Subordination of p-harmonic mappings*

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Abstract

A $2p$ $(p \geq 1)$ times continuously differentiable complex-valued function $F = u + iv$ in a domain $D \subseteq \mathbb{C}$ is *p-harmonic* if F satisfies the p-harmonic equation $\Delta^p F = \Delta(\Delta^{p-1})F = 0$, where Δ represents the complex Laplacian operator

$$\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

In this paper, the main aim is to investigate the subordination of p-harmonic mappings. First, the characterization for p-harmonic mappings to be subordinate are obtained. Second, we get two results on the relation of integral means of subordinate p-harmonic mappings. Finally, we discuss the existence of extreme points for subordination families of p-harmonic mappings. Two sufficient conditions for p-harmonic mappings to be extreme points of the closed convex hulls of the corresponding subordination families are established.

1 Introduction

A $2p$ $(p \geq 1)$ times continuously differentiable complex-valued function $F = u + iv$ in a domain $D \subseteq \mathbb{C}$ is *p-harmonic* if F satisfies the p-harmonic equation $\Delta^p F = \Delta(\Delta^{p-1})F = 0$, where Δ represents the complex Laplacian operator

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$$\Delta = 4\frac{\partial^2}{\partial z \partial \bar{z}} := \frac{\partial^2}{\partial x^2} + \frac{\partial^2}{\partial y^2}.$$

A mapping F is p-harmonic in a simply connected domain D if and only if F has the following representation:

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z),$$

where each G_{p-k+1} is harmonic, i.e., $\Delta G_{p-k+1}(z) = 0$ for $k \in \{1, \cdots, p\}$ (cf. [7, Proposition 1]).

Obviously, when $p = 1$ (resp. 2), F is harmonic (resp. biharmonic). The properties of harmonic and biharmonic mappings have been investigated by many authors, see [1, 2, 3, 8, 11] etc.

Throughout this paper we consider p-harmonic mappings in the unit disk $\mathbb{D} = \{z \in \mathbb{C} : |z| < 1\}$.

It is known that every composition of a harmonic mapping with an analytic function is harmonic, but this useful fact does not always hold for p-harmonic mappings $(p > 1)$. For example, let $F(z) = |z|^{2(p-1)}z$ and $\varphi(z) = z^2$. Then F is p-harmonic and φ is analytic in $\mathbb{D}$. It is easy to show that $F \circ \varphi$ is not p-harmonic. Thus we can not give the definition of subordination of p-harmonic mappings by composition with a Schwarz function as those in the cases of analytic functions and harmonic mappings. Now we introduce the following definition.

Definition 1.1. Let

$$f(z) = \sum_{k=1}^{p} |z|^{2(k-1)} g_{p-k+1}(z) \quad \text{and} \quad F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$$

be two p-harmonic mappings of $\mathbb{D}$. We will say that f is *subordinate* to F and write $f \prec F$ or $f(z) \prec F(z)$ if there exists a Schwarz function φ of $\mathbb{D}$, that is, φ is analytic, $\varphi(0) = 0$ and $|\varphi(z)| \leq |z|$ for $z \in \mathbb{D}$, such that

$$\sum_{k=1}^{p} |z|^{2(k-1)} g_{p-k+1}(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(\varphi(z)). \tag{1.1}$$

Obviously, when $p = 1$, this is the same as the case of harmonic mappings, see [6, 11, 21] for the details.

Note that for analytic functions f and F, it is known that if F is univalent, then $f(\mathbb{D}) \subset F(\mathbb{D})$ if and only if $f \prec F$ (cf. [10]). This useful property is not valid for p-harmonic mappings, even for the case $p = 1$ (cf. [21]). We study this property further. In Section 3, we establish the characterization for p-harmonic mappings to be subordinate, which is stated as Theorem 3.1. The idea of the method used in the proof of our main result comes from [12] which is about the decomposition of harmonic mappings.

In [21], Schaubroeck considered the relation of the integral means of subordinate harmonic mappings. The following is one of the main results in [21], which is a generalization of the corresponding one in [17].

Theorem A. ([21, Theorem 2.4]) *Let f and F be harmonic in $\mathbb{D}$. If $f \prec F$, that is $f(z) = F(\varphi(z))$ for some Schwarz function φ, then $M_s(r, f) \le M_s(r, F)$ for $s \ge 1$ and $0 \le r < 1$, where $M_s(r, f)$ is the integral mean $(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^s d\theta)^{\frac{1}{s}}$. Equality occurs for $0 < r < 1$ only when F is constant or when φ is a rotation, i.e., $\varphi(z) = e^{i\theta}z$.*

In the proof of Theorem A, the fact that the harmonicity of f implies the subharmonicity of $|f|$ plays an important role in the corresponding discussions. But this useful property does not always hold for p-harmonic mappings when $p \ge 2$, which can be seen from the following example. Let $F(z) = |z|^2 - \bar{z}^2$. Then $|F|$ is not subharmonic in $\mathbb{D}$. In Section 4, by using a different method, we consider the relation of the integral means of subordinate p-harmonic mappings. Our main results are Theorems 4.1 and 4.2, where Theorem 4.1 is a generalization of Theorem A for p-harmonic mappings and Theorem 4.2 is a generalization of [19, Theorem 1].

