appear to be equivalent to the compact embedding of certain weighted Sobolev and Lebesgue spaces. Every eigenvalue allows the variational characterization of Ljusternik–Schnirelmann type.

1 Introduction

We study the Sturm–Liouville problem for half–linear equations (equations of the $p-$Laplacian type) with weights subject to the Neumann–Dirichlet boundary conditions

$$\begin{cases} \left(\rho(t)|u'(t)|^{p-2}u'(t)\right)' + \lambda\sigma(t)|u(t)|^{p-2}u(t) = 0, \quad t \in (a,\, b), \\ \lim\limits_{t\to a+} \rho(t)|u'(t)|^{p-2}u'(t) = \lim\limits_{t\to b-} u(t) = 0. \end{cases} \tag{1.1}$$

Here, $p > 1$ is a real number, $-\infty \le a < b \le \infty$, $\rho = \rho(t)$, $\sigma = \sigma(t)$ are continuous positive functions in $(a,\, b)$. As for ρ and σ we assume that for any $x \in (a,\, b)$ we have $\sigma \in L^1(a,\, x)$ and $\rho^{1-p'} \in L^1(x,\, b)$, where $\frac{1}{p} + \frac{1}{p'} = 1$. We emphasize that we *do not assume* $\sigma, \rho^{1-p'} \in L^1(a,\, b)$ in general! We will use later the convention $u(b) \overset{def}{=} \lim\limits_{t\to b-} u(t)$.

Received by the editors May 2011.

Communicated by J. Mawhin.

2000 *Mathematics Subject Classification :* Primary: 34L30, 34B24 Secondary: 34B40, 35J92.

Key words and phrases : Hardy inequality, weighted spaces, Sturm–Liouville problem,variational eigenvalues, oscillatory theory.

In particular, the radial eigenvalue problem for the $p-$Laplacian

$$\begin{cases} -\Delta_p u = \lambda\eta(|x|)|u|^{p-2}u & \text{in} \quad \mathbb{R}^N, \\ \lim\limits_{|x|\to+\infty} u(x) = 0 \end{cases}$$

with suitable radial weight function $\eta = \eta(|x|)$ reduces to (1.1) as we shall see later on.

For the sake of brevity we introduce a function $\varphi : \mathbb{R} \to \mathbb{R}$ defined by $\varphi(s) \stackrel{def}{=} |s|^{p-2}s$ for $s \neq 0$ and $\varphi(0) \stackrel{def}{=} 0$.

By a *solution* of (1.1) we understand a function $u \in C^1(a,b)$ such that $\rho\varphi(u') \in C^1(a,b)$, the equation in (1.1) holds at every point, the boundary conditions are satisfied and the Dirichlet integral $\int_a^b \rho(t)|u'(t)|^p \,\mathrm{d}t$ is finite.

The parameter λ is called an *eigenvalue* of (1.1) if this problem has a nontrivial (i.e. nonzero) solution. This solution is then called an *eigenfunction* of (1.1) associated with λ.

We say that the (S.L.) *Property for* (1.1) is satisfied if "the set of all eigenvalues of (1.1) forms an increasing sequence $\{\lambda_n\}_{n=1}^{\infty}$ such that $\lambda_1 > 0$ and $\lim\limits_{n\to+\infty} \lambda_n = +\infty$; every eigenvalue λ_n, $n = 1, 2, ...,$ is simple in the sense that all eigenfunctions associated with λ_n are mutually proportional; the eigenfunction u_{λ_n} associated with λ_n has precisely $n-1$ zeros in (a,b); for $n \geq 3$ zero points of $u_{\lambda_{n-1}}$ separate zero points of u_{λ_n}."

Theorem 1.1. *The* (S.L.) *Property for* (1.1) *is satisfied if and only if the following two conditions hold:*

$$\lim_{t\to a+} \left(\int_a^t \sigma(\tau)\,\mathrm{d}\tau\right)\left(\int_t^b \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{p-1} = 0, \tag{1.2}$$

$$\lim_{t\to b-} \left(\int_a^t \sigma(\tau)\,\mathrm{d}\tau\right)\left(\int_t^b \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{p-1} = 0. \tag{1.3}$$

Remark 1.2. Conditions (1.2) and (1.3) are equivalent with the *compact embedding*

$$W_b^{1,p}(\rho) \hookrightarrow\hookrightarrow L^p(\sigma), \tag{1.4}$$

where $L^p(\sigma)$ is the weighted Lebesgue space of all functions $u = u(t)$ defined on (a,b), for which

$$\|u\|_{p;\sigma} \stackrel{def}{=} \left(\int_a^b \sigma(t)|u(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p}} < \infty;$$

$W_b^{1,p}(\rho)$ is the weighted Sobolev space of all functions u which are absolutely continuous on every compact subinterval of $(a,\, b)$, such that $u(b) = 0$, and

$$\|u\|_{1,p;\rho} \stackrel{def}{=} \left(\int_a^b \rho(t)|u'(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p}} < \infty.$$

Note that $L^p(\sigma)$ and $W_b^{1,p}(\rho)$ equipped with the norms $\|\cdot\|_{p;\sigma}$ and $\|\cdot\|_{1,p;\rho}$, respectively, are uniformly convex Banach spaces.