Extreme points of analytic functions and harmonic mappings play an important role in solving extremal problems. Many references have been in literature, see [4, 5, 13, 14, 18] etc. As the third aim of this paper, we study the sufficient conditions for the extreme points of the closed convex hulls of subordination families of p-harmonic mappings. Two results are obtained, which are Theorems 5.1 and 5.2.

Several useful lemmas will be proved in Section 2.

2 Several lemmas

In this section, we will prove several lemmas which are useful for the following discussions.

Lemma 2.1. *Let $F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$ with $p \ge 2$ be a p-harmonic mapping. If there is some $0 < p_1 \le p-1$ such that*

$$\Delta^{p_1} F(z) = 0,$$

then for each $k \in \{p_1 + 1, \cdots, p\}$, $G_{p-k+1} \equiv 0$.

Proof. First we consider the case $p = 2$. Then $p_1 = 1$.

For any biharmonic mapping F, assume $F(z) = |z|^2 G_1(z) + G_2(z)$. If $\Delta F(z) = 0$, then

$$G_1(z) + z(G_1)_z(z) + \bar{z}(G_1)_{\bar{z}}(z) = 0.$$

Thus $G_1 \equiv 0$. The proof for this case is complete.

In the following we come to consider the case $p > 2$. For any p-harmonic mapping F, let $F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$ with $G_{p-k+1} = h_{1,p-k+1} + \overline{h}_{2,p-k+1}$. Assume there exists some $0 < p_1 \le p-1$ such that $\Delta^{p_1} F(z) = 0$. Then it follows that

$$\Delta^{p_1}\Big(\sum_{k=p_1+1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)\Big) = 0.$$

By straight computations,

$$
\begin{aligned}
&\Delta^{p_1}\Big(\sum_{k=p_1+1}^{p}|z|^{2(k-1)}G_{p-k+1}(z)\Big)\\
=\ &4^{p_1}\Big(\sum_{k=p_1+1}^{p}\frac{\partial^{2p_1}\big(|z|^{2(k-1)}h_{1,p-k+1}(z)\big)}{\partial z^{p_1}\partial\bar{z}^{p_1}}+\sum_{k=p_1+1}^{p}\frac{\partial^{2p_1}\big(|z|^{2(k-1)}\overline{h}_{2,p-k+1}(z)\big)}{\partial z^{p_1}\partial\bar{z}^{p_1}}\Big)\\
=\ &4^{p_1}\Big(\sum_{k=p_1+1}^{p}(k-1)\cdots(k-p_1)\bar{z}^{k-p_1-1}\frac{\partial^{p_1}\big(z^{k-1}h_{1,p-k+1}(z)\big)}{\partial z^{p_1}}\\
&+\sum_{k=p_1+1}^{p}(k-1)\cdots(k-p_1)z^{k-p_1-1}\frac{\partial^{p_1}\big(\bar{z}^{k-1}\overline{h}_{2,p-k+1}(z)\big)}{\partial\bar{z}^{p_1}}\Big)\\
=\ &4^{p_1}\sum_{k=1}^{p-p_1}|z|^{2(k-1)}\big(h^*_{1,p-p_1-k+1}(z)+\overline{h}^*_{2,p-p_1-k+1}(z)\big),
\end{aligned}
$$

where for each $k\in\{1,\cdots,p-p_1\}$,

$$
h^*_{1,p-p_1-k+1}(z)=\begin{cases}(k+p_1-1)\cdots k\Big(\dfrac{\partial^{p_1}\big(z^{k+p_1-1}h_{1,p-p_1-k+1}(z)\big)}{\partial z^{p_1}}\Big)/z^{k-1} & \text{if } z\neq 0\\ (k+p_1-1)\cdots kh_{1,p-p_1-k+1}(0) & \text{if } z=0\end{cases}
\tag{2.1}
$$

and

$$
h^*_{2,p-p_1-k+1}(z)=\begin{cases}(k+p_1-1)\cdots k\Big(\dfrac{\partial^{p_1}\big(z^{k+p_1-1}h_{2,p-p_1-k+1}(z)\big)}{\partial z^{p_1}}\Big)/z^{k-1} & \text{if } z\neq 0\\ (k+p_1-1)\cdots kh_{2,p-p_1-k+1}(0) & \text{if } z=0\end{cases}.
\tag{2.2}
$$