The proof that (1.2) and (1.3) are equivalent with (1.4) can be found e.g. in Opic and Kufner [1, Theorem 7.4]. Together with Theorem 1.1 we have the following "round about theorem":

Theorem 1.3. *The following statements are equivalent*

(i) *The* (S.L.) *Property for* (1.1) *is satisfied.*

(ii) *Compact embedding* (1.4) *holds.*

(iii) *Conditions* (1.2) *and* (1.3) *hold.*

Let us mention the pioneering work in this direction by Nečas [2], where the discreteness of the spectrum of half–linear Sturm–Liouville problem is considered for the first time. We also quote the work of Elbert [3] which has not been recognized even several years after its publication.
Note that the special case of problem (1.1) is studied in Drábek and Kufner [4], [5], where $a = 0$, $b = \infty$ and ρ, σ are continuous and positive functions in $[0, \infty)$. In this special case (1.2) and (1.3) reduce just to one condition

$$\lim_{t\to+\infty}\left(\int_0^t \sigma(\tau)\,\mathrm{d}\tau\right)\left(\int_t^{+\infty}\rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{p-1} = 0.$$

Similar problem to (1.1) is studied in Binding and Drábek [6], where $a = 0$ and $b > 0$ is a finite number and the weight functions are supposed to satisfy $\rho^{1-p'} \in L^1(0, b)$ and $\sigma \in L^1(0, b)$.

The results of this paper generalize those mentioned above and fit with the linear theory obtained by Lewis [7] for $p = 2$ as well.

This paper is organized as follows. In Section 2 we define the weak solution, discuss its regularity and give an illustrative application of our results to the radial problem. Section 3 is devoted to the construction of variational eigenvalues of our problem. We prove some oscillatory and nonoscillatory results for our problem in Section 4. This topic is well elaborated in the literature but we prefer to give our straightforward proof based on the Hardy inequality. In fact, nonoscillatory result from Proposition 4.3 is not needed in the proof of our main result. It is a by–product of our paper of its own and independent interest. Section 5 is the proof of Theorem 1.1.

We point out that as another by–product of our result we obtain that *every eigenvalue* of (1.1) *allows variational characterization* of Ljusternik–Schnirelmann type.

2 Weak solution

A function $u \in W_b^{1,p}(\rho)$ is called a *weak solution* of (1.1) if the integral identity

$$\int_a^b \rho(t)\varphi(u'(t))v'(t)\,\mathrm{d}t = \lambda\int_a^b \sigma(t)\varphi(u(t))v(t)\,\mathrm{d}t \tag{2.1}$$

holds for all $v \in W_b^{1,p}(\rho)$ (with both integrals being finite).

It is clear that every solution of (1.1) is also its weak solution. Converse is true as well. Indeed, take arbitrary $v \in C_0^\infty(a, b)$ (smooth functions with compact support in (a, b)) as a test function in (2.1) and integrate by parts. We get that there is a constant $c \in \mathbb{R}$ such that

$$\rho(t)\varphi(u'(t)) + \int_a^t \lambda\sigma(\tau)\varphi(u(\tau))\,\mathrm{d}\tau = c \tag{2.2}$$

for a.e. $t \in (a, b)$. Hence, continuity of $\tau \longmapsto \sigma(\tau)\varphi(u(\tau))$ in (a, b) implies that $\rho\varphi(u') \in C^1(a, b)$ and (2.2) (and thus also the equation (1.1)) holds at every point $t \in (a, b)$. Now, testing (2.1) with $v \in W_b^{1,p}(\rho)$, $v(a) \neq 0$, $v \equiv 0$ in the left neighborhood of b, and integrating by parts we arrive at $\lim\limits_{t\to a+} \rho(t)\varphi(u'(t)) = 0$. Since we have $u(b) = 0$ by $u \in W_b^{1,p}(\rho)$, a weak solution u is a solution in the sense of our definition from Section 1 at the same time.

Remark 2.1. Since $\lim\limits_{t\to a+} \int_a^t \sigma(\tau)\,\mathrm{d}\tau = 0$ by our assumptions, for a weak solution $u \in W_b^{1,p}(\rho)$ of (1.1), we get

$$\left|\int_a^t \lambda\sigma(\tau)\varphi(u(\tau))\,\mathrm{d}\tau\right| \leq |\lambda| \left(\int_a^t \sigma(\tau)|u(\tau)|^p\,\mathrm{d}\tau\right)^{\frac{1}{p'}} \left(\int_a^t \sigma(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{p}} \to 0$$

as $t \to a+$. This fact together with $\lim\limits_{t\to a+} \rho(t)\varphi(u'(t)) = 0$ yields $c = 0$ in (2.2). Assume, moreover, that

$$\lim_{t\to a+} \frac{1}{\rho(t)} \left(\int_a^t \sigma(\tau)\,\mathrm{d}\tau\right)^{\frac{1}{p}} < +\infty. \tag{2.3}$$

Let $a \in \mathbb{R}$. Then (2.2) (with $c = 0$) implies $u'(a) = 0$. Hence, under condition (2.3) we obtain a classical Neumann boundary condition at the point a. This is the case when, e.g., ρ and σ are continuous and positive in $[a, b)$. (cf. [4]).