Let

$$
F^*(z)=\sum_{k=1}^{p_2}|z|^{2(k-1)}\big(h^*_{1,p_2-k+1}(z)+\overline{h}^*_{2,p_2-k+1}(z)\big),
$$

$$
H_{p_2}(z)=|z|^{2(p_2-1)}\big(h^*_{1,1}(z)+\overline{h}^*_{2,1}(z)\big)
$$

and

$$
H_{p_2-1}(z)=H_{p_2}(z)-F^*(z)=-\sum_{k=1}^{p_2-1}|z|^{2(k-1)}\big(h^*_{1,p_2-k+1}(z)+\overline{h}^*_{2,p_2-k+1}(z)\big),
$$

where $p_2=p-p_1$. Then $F^*\equiv 0$. It follows that

$$
H_{p_2}=H_{p_2-1}
$$

and H_{p_2} is (p_2-1)-harmonic. Thus

$$\begin{aligned}\Delta^{p_2-1}H_{p_2}(z) &= 4^{p_2-1}\Big(\frac{\partial^{2(p_2-1)}\big(|z|^{2p_2}h^*_{1,1}(z)\big)}{\partial z^{p_2-1}\partial\bar{z}^{p_2-1}}+\frac{\partial^{2(p_2-1)}\big(|z|^{2p_2}\overline{h}^*_{2,1}(z)\big)}{\partial z^{p_2-1}\partial\bar{z}^{p_2-1}}\Big)\\ &= 4^{p_2-1}p_2!\Big(\bar{z}\frac{\partial^{p_2-1}\big(z^{p_2}h^*_{1,1}(z)\big)}{\partial z^{p_2-1}}+z\frac{\partial^{p_2-1}\big(\bar{z}^{p_2}\overline{h}^*_{2,1}(z)\big)}{\partial\bar{z}^{p_2-1}}\Big)\\ &= 0.\end{aligned}$$

Hence $h^*_{1,1}\equiv 0$, $h^*_{2,1}\equiv 0$ and $F^*(z)=\sum_{k=1}^{p_2-1}|z|^{2(k-1)}\big(h^*_{1,p_2-k+1}(z)+\overline{h}^*_{2,p_2-k+1}(z)\big)\equiv 0$. Similarly, we have $h^*_{1,p_2-k+1}\equiv 0$ and $h^*_{2,p_2-k+1}\equiv 0$ for each $k\in\{1,\cdots,p_2-1\}$. Equations (2.1) and (2.2) show that $h_{1,p-p_1-k+1}\equiv 0$ and $h_{2,p-p_1-k+1}\equiv 0$ for all $k\in\{1,\cdots,p-p_1\}$. Hence $G_{p-k+1}\equiv 0$ for each $k\in\{p_1+1,\cdots,p\}$. ■

By using the similar proof method as in Lemma 2.1, we have the following uniqueness of p-harmonic mappings.

Lemma 2.2. *Let*

$$f(z)=\sum_{k=1}^{p}|z|^{2(k-1)}g_{p-k+1}(z)$$

and

$$F(z)=\sum_{k=1}^{p}|z|^{2(k-1)}G_{p-k+1}(z)$$

be two p-harmonic mappings. Then $f=F$ if and only if $g_{p-k+1}=G_{p-k+1}$ for all $k\in\{1,\cdots,p\}$.

By Lemma 2.2, the following useful result easily follows.

Lemma 2.3. *Let*

$$f(z)=\sum_{k=1}^{p}|z|^{2(k-1)}g_{p-k+1}(z)$$

and

$$F(z)=\sum_{k=1}^{p}|z|^{2(k-1)}G_{p-k+1}(z)$$

be two p-harmonic mappings. Then $f\prec F$ if and only if there exists some Schwarz function φ such that $g_{p-k+1}(z)=G_{p-k+1}(\varphi(z))$ for each $k\in\{1,\cdots,p\}$.

3 Characterization for subordination

In this section, using the decomposition of harmonic mappings and the relation of regions for subordinate harmonic mappings, we give a characterization of subordination for p-harmonic mappings.

In [12], authors considered the decomposition of harmonic mappings and the main tool in the proofs was the theory about the existence and uniqueness of solutions to Beltrami equation with a given dilatation. In the theory of quasiconformal mappings, it is known that for any measurable function μ with $\|\mu\|_\infty<1$, the Beltrami equation $f_{\bar{z}} = \mu f_z$ admits a homeomorphic solution F, and every solution has the form $f = \psi \circ F$ for some analytic function ψ (cf. [16]). A complex-valued harmonic mapping with positive Jacobian in $\mathbb{D}$ is known to satisfy the Beltrami equation of second kind $\overline{f_{\bar{z}}} = af_z$, where a is an analytic function with the property $|a(z)| < 1$ in $\mathbb{D}$. On the other hand, every solution of such an equation is harmonic. Moreover, if $\|a\|_\infty < 1$, then the equation admits homeomorphic solutions (cf. [15]).

We assume that all harmonic mappings mentioned in this section are sense-preserving, i.e. have a positive Jacobian.

In [12], authors proved

Theorem B. *Let f be a complex-valued nonconstant harmonic mapping defined on a domain $D \subset \mathbb{C}$ and let a be its dilatation function. Then in order that f have a decomposition $f = F \circ \varphi$ for some function φ analytic in D and some univalent harmonic mapping F defined on $\varphi(D)$, it is necessary and sufficient that $|a(z)| \neq 1$ on D and $a(z_1) = a(z_2)$ wherever $f(z_1) = f(z_2)$. Under these conditions the representation is unique up to conformal mappings; any other representation $f = \tilde{F} \circ \tilde{\varphi}$ has the form $\tilde{F} = F \circ \psi^{-1}$ and $\tilde{\varphi} = \psi \circ \varphi$ for some conformal mapping ψ defined on $\varphi(D)$.*

Using Theorem B, we obtain the following lemma.

Lemma 3.1. *Let f and g be two harmonic mappings of $\mathbb{D}$, where $f(0) = g(0)$ and g is univalent. Then $f \prec g$ if and only if $f(\mathbb{D}) \subset g(\mathbb{D})$, $|a_f(z)| \neq 1$, $a_f(z_1) = a_f(z_2)$ wherever $f(z_1) = f(z_2)$ and $(g^{-1})_{\bar{w}} = \mu(g^{-1})_w$, where a_f is the dilatation of f and $\mu = -\overline{a_f \circ f^{-1}}$.*

Proof. Assume $f \prec g$. Then there is some Schwarz function φ such that $f = g \circ \varphi$. The necessity follows from Theorem B.

Now, we come to prove the sufficiency. By the assumptions and Theorem B, we see that there is an univalent harmonic mapping f_1 and an analytic function φ_1 such that $f = f_1 \circ \varphi_1$. Since $(g^{-1})_{\bar{w}} = \mu(g^{-1})_w$, by the uniqueness of quasiconformal mappings with a prescribed complex dilatation, we conclude that $f_1 = g \circ \psi$ for some conformal mapping ψ (cf. [16]). Thus

$$f = f_1 \circ \varphi_1 = (g \circ \psi) \circ \varphi_1 = g \circ \varphi,$$

where $\varphi = \psi \circ \varphi_1$. From $f(\mathbb{D}) \subset g(\mathbb{D})$, we deduce that φ is a Schwarz function and then $f \prec g$. ∎

Now, we are ready to state our main result of this section.

Theorem 3.1. *Let*

$$f(z) = \sum_{k=1}^{p} |z|^{2(k-1)} g_{p-k+1}(z)$$

and

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$$

be two p-harmonic mappings with $g_{p-k+1}(0) = G_{p-k+1}(0)$ for $k \in \{1, \cdots, p\}$. If all G_{p-k+1} $(k \in \{1, \cdots, p\})$ are univalent harmonic mappings, then $f \prec F$ if and only if *for each* $k \in \{1, \cdots, p\}$, $g_{p-k+1}(\mathbb{D}) \subset G_{p-k+1}(\mathbb{D})$, $|a_{p-k+1}(z)| \neq 1$, $a_{p-k+1}(z_1) = a_{p-k+1}(z_2)$ *wherever* $g_{p-k+1}(z_1) = g_{p-k+1}(z_2)$, $(G^{-1}_{p-k+1})_{\overline{w}} = \mu_{p-k+1}(G^{-1}_{p-k+1})_w$, *where* a_{p-k+1} *is the dilatation of* g_{p-k+1}, $\mu_{p-k+1} = -\overline{a}_{p-k+1} \circ \overline{g}^{-1}_{p-k+1}$ *and* $G_1^{-1} \circ g_1 = \cdots = G_p^{-1} \circ g_p$.

The proof easily follows from Lemmas 2.3 and 3.1.

4 Integral means

In [21], Schaubroeck studied the relation of the integral means for subordinate harmonic mappings. By using a different method, we generalize one of the main results in [21], i.e., Theorem A, to the case of p-harmonic mappings. Our result is as follows.

Theorem 4.1. *Let*

$$f(z) = \sum_{k=1}^{p} |z|^{2(k-1)} g_{p-k+1}(z) \quad \text{and} \quad F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$$

be two p-harmonic mappings of $\mathbb{D}$. *Suppose* $f \prec F$. *Then for any* $s \geq 1$ *and* $0 \leq r < 1$, $M_s(r, f) \leq M_s(r, F)$, *where* $M_s(r, f)$ *is the integral mean* $\left(\frac{1}{2\pi}\int_0^{2\pi} |f(re^{i\theta})|^s d\theta\right)^{\frac{1}{s}}$. *Equality occurs for* $0 < r < 1$ *only when* G_{p-k+1} *is constant for each* $k \in \{1, \cdots, p\}$ *or when* φ *is a rotation* $\varphi(z) = e^{i\theta}z$ *for some* $\theta \in [0, 2\pi]$.

Proof. Since $f \prec F$, it follows that $f(0) = F(0)$, and then $|f(0)|^s = |F(0)|^s$.
Fix $r \in (0, 1)$ and let

$$f_r(z) = \sum_{k=1}^{p} r^{2(k-1)} g_{p-k+1}(z) \quad \text{and} \quad F_r(z) = \sum_{k=1}^{p} r^{2(k-1)} G_{p-k+1}(z).$$