Let us consider the *radial eigenvalue problem* for the $p-$Laplacian on the entire $\mathbb{R}^N$:

$$\begin{cases} -\Delta_p u = \frac{\lambda}{1+|x|^\gamma}|u|^{p-2}u & \text{in} \quad \mathbb{R}^N, \\ \lim\limits_{|x|\to+\infty} u(x) = 0. \end{cases} \tag{2.4}$$

This problem reduces to the one–dimensional equation

$$-(r^{N-1}\varphi(u'(r)))' = \lambda\frac{r^{N-1}}{1+r^\gamma}\varphi(u(r)), \qquad r \in (0, +\infty), \tag{2.5}$$

where $r = |x|$. For $1 < p < N$ and $\gamma > p$ the weights $\rho(r) = r^{N-1}$ and $\sigma(r) = \frac{r^{N-1}}{1+r^\gamma}$ satisfy (1.2) and (1.3). Hence the solution of (2.5) is also forced to satisfy the Neumann–Dirichlet homogenous boundary conditions

$$\lim_{r\to 0+} r^{N-1}\varphi(u'(r)) = u(+\infty) = 0. \tag{2.6}$$

It follows from Theorem 1.1 that (S.L.) Property for (2.5), (2.6) is satisfied. In particular, we have the following assertion:

Theorem 2.2. *Let $1 < p < N$ and $\gamma > p$. Then the eigenvalues of the radial eigenvalue problem* (2.4) *exhaust the sequence $\{\lambda_n\}_{n=1}^{\infty}$, $0 < \lambda_1 < \lambda_2 < \ldots \to +\infty$ with all λ_n being simple. A normalized eigenfunction u_{λ_n} associated with λ_n, $n \geq 1$, has precisely n nodal domains (maximal connected sets on which u_{λ_n} is of definite sign) in $\mathbb{R}^N$. The nodal lines of u_{λ_n} are spheres in $\mathbb{R}^N$ centered at the origin. The nodal lines of $u_{\lambda_{n-1}}$ separate those of u_{λ_n}.*

Of course, more general radial eigenvalue problems for the $p-$Laplacian with weights can be considered and reduced to problem (1.1).

3 Variational eigenvalues

In this section we assume that (1.2) and (1.3) hold. The following assertion is a standard consequence of the Lagrange multiplier method and compactness of the embedding (1.4).

Lemma 3.1. *Let us assume* (1.2) *and* (1.3)*. Then* (1.1) *has the least (principal) eigenvalue $\lambda_1 > 0$ characterized by*

$$\lambda_1 = \min \frac{\int_a^b \rho(t)|u'(t)|^p \,\mathrm{d}t}{\int_a^b \sigma(t)|u(t)|^p \,\mathrm{d}t}, \tag{3.1}$$

where the minimum is taken over all $u \in W_b^{1,p}(\rho)$, $u \neq 0$.

For $a < c \leq b$ we define auxiliary Sobolev function space $W_c^{1,p}(\rho)$ and for $a \leq c < d \leq b$ the space $W_{c,d}^{1,p}(\rho)$ as follows:
$W_c^{1,p}(\rho)$ is defined as $W_b^{1,p}(\rho)$ with b replaced by c;

$$W_{c,d}^{1,p}(\rho) \stackrel{def}{=} \Big\{u = u(t) \text{ is absolutely continuous in every compact subinterval of } (c, d),\ u(c) = u(d) = 0, |||u||| < \infty\Big\}, \text{ where}$$

$$|||u||| \stackrel{def}{=} \left(\int_c^d \rho(t)|u'(t)|^p \,\mathrm{d}t\right)^{\frac{1}{p}}.$$

The space $L_{c,d}^p(\sigma)$ is defined as $L^p(\sigma)$ with a and b replaced by c and d, respectively, where $a \leq c < d \leq b$. It follows from (1.2), (1.3), continuity and positivity of ρ and σ in (a, b) that

(i) $W_c^{1,p}(\rho) \hookrightarrow\hookrightarrow L_{a,c}^p(\sigma)$ if $a < c$.

(ii) $W_{c,d}^{1,p}(\rho) \hookrightarrow\hookrightarrow L_{c,d}^p(\sigma)$.

(iii) For $a < c < d < b$ we have $L_{c,d}^p(\sigma) = L^p(c, d)$, where $L^p(c, d)$ is the usual Lebesgue space on (c, d).

(iv) For $a < c < d < b$ we have $W^{1,p}_{c,d}(\rho) = W^{1,p}_0(c,d)$, where $W^{1,p}_0(c,d)$ is the usual Sobolev space of functions vanishing at c and d.

In particular, it follows from (iii) and (iv) that also

(v) $W^{1,p}_{c,d}(\rho) \hookrightarrow\hookrightarrow L^p_{c,d}(\sigma)$ if $a < c < d < b$.