It is obvious that f_r and F_r are harmonic mappings. Since $f \prec F$, Lemma 2.3 implies that there exists some Schwarz function φ such that $g_{p-k+1}(z) = G_{p-k+1}(\varphi(z))$ for each $k \in \{1, \cdots, p\}$. It follows that $f_r \prec F_r$. By Theorem A, we have $M_s(r_1, f_r) \leq M_s(r_1, F_r)$ for any $r_1 \in (0, 1)$. Hence

$$\begin{aligned} \int_0^{2\pi} |f_r(r_1 e^{i\theta})|^s d\theta &= \int_0^{2\pi} |\sum_{k=1}^{p} r^{2(k-1)} g_{p-k+1}(r_1 e^{i\theta})|^s d\theta \\ &\leq \int_0^{2\pi} |F_r(r_1 e^{i\theta})|^s d\theta \\ &= \int_0^{2\pi} |\sum_{k=1}^{p} r^{2(k-1)} G_{p-k+1}(r_1 e^{i\theta})|^s d\theta. \end{aligned}$$

Let $r_1 = r$. Then

$$\int_0^{2\pi} |\sum_{k=1}^{p} r^{2(k-1)} g_{p-k+1}(re^{i\theta})|^s d\theta \leq \int_0^{2\pi} |\sum_{k=1}^{p} r^{2(k-1)} G_{p-k+1}(re^{i\theta})|^s d\theta,$$

so

$$M_s(r, f) \leq M_s(r, F).$$

Assume that

$$M_s(r, f) = M_s(r, F),$$

that is,

$$M_s(r, f_r) = M_s(r, F_r).$$

A similar argument as in the proof of Theorem A shows that F_r is constant or φ is a rotation $\varphi(z) = e^{i\theta} z$ for some $\theta \in [0, 2\pi]$. It is easy to show that $F_r \equiv c_r$ for each $r \in (0, 1)$ if and only if $G_{p-k+1} = c_{p-k+1}$ for $k \in \{1, \cdots, p\}$, where c_r and c_{p-k+1} are constants depending only on r and k, respectively. The proof is complete. ∎

In [19], Nunokawa, Saitoh, Owa and Takahashi discussed the relation of subordination and integral means of real harmonic mappings. Their main result is [19, Theorem 1]. In the following, we find an analogue of [19, Theorem 1] for p-harmonic mappings.

Theorem 4.2. *Suppose that*

$$f(z) = \sum_{k=1}^{p} |z|^{2(k-1)} g_{p-k+1}(z) \quad \textit{and} \quad F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$$

are two p-harmonic mappings of $\mathbb{D}$. If $f \prec F$, then for any r with $0 \leq r < 1$ and any s with $s \geq 1$,

$$\int_0^{2\pi} |\mathrm{Re} f(re^{i\theta})|^s d\theta \leq \int_0^{2\pi} |\mathrm{Re} F(re^{i\theta})|^s d\theta. \tag{4.1}$$

Proof. It follows from $f \prec F$ that $\mathrm{Re} f(0) = \mathrm{Re} F(0)$, and then $|\mathrm{Re} f(0)|^s = |\mathrm{Re} F(0)|^s$.

Fix $r \in (0, 1)$ and let

$$f_r(z) = \sum_{k=1}^{p} r^{2(k-1)} g_{p-k+1}(z) \text{ and } F_r(z) = \sum_{k=1}^{p} r^{2(k-1)} G_{p-k+1}(z).$$

Then both f_r and F_r are harmonic mappings. By Lemma 2.3, $f_r \prec F_r$ and then $f_r(0) = F_r(0)$.

Let $u_r(z) = \mathrm{Re} f_r(z)$ and $U_r(z) = \mathrm{Re} F_r(z)$. Since $f_r \prec F_r$, we know $u_r(z) = U_r(\varphi(z))$ for some Schwarz function φ. Let q_r, Q_r be analytic functions whose real parts are u_r and U_r, respectively, and $q_r(0) = Q_r(0)$. Then

$$u_r(z) = \frac{q_r(z) + \overline{q}_r(z)}{2} \quad \text{and} \quad U_r(z) = \frac{Q_r(z) + \overline{Q}_r(z)}{2}.$$

Obviously, $u_r(z) = U_r(\varphi(z))$ if and only if $q_r(z) + \overline{q}_r(z) = Q_r(\varphi(z)) + \overline{Q}_r(\varphi(z))$. Thus $q_r(z) = Q_r(\varphi(z))$ which implies that $q_r \prec Q_r$. By [19, Theorem 1], for any $r_1 \in (0,1)$ and $s \geq 1$,

$$\int_0^{2\pi} |\mathrm{Re} q_r(r_1 e^{i\theta})|^s d\theta \leq \int_0^{2\pi} |\mathrm{Re} Q_r(r_1 e^{i\theta})|^s d\theta.$$

That is,

$$\int_0^{2\pi} |\mathrm{Re} f_r(r_1 e^{i\theta})|^s d\theta \leq \int_0^{2\pi} |\mathrm{Re} F_r(r_1 e^{i\theta})|^s d\theta$$

and, thus for $r_1 = r$ and $s \geq 1$,

$$\int_0^{2\pi} |\mathrm{Re} \sum_{k=1}^{p} r^{2(k-1)} g_{p-k+1}(re^{i\theta})|^s d\theta \leq \int_0^{2\pi} |\mathrm{Re} \sum_{k=1}^{p} r^{2(k-1)} G_{p-k+1}(re^{i\theta})|^s d\theta.$$

The proof is complete. ∎

We remark that the inequality (4.1) does not hold in general for p-harmonic mappings f and F when $0 < s < 1$. This can be seen from the following result.