Now, using (i), (ii) and (v), we can define the principal eigenvalue $\lambda_1(c,d)$ as in (3.1):

$$\lambda_1(c,d) = \min \frac{\int_c^d \rho(t)|u'(t)|^p \, dt}{\int_c^d \sigma(t)|u(t)|^p \, dt}. \tag{3.2}$$

We then have

(α) if $a = c$, $d \le b$ then $\lambda(c,d)$ is a principal eigenvalue of

$$\big(\rho(t)\varphi(u'(t))\big)' + \lambda\sigma(t)\varphi(u(t)) = 0, \quad t \in (c,\, d), \tag{3.3}$$

with boundary conditions

$$\lim_{t\to a+} \rho(t)\varphi(u'(t)) = u(d) = 0; \tag{3.4}$$

(β) if $a < c$, $d \le b$ then $\lambda_1(c,d)$ is a principal eigenvalue of (3.3) with boundary conditions

$$u(c) = u(d) = 0. \tag{3.5}$$

Following literally the proof of Theorem 1.3 from Lindqvist [8], we find that λ_1 (and $\lambda_1(c,d)$) is a simple eigenvalue, i.e., the functions in which the minimum in (3.1) (and (3.2)) is achieved are merely constant multiple of each other and they are either strictly positive or strictly negative in $(c,\, d)$.

In order to get the higher eigenvalues of (1.1) we employ the variational argument. Let

$$\mathcal{S} \overset{def}{=} \Big\{ u \in W^{1,p}_b(\rho) : \quad \|u\|_{p;\sigma} = 1 \Big\}$$

and let $\mathcal{S}^{k-1}$ be the unit sphere in $\mathbb{R}^k$, $k \in \mathbb{N}$. We consider the family of sets

$$\mathcal{F}_k \overset{def}{=} \Big\{ \mathcal{A} \subset \mathcal{S} : \mathcal{A} \quad \text{is the image of a continuous odd function} \quad h : \mathcal{S}^{k-1} \to \mathcal{S} \Big\}.$$

Define

$$\lambda_k \overset{def}{=} \inf_{\mathcal{A}\in\mathcal{F}_k} \sup_{u\in\mathcal{A}} \|u\|^p_{1,p;\rho}. \tag{3.6}$$

Following literally the proofs from Drábek and Robinson [9, Section 3], one can show (due to the compactness of the embedding (1.4)) that λ_k is an eigenvalue of (1.1) and, moreover,

$$\lim_{k\to+\infty} \lambda_k = +\infty.$$

Note that every $\mathcal{A} \in \mathcal{F}_1$ is formed by two antipodal points from $\mathcal{S}$ and so, for $k = 1$, (3.6) coincides with (3.1). It is not clear at this moment whether $\{\lambda_k\}_{k=1}^\infty$ defined by (3.6) exhausts the set of all eigenvalues of (1.1).

Following literally the proof of Proposition A.2 from Drábek and Kufner [4] one can prove that $u_\lambda = u_\lambda(t)$ has at least one zero in (a, b) whenever u_λ is an eigenfunction associated with an eigenvalue λ of (1.1) with $\lambda > \lambda_1$.

It also follows from Drábek and Robinson [10, Theorem 3.2] that an eigenfunction u_λ associated with an eigenvalue (not necessarily characterized by (3.6)) $\lambda < \lambda_k$ has at most $k-2$ zeros in (a,b), $k = 2,3,\ldots$, and from [10, Theorem 3.1] that an eigenfunction u_{λ_k} associated with λ_k, $k = 2,3,\ldots$, has at most $k-1$ zeros in (a,b). We note here that the unique continuation property is an essential assumption of [10, Theorem 3.1]. It is satisfied due to the strict positivity of weights in (a, b) and the uniqueness result for the initial value problem associated with the equation in (1.1) (cf. Došlý [11, Theorem 1.1]).

In particular, it follows from above

Lemma 3.2. *If λ is an eigenvalue of* (1.1) *and $\lambda < \lambda_2$, then $\lambda = \lambda_1$. An eigenfunction associated with λ_1 has no zero point in (a, b) and an eigenfunction associated with λ_2 has exactly one zero point in (a, b).*

4 Oscillatory and nonoscillatory results

A solution $u = u(t)$ of the equation in (1.1) is called *nonoscillatory solution*, if there exist $\bar{a}, \bar{b} \in (a, b)$ such that $u(t) \neq 0$ for all $t \in (a, \bar{a}) \cup (\bar{b}, b)$. Otherwise, a solution is called *oscillatory solution*. The *equation* in (1.1) is called *nonoscillatory*, if every its solution is nonoscillatory. We note here that a nonzero and nonoscillatory solution u has only a finite number of zero points in (a, b). Indeed, if there were infinitely many zeros of u in (a, b), there should be a $t_0 \in (a, b)$ such that $u(t_0) = u'(t_0) = 0$. The uniqueness result for the initial value problem, see [11, Theorem 1.1] then forces $u \equiv 0$ in (a, b), a contradiction. On the other hand, a nonzero and oscillatory solution has infinitely many zeros in (a, b) with a and/or b as the only possible limit point(s). If a (or b) is the limit point of zeros of an oscillatory solution u, then we say that u is *oscillatory in the right neighborhood of a* (or *the left neighborhood of b*, respectively).