Theorem 4.3. *Let $f(z) = |z|^{2(p-1)} g(z)$ and $F(z) = |z|^{2(p-1)} G(z)$, where g and G are analytic functions such that $\mathrm{Re}\, G(z) > 0$ for any $z \in \mathbb{D}$. Suppose $f \prec F$. Then for any r with $0 \leq r < 1$ and any s with $0 < s < 1$,*

$$\int_0^{2\pi} |\mathrm{Re} f(re^{i\theta})|^s d\theta \geq \int_0^{2\pi} |\mathrm{Re} F(re^{i\theta})|^s d\theta.$$

Proof. It follows from $f(0) = F(0) = 0$ that $|\mathrm{Re} f(0)|^s = |\mathrm{Re} F(0)|^s$.

Since $\mathrm{Re}\, G(z) > 0$ and $f \prec F$, for any r_1 and r with $0 < r_1 < r < 1$, we have for some Schwarz function φ,

$$\begin{aligned}
\mathrm{Re} f(r_1 e^{i\theta}) &= |\mathrm{Re} f(r_1 e^{i\theta})| \\
&= r_1^{2(p-1)} \mathrm{Re}\, G(\varphi(r_1 e^{i\theta})) \\
&= \frac{r_1^{2(p-1)}}{2\pi} \int_0^{2\pi} \mathrm{Re}\, G(re^{i\nu}) \mathrm{Re} \frac{re^{i\nu} + \varphi(r_1 e^{i\theta})}{re^{i\nu} - \varphi(r_1 e^{i\theta})} d\nu \\
&= \frac{r_1^{2(p-1)}}{2\pi r^{2(p-1)}} \int_0^{2\pi} |\mathrm{Re} F(re^{i\nu})|\, \mathrm{Re} \frac{re^{i\nu} + \varphi(r_1 e^{i\theta})}{re^{i\nu} - \varphi(r_1 e^{i\theta})} d\nu.
\end{aligned}$$

By Hölder's inequality, we obtain for any s with $0 < s < 1$,

$$\begin{aligned}
&\frac{1}{2\pi}\int_0^{2\pi}|\mathrm{Re}\,F(re^{i\nu})|^s\mathrm{Re}\,\frac{re^{i\nu}+\varphi(r_1e^{i\theta})}{re^{i\nu}-\varphi(r_1e^{i\theta})}\,d\nu\\
=\;&\frac{1}{2\pi}\int_0^{2\pi}|\mathrm{Re}\,F(re^{i\nu})|^s\Big(\mathrm{Re}\,\frac{re^{i\nu}+\varphi(r_1e^{i\theta})}{re^{i\nu}-\varphi(r_1e^{i\theta})}\Big)^s\Big(\mathrm{Re}\,\frac{re^{i\nu}+\varphi(r_1e^{i\theta})}{re^{i\nu}-\varphi(r_1e^{i\theta})}\Big)^{1-s}d\nu\\
\le\;&\Big(\frac{1}{2\pi}\int_0^{2\pi}|\mathrm{Re}\,F(re^{i\nu})|\,\mathrm{Re}\,\frac{re^{i\nu}+\varphi(r_1e^{i\theta})}{re^{i\nu}-\varphi(r_1e^{i\theta})}\,d\nu\Big)^s\times\Big(\frac{1}{2\pi}\int_0^{2\pi}\mathrm{Re}\,\frac{re^{i\nu}+\varphi(r_1e^{i\theta})}{re^{i\nu}-\varphi(r_1e^{i\theta})}\,d\nu\Big)^{1-s}\\
=\;&\Big(\frac{1}{2\pi}\int_0^{2\pi}|\,\mathrm{Re}F(re^{i\nu})|\,\mathrm{Re}\,\frac{re^{i\nu}+\varphi(r_1e^{i\theta})}{re^{i\nu}-\varphi(r_1e^{i\theta})}\,d\nu\Big)^s,
\end{aligned}$$

and then

$$\begin{aligned}
\int_0^{2\pi}|\mathrm{Re}f(r_1e^{i\theta})|^s d\theta &= \frac{r_1^{2s(p-1)}}{r^{2s(p-1)}}\int_0^{2\pi}\Big(\frac{1}{2\pi}\int_0^{2\pi}|\mathrm{Re}F(re^{i\nu})|\,\mathrm{Re}\frac{re^{i\nu}+\varphi(r_1e^{i\theta})}{re^{i\nu}-\varphi(r_1e^{i\theta})}d\nu\Big)^s d\theta\\
&\ge \frac{r_1^{2s(p-1)}}{r^{2s(p-1)}}\int_0^{2\pi}\Big(\frac{1}{2\pi}\int_0^{2\pi}|\mathrm{Re}F(re^{i\nu})|^s\mathrm{Re}\frac{re^{i\nu}+\varphi(r_1e^{i\theta})}{re^{i\nu}-\varphi(r_1e^{i\theta})}d\theta\Big)d\nu\\
&= \frac{r_1^{2s(p-1)}}{r^{2s(p-1)}}\int_0^{2\pi}|\mathrm{Re}F(re^{i\nu})|^s d\nu.
\end{aligned}$$