Let $a \leq c < d \leq b$ and let us introduce the following functional

$$\mathcal{F}(v; c, d) := \int_c^d \Big(\rho(t)|v'(t)|^p - \lambda\sigma(t)|v(t)|^p\Big)\,\mathrm{d}t. \tag{4.1}$$

Proposition 4.1. (cf. Došlý [11, Theorem 2.2]) *The equation in* (1.1) *is nonoscillatory if there exist $\bar{a}, \bar{b} \in (a, b)$ $(\bar{a} < \bar{b})$ such that*

$$\mathcal{F}(v; a, \bar{a}) > 0 \qquad \textit{and} \qquad \mathcal{F}(v; \bar{b}, b) > 0 \tag{4.2}$$

for all $v \neq 0$, $v \in W^{1,p}_{\bar{a}}(\rho)$ and $v \in W^{1,p}_{\bar{b},b}(\rho)$, respectively.

Proof. We argue by contradiction. Let u be an oscillatory solution of the equation in (1.1). Then it is oscillatory either in the right neighborhood of a or in the left neighborhood of b. Without loss of generality we assume that for arbitrary $\bar{a} \in$

$(a,\, b)$ there exist $t_1,\, t_2 \in (a,\, \bar{a})$ such that $t_1 < t_2$ and $u(t_1) = u(t_2) = 0$. The other case is treated analogously. Let us construct the function v as follows

$$v(t) = \begin{cases} u(t) & t \in (t_1,\, t_2), \\ 0 & t \notin (t_1,\, t_2). \end{cases}$$

Then $v \in W^{1,p}_{\bar{a}}(\rho)$, and

$$\begin{aligned} \mathcal{F}(v; a, \bar{a}) &= \int\limits_a^{\bar{a}} \Big(\rho(t)|v'(t)|^p - \lambda\sigma(t)|v(t)|^p\Big)\,\mathrm{d}t \\ &= \int\limits_{t_1}^{t_2} \Big((\rho(t)\varphi(u'(t)))' - \lambda\sigma(t)\varphi(u(t))\Big)\,u(t)\,\mathrm{d}t = 0, \end{aligned}$$

a contradiction with (4.2). ■

The following result is a special case of [1, Theorem 6.2].

Proposition 4.2. *If* $1 < p < \infty$, *then inequality*

$$\int_c^d \left(\int_x^d w(t)\,\mathrm{d}t\right)^p \sigma(x)\,\mathrm{d}x \le C \int_c^d w^p(x)\rho(x)\,\mathrm{d}x \tag{4.3}$$

holds for all measurable $w(x) \ge 0$ *on* $(c,\, d)$ *if and only if*

$$B(c,\, d) \overset{def}{=} \sup_{t \in (c,d)} \left(\int_c^t \sigma(\tau)\,\mathrm{d}\tau\right)\left(\int_t^d \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{p-1} < \infty. \tag{4.4}$$

Moreover, the best constant $C = C(c,d)$ *in* (4.3) *satisfies*

$$B(c,d) \le C(c,d) \le \frac{p^p}{(p-1)^{p-1}} B(c,d). \tag{4.5}$$

Proposition 4.3. (cf. Došlý [12, Theorems 6 and 4]) *Let*

$$\limsup_{t \to a+} \left(\int_a^t \sigma(\tau)\,\mathrm{d}\tau\right)\left(\int_t^b \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{p-1} < \frac{(p-1)^{p-1}}{\lambda p^p} \tag{4.6}$$

and

$$\limsup_{t \to b-} \left(\int_a^t \sigma(\tau)\,\mathrm{d}\tau\right)\left(\int_t^b \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{p-1} < \frac{(p-1)^{p-1}}{\lambda p^p}. \tag{4.7}$$

Then the equation in (1.1) *is nonoscillatory.*

Let

$$\limsup_{t \to a+} \left(\int_a^t \sigma(\tau)\,\mathrm{d}\tau\right)\left(\int_t^b \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{p-1} > \frac{1}{\lambda} \tag{4.8}$$

or

$$\limsup_{t \to b-} \left(\int_a^t \sigma(\tau)\,\mathrm{d}\tau\right)\left(\int_t^b \rho^{1-p'}(\tau)\,\mathrm{d}\tau\right)^{p-1} > \frac{1}{\lambda}. \tag{4.9}$$

Then every solution of problem (1.1) *is oscillatory.*

Proof. First we show that (4.6) implies the existence of $\bar{a} \in (a, b)$ such that any solution $u = u(t)$ of the equation in (1.1) satisfies $u(t) \neq 0$ for all $t \in (a, \bar{a})$. For this purpose we use Proposition 4.1, i.e., we show that $\mathcal{F}(v; a, \bar{a}) > 0$ for all $v \in W^{1,p}_{\bar{a}}(\rho)$.

Indeed, from (4.6) we obtain that there exists $\bar{a} \in (a, b)$ such that

$$B(a, \bar{a}) = \sup_{t \in (a, \bar{a})} \left(\int_a^t \sigma(\tau)\, \mathrm{d}\tau \right) \left(\int_t^{\bar{a}} \rho^{1-p'}(\tau)\, \mathrm{d}\tau \right)^{p-1} < \frac{(p-1)^{p-1}}{\lambda p^p}. \tag{4.10}$$

Let $v \in W^{1,p}_{\bar{a}}(\rho)$ be arbitrary but fixed. Then using inequality (4.3) with $w = |v'|$, $c = a$, $d = \bar{a}$ we estimate the following integral

$$\begin{aligned} \int_a^{\bar{a}} \sigma(t) |v(t)|^p \, \mathrm{d}t &= \int_a^{\bar{a}} \sigma(t) \left| \int_t^{\bar{a}} v'(s)\, \mathrm{d}s \right|^p \mathrm{d}t \\ &\le \int_a^{\bar{a}} \sigma(t) \left(\int_t^{\bar{a}} \left| v'(s) \right| \, \mathrm{d}s \right)^p \mathrm{d}t \\ &\le C(a, \bar{a}) \int_a^{\bar{a}} \rho(t) |v'(t)|^p \, \mathrm{d}t \\ &< \frac{1}{\lambda} \int_a^{\bar{a}} \rho(t) |v'(t)|^p \, \mathrm{d}t, \end{aligned} \tag{4.11}$$

i.e.

$$\mathcal{F}(v; a, \bar{a}) = \lambda \left[\frac{1}{\lambda} \int_a^{\bar{a}} \rho(t) |v'(t)|^p \, \mathrm{d}t - \int_a^{\bar{a}} \sigma(t) |v(t)|^p \, \mathrm{d}t \right] > 0.$$

To get (4.11) from (4.10) we used (4.5) for the constant $C(a, \bar{a})$:

$$\begin{aligned} C(a, \bar{a}) &\le p(p')^{p-1} B(a, \bar{a}) \\ &< p(p')^{p-1} \frac{(p-1)^{p-1}}{\lambda p^p} = \frac{1}{\lambda}. \end{aligned}$$

The fact that (4.7) implies the existence of $\bar{b} \in (a, b)$ such that any solution $u = u(t)$ of the equation in (1.1) satisfies $u(t) \neq 0$ for $t \in (\bar{b}, b)$ is proved analogously.

We prove the second part of the theorem by contradiction, i.e., we suppose that condition (4.8) holds, but there exists a nonoscillatory solution of (1.1). Then there exists an interval $(a, \bar{b}) \subset (a, b)$, such that the function $u(t)$ does not change the sign in $(a, \bar{b})$ and $u(\bar{b}) = 0$. Indeed, either u is of definite sign in the entire (a, b) and then we choose $\bar{b} = b$ or else $\bar{b}$ is the first zero of u in (a, b). We suppose that the solution u is positive in $(a, \bar{b})$. The other case is treated similarly. In Section 2 we derived formula

$$\rho(t)\varphi(u'(t)) = -\int_a^t \lambda \sigma(\tau) \varphi(u(\tau))\, \mathrm{d}\tau. \tag{4.12}$$

From this we find

$$u'(t) = -\lambda^{p'-1} \rho(t)^{1-p'} \varphi^{-1} \left(\int_a^t \sigma(\tau) \varphi(u(\tau))\, \mathrm{d}\tau \right), \qquad t \in (a, \bar{b}),$$

where φ^{-1} denotes the inverse function of φ. Now, for $a < x < t < \bar{b}$, integrating over the interval $(x, \bar{b})$ with respect to t, we get

$$u(x) = -\int_x^{\bar{b}} u'(t)\,\mathrm{d}t = \lambda^{p'-1} \int_x^{\bar{b}} \rho(t)^{1-p'} \left(\int_a^t \sigma(\tau)\,(u(\tau))^{p-1}\,\mathrm{d}\tau \right)^{\frac{1}{p-1}} \mathrm{d}t.$$

Using the positivity of the solution and monotonicity of the inner integral, we obtain

$$u(x) \geq \lambda^{p'-1} \left(\int_x^{\bar{b}} \rho(t)^{1-p'}\,\mathrm{d}t \right) \left(\int_a^x \sigma(\tau)\,(u(\tau))^{p-1}\,\mathrm{d}\tau \right)^{\frac{1}{p-1}}.$$

From the positivity of the solution in the interval $(a, \bar{b})$ and (4.12) we obtain that the function u is strictly decreasing in $(a, \bar{b})$, which yields

$$u(x) \geq \lambda^{p'-1} \left(\int_x^{\bar{b}} \rho(t)^{1-p'}\,\mathrm{d}t \right) \left(\int_a^x \sigma(\tau)\,\mathrm{d}\tau \right)^{\frac{1}{p-1}} u(x),$$

i.e.

$$\limsup_{x\to a+} \left(\int_x^{\bar{b}} \rho(t)^{1-p'}\,\mathrm{d}t \right) \left(\int_a^x \sigma(\tau)\,\mathrm{d}\tau \right)^{\frac{1}{p-1}} \leq \lambda^{1-p'}.$$