Letting $r_1 \to r$ implies

$$\int_0^{2\pi}|\mathrm{Re}f(re^{i\theta})|^s d\theta \ge \int_0^{2\pi}|\mathrm{Re}F(re^{i\nu})|^s d\nu = \int_0^{2\pi}|\mathrm{Re}F(re^{i\theta})|^s d\theta. \qquad \blacksquare$$

Remark 4.1. Let $f(z) = |z|^{2(p-1)}(1-z)$ and $F(z) = |z|^{2(p-1)}(1-z^n)$ for large enough n in Theorem 4.3. Then both f and F are p-harmonic and

$$\int_0^{2\pi}|\mathrm{Re}f(re^{i\theta})|^s d\theta > \int_0^{2\pi}|\mathrm{Re}F(re^{i\theta})|^s d\theta$$

for $0 < s < 1$, which shows that the requirement "$s \ge 1$" in Theorem 4.2 is necessary.

5 Extreme points of closed convex hulls of subordination families

Before the statement of the main results, we first introduce the following concept.

Definition 5.1. Let X be a topological vector space over the field of complex numbers and D a set of X. A point $x \in D$ is called an *extreme point* of D if it has no representation of the form $x = ty + (1-t)z$ $(0 < t < 1)$ as a proper convex combination of two distinct points y and z in D.

We denote by ED the set of all extreme points of D and by HD the *closed convex hull* of D, that is, the smallest closed convex set containing D (cf. [10, P_{281}]).

In [14], the authors proved two results on the extreme points of the family of functions subordinate to a fixed analytic function. The main aim of this section is to generalize these results to the case of p-harmonic mappings.

Theorem 5.1. *Let $F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$ be p-harmonic in $\mathbb{D}$ with*

$$G_{p-k+1}(z) = G^{(1)}_{p-k+1}(z) + \overline{G}^{(2)}_{p-k+1}(z)$$

such that $G^{(1)}_{p-k+1}(0) = 0$ and $G^{(2)}_{p-k+1}(0) = 0$, and $s(F)$ be the family of p-harmonic mappings subordinate to F. Then each mapping $f(z) = F(xz)$ with $|x| = 1$ belongs to $EHs(F)$.

Proof. Suppose, on the contrary, that $f(z) = F(xz)$ doesn't belong to $EHs(F)$ for some x with $|x| = 1$. Then there exist f_1 and $f_2 \in Hs(F)$ such that $f_1 \neq f_2$ and

$$f(z) = F(xz) = tf_1(z) + (1-t)f_2(z),$$

where $0 < t < 1$,

$$f_1(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{1,p-k+1}(z) = \sum_{k=1}^{p} |z|^{2(k-1)} \big(G^{(1)}_{1,p-k+1}(z) + \overline{G}^{(2)}_{1,p-k+1}(z)\big)$$

and

$$f_2(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{2,p-k+1}(z) = \sum_{k=1}^{p} |z|^{2(k-1)} \big(G^{(1)}_{2,p-k+1}(z) + \overline{G}^{(2)}_{2,p-k+1}(z)\big).$$

Obviously, $G^{(1)}_{1,p-k+1}(0) = G^{(2)}_{1,p-k+1}(0) = G^{(1)}_{2,p-k+1}(0) = G^{(2)}_{2,p-k+1}(0) = 0$ for each $k \in \{1, \cdots, p\}$.

By Lemma 2.2, we get

$$G_{p-k+1}(xz) = tG_{1,p-k+1}(z) + (1-t)G_{2,p-k+1}(z)$$

for each $k \in \{1, \cdots, p\}$. And then, using $G^{(1)}_{1,p-k+1}(0) = G^{(2)}_{1,p-k+1}(0) = G^{(1)}_{2,p-k+1}(0) = G^{(2)}_{2,p-k+1}(0) = 0$, we have

$$G^{(1)}_{p-k+1}(xz) = tG^{(1)}_{1,p-k+1}(z) + (1-t)G^{(1)}_{2,p-k+1}(z)$$

and

$$G^{(2)}_{p-k+1}(xz) = tG^{(2)}_{1,p-k+1}(z) + (1-t)G^{(2)}_{2,p-k+1}(z)$$

for each $k \in \{1, \cdots, p\}$. Hence either $G^{(1)}_{p-k+1}(xz)$ does not belong to $EHs(G^{(1)}_{p-k+1})$ or $G^{(2)}_{p-k+1}(xz)$ does not belong to $EHs(G^{(2)}_{p-k+1})$ for each $1 \leq k \leq p$, which contradicts [14, Theorem 6]. Hence each $f(z) = F(xz)$ ($|x| = 1$) belongs to $EHs(F)$. ∎