Consequently, due to the assumptions on the weights, we obtain

$$\begin{aligned}
&\limsup_{x\to a+} \left(\int_a^x \sigma(\tau)\,\mathrm{d}\tau \right) \left(\int_x^b \rho(t)^{1-p'}\,\mathrm{d}t \right)^{p-1} \\
=\ &\limsup_{x\to a+} \left(\int_a^x \sigma(\tau)\,\mathrm{d}\tau \right) \left(\int_x^{\bar{b}} \rho(t)^{1-p'}\,\mathrm{d}t + \int_{\bar{b}}^b \rho(t)^{1-p'}\,\mathrm{d}t \right)^{p-1} \\
=\ &\limsup_{x\to a+} \Bigg(\left(\int_a^x \sigma(\tau)\,\mathrm{d}\tau \right)^{\frac{1}{p-1}} \int_x^{\bar{b}} \rho(t)^{1-p'}\,\mathrm{d}t \\
&\qquad\qquad + \left(\int_a^x \sigma(\tau)\,\mathrm{d}\tau \right)^{\frac{1}{p-1}} \int_{\bar{b}}^b \rho(t)^{1-p'}\,\mathrm{d}t \Bigg)^{p-1} \\
=\ &\limsup_{x\to a+} \left(\int_a^x \sigma(\tau)\,\mathrm{d}\tau \right) \left(\int_x^{\bar{b}} \rho(t)^{1-p'}\,\mathrm{d}t \right)^{p-1} \leq \frac{1}{\lambda}
\end{aligned}$$

a contradiction with (4.8).

Let us suppose that condition (4.9) holds but there exists a nonoscillatory solution $u = u(t)$ of (1.1). Then either u does not change the sign in (a, b) or else u has a zero point in (a, b). Then there exists an interval $(\bar{a}, b) \subset (a, b)$, where the function u does not change the sign and $u'(\bar{a}) = 0$. Indeed, u has the largest zero $\tilde{a} \in (a, b)$ and it follows from the Lagrange intermediate value theorem that there is $\bar{a} \in (\tilde{a}, b)$ such that $u'(\bar{a}) = 0$. We can write (4.12) in the form:

$$\begin{aligned}
\rho(t)\varphi(u'(t)) &= -\int_a^t \lambda\sigma(\tau)\varphi(u(\tau))\,\mathrm{d}\tau \\
&= -\int_a^{\bar{a}} \lambda\sigma(\tau)\varphi(u(\tau))\,\mathrm{d}\tau - \int_{\bar{a}}^t \lambda\sigma(\tau)\varphi(u(\tau))\,\mathrm{d}\tau \\
&= \rho(\bar{a})\varphi(u'(\bar{a})) - \int_{\bar{a}}^t \lambda\sigma(\tau)\varphi(u(\tau))\,\mathrm{d}\tau \\
&= -\int_{\bar{a}}^t \lambda\sigma(\tau)\varphi(u(\tau))\,\mathrm{d}\tau.
\end{aligned}$$

If u does not change the sign in (a, b), we just set $\bar{a} = a$. Repeating similar calculations as above, we obtain

$$\limsup_{x\to b-} \left(\int_{\bar{a}}^x \sigma(\tau)\,\mathrm{d}\tau\right)\left(\int_x^b \rho(t)^{1-p'}\,\mathrm{d}t\right)^{p-1} \le \frac{1}{\lambda},$$

a contradiction with (4.9). The proof is complete. ∎

From Proposition 4.3 we immediately get

Corollary 4.4. *Let* (1.2) *and* (1.3) *hold. Then the equation in* (1.1) *is nonoscillatory for an arbitrary* $\lambda \in \mathbb{R}$. *In particular, every eigenfunction of* (1.1) *has a finite number of zeros in* (a, b). *On the other hand, if either* (1.2) *or* (1.3) *is violated, then there exists* $\lambda_0 > 0$ *such that every solution of problem* (1.1) *is oscillatory for* $\lambda \ge \lambda_0$ *either in the right neighborhood of* a *or in the left neighborhood of* b, *respectively.*

5 Proof of Theorem 1.1

Necessity of (1.2) *and* (1.3). We proceed via contradiction. Assume that (1.2) is violated. It follows from Corollary 4.4 that there exists $\lambda_0 > 0$ such that for all $\lambda \ge \lambda_0$ any solution of problem (1.1) is oscillatory in the right neighborhood of a. In particular, this is true also for $\lambda = \lambda_k$ with k large enough. Since an eigenfunction associated with λ_k has at most $k-1$ zero points (cf. Section 3), we have a contradiction.

The case when (1.3) is violated can be handled similarly.

Sufficiency of (1.2) *and* (1.3). By Lemma 3.2, we have that $\lambda_1 < \lambda_2$, there is no eigenvalue $\lambda \in (-\infty, \lambda_1) \cup (\lambda_1\,\lambda_2)$, an eigenfunction u_{λ_1} associated with λ_1 has no zero in (a, b) and an eigenfunction u_{λ_2} associated with λ_2 has exactly one zero $t_1^2 \in (a, b)$. Let $\lambda > \lambda_2$ be an eigenvalue of (1.1) (not necessarily characterized by (3.6)) and associated eigenfunction u_λ has just one zero $\bar{t}_1 \in (a, b)$. Then $\lambda = \lambda_1(a, \bar{t}_1) = \lambda_1(\bar{t}_1, b)$ and $\lambda_2 = \lambda_1(a, t_1) = \lambda_1(t_1, b)$ (the principal eigenfunction is then restriction of u_λ and u_{λ_2} to the corresponding interval, respectively) the inequality $\lambda > \lambda_2$ and the variational characterization of the principal eigenvalue force $\bar{t}_1 < t_1$ and $t_1 < \bar{t}_1$ to hold simultaneously. This is a contradiction. Hence there exist at least two points $t_1, t_2 \in (a, b)$, $t_1 < t_2$, such that

$u_\lambda(t_1) = u_\lambda(t_2) = 0$. It follows from the discussion in Section 3 that the least eigenvalue λ with this property is λ_3 and u_{λ_3} is thus forced to have precisely two zeros $t_1^3 < t_2^3$, $t_1^3, t_2^3 \in (a, b)$. Similarly as above we show that an eigenfunction u_λ associated with an eigenvalue $\lambda > \lambda_3$ (not necessarily characterized by (3.6)) possesses at least three zeros $t_1 < t_2 < t_3$ in (a, b). Moreover, the least eigenvalue with this property is λ_4 with u_{λ_4} with precisely three zeros $t_1^4 < t_2^4 < t_3^4$ in (a, b) such that $t_1^4 < t_1^3 < t_2^4 < t_2^3 < t_3^4$. By the induction we can proceed further to find out that $\{\lambda_k\}_{k=1}^\infty$ are the only eigenvalues of (1.1) and zero points of corresponding eigenfunctions have the properties started in Theorem 1.1. It remains to prove the simplicity of every λ_k, $k = 1, 2, \ldots$. The case $k = 1$ is discussed already in Section 3. Let $k \geq 2$ be fixed. Denote by t_l^k, $l = 1, .., k-1$, the zeros of u_{λ_k}, $a < t_1^k < \ldots < t_{k-1}^k < b$. Then $\lambda_k = \lambda_1(a, t_1^k) = \lambda_1(t_l^k, t_{l+1}^k) = \lambda_1(t_{k-1}^k, b)$, $l = 1, \ldots, k-2$, and the restrictions of u_{λ_k} to the corresponding intervals are associated principal eigenfunctions. The simplicity of the principal eigenvalue stated in Section 3 implies that these restrictions of u_{λ_k} on every nodal interval $(a, t_1^k), \ldots, (t_{k-1}^k, b)$ are merely constant multiple of each other. The regularity of $u_{\lambda_k} \in C^1(a, b)$ yields that the multiplicative constants mutually depend on each other. It then follows that any two eigenfunctions associated with λ_k are proportional.

Acknowledgement. The authors were supported by the Research Plan MSM 4977751301 of the Ministry of Education, Youth and Sports of the Czech Republic. The first author was also supported by KONTAKT ME 10093.

References

[1] B. Opic and A. Kufner: *Hardy–Type Inequalities*, Pitman Research Notes in Mathematics Series 279, Longman Scientific and Technical, Harlow 1990.

[2] J. Nečas: *On the discreteness of the spectrum of nonlinear Sturm–Liouville equation* (Russian), Dokl. Akad. Nauk SSSR 201 (1971), 1045-1048. English translation: Soviet Math. Dokl. 12 (1971), 1779-1783.

[3] A. Elbert: *A half–linear second order differential equation, in: Qualitative theory of differential equations*, Coll. Math. Soc. J. Bolyai 30 (Szeget, 1979), 153-179.

[4] P. Drábek and A. Kufner: *Hardy inequality and properties of the quasilinear Sturm–Liouville problem*, Atti Accad. Naz. Lincei Cl. Sci. Fis. Mat. Natur. Rend. Lincei (9) Mat. Appl. 18 (2007), no. 2, 125-138

[5] P. Drábek and A. Kufner: *Discretness and simplicity of the spectrum of a quasilinear Sturm–Liouville-type problem on an infinite interval*, Proc. Amer. Math. Soc. 134, No 1 (2006), 235-242.

[6] P. Binding and P. Drábek: *Sturm–Liouville theory for the p-Laplacian.* Studia Sci. Math. Hungar. 40 (2003), no. 4, 375-396.

[7] R. T. Lewis: *The discretness of the spectrum of self adjoint, oven order, one term, differential operators*, Proc. Amer. Math. Soc. 42 (1974), 480-482.

[8] P. Lindqvist, *On the equation* $\mathrm{div}(|\nabla u|^{p-2}\nabla u) + \lambda|u|^{p-2}u = 0$, Proc. Amer. Math. Soc., 109 (1) (1990), 157–164.

[9] P. Drábek and S. B. Robinson: *Resonance problems for the* $p-$*Laplacian*, J. Functional Analysis, 169 (1999), 189–200.

[10] P. Drábek and S. B. Robinson: *On the generalization of the Courant nodal domain theorem*, J. Differential Equations 181 (2002), 58-71.

[11] O. Došlý: *Half–linear differential equations*, in: Handbook of Differential Equations, Ordinary Differential Equations, Vol. 1 (A. Cañada, P. Drábek and A. Fonda eds.), Elsevier, North Holland, Amsterdam 2004, pp. 161-357.

[12] O. Došlý: *Methods of oscillation theory of half–linear second order differential equations*, Czech. Math. J. 50 (125), (2000), 657-671.

Department of Mathematics
University of West Bohemia
Univerzitní 22CZ-306 14 Plzeň, Czech Republic
e-mail: `pdrabek@kma.zcu.cz`, `komil@kma.zcu.cz`