Denotes by $\mathcal{H}^s$ $(0 < s \leq \infty)$ the class of p-harmonic mappings in $\mathbb{D}$ subject to the condition:

$$\mathcal{M}_s(r, F) = \Big(\frac{1}{2\pi}\int_0^{2\pi} |F(re^{i\theta})|^s d\theta\Big)^{\frac{1}{s}}$$

remains bounded as $r = |z| \to 1$. The norm is defined as

$$\|F\|_s = \lim_{r\to 1} \mathcal{M}_s(r, F).$$

It is evident that $\mathcal{H}^{s_1} \supset \mathcal{H}^{s_2}$ if $0 < s_1 < s_2 \leq \infty$. Obviously, if

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z) = \sum_{k=1}^{p} |z|^{2(k-1)} \big(G^{(1)}_{p-k+1}(z) + \overline{G}^{(2)}_{p-k+1}(z)\big),$$

then

$$\|F\|_2^2 = \|\sum_{k=1}^{p} G^{(1)}_{p-k+1}\|_2^2 + \|\sum_{k=1}^{p} G^{(2)}_{p-k+1}\|_2^2,$$

that is, $\sum_{k=1}^{p} G^{(1)}_{p-k+1}$ and $\sum_{k=1}^{p} G^{(2)}_{p-k+1}$ belong to the space H^2 for analytic functions, see [9].

In order to state the next result, we introduce a concept.

Definition 5.2. An *inner function* is an analytic function φ in $\mathbb{D}$ with $|\varphi(z)| \leq 1$ and $|\varphi(e^{i\theta})| = 1$ for almost all θ (cf. [9]).

Theorem 5.2. *Let*

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$$

be p-harmonic in $\mathbb{D}$ and $s(F)$ be the family of mappings subordinate to F. Suppose that $F \in \mathcal{H}^s$, where $2 \leq s < \infty$. If φ is an inner function with $\varphi(0) = 0$, then $f(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(\varphi(z)) \in EHs(F)$.

Proof. The proof of Theorem 5.2 easily follows from the similar reasoning as in the proof of [14, Theorem 7] and the following lemma. ■

Lemma 5.1. *Let $f(z) = \sum_{k=1}^{p} |z|^{2(k-1)} g_{p-k+1}(z)$ and $F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z)$ be two p-harmonic mappings. Suppose $f \prec F$ and $F \in \mathcal{H}^2$. Then $\|f\|_2 = \|F\|_2$ if and only if there is some inner function φ with $\varphi(0) = 0$ such that*

$$\sum_{k=1}^{p} |z|^{2(k-1)} g_{p-k+1}(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(\varphi(z)).$$

Proof. Suppose

$$f(z) = \sum_{k=1}^{p} |z|^{2(k-1)} g_{p-k+1}(z) = \sum_{k=1}^{p} |z|^{2(k-1)} \big(g^{(1)}_{p-k+1}(z) + \overline{g}^{(2)}_{p-k+1}(z)\big)$$

and

$$F(z) = \sum_{k=1}^{p} |z|^{2(k-1)} G_{p-k+1}(z) = \sum_{k=1}^{p} |z|^{2(k-1)} \big(G^{(1)}_{p-k+1}(z) + \overline{G}^{(2)}_{p-k+1}(z)\big).$$

Since

$$\|f\|_2^2 = \| \sum_{k=1}^{p} g^{(1)}_{p-k+1}\|_2^2 + \| \sum_{k=1}^{p} g^{(2)}_{p-k+1}\|_2^2$$

and

$$\|F\|_2^2 = \| \sum_{k=1}^{p} G^{(1)}_{p-k+1}\|_2^2 + \| \sum_{k=1}^{p} G^{(2)}_{p-k+1}\|_2^2,$$

we know that $\|f\|_2 = \|F\|_2$ if and only if

$$\| \sum_{k=1}^{p} g^{(1)}_{p-k+1}\|_2^2 + \| \sum_{k=1}^{p} g^{(2)}_{p-k+1}\|_2^2 = \| \sum_{k=1}^{p} G^{(1)}_{p-k+1}\|_2^2 + \| \sum_{k=1}^{p} G^{(2)}_{p-k+1}\|_2^2.$$

It follows from $f \prec F$ and Lemma 2.3 that $\sum_{k=1}^{p} g^{(1)}_{p-k+1} \prec \sum_{k=1}^{p} G^{(1)}_{p-k+1}$ and $\sum_{k=1}^{p} g^{(2)}_{p-k+1} \prec \sum_{k=1}^{p} G^{(2)}_{p-k+1}$. By [10, Theorem 6.3], we have

$$\| \sum_{k=1}^{p} g^{(1)}_{p-k+1}\|_2 \le \| \sum_{k=1}^{p} G^{(1)}_{p-k+1}\|_2$$

and

$$\| \sum_{k=1}^{p} g^{(2)}_{p-k+1}\|_2 \le \| \sum_{k=1}^{p} G^{(2)}_{p-k+1}\|_2.$$

Hence

$$\| \sum_{k=1}^{p} g^{(1)}_{p-k+1}\|_2 = \| \sum_{k=1}^{p} G^{(1)}_{p-k+1}\|_2$$

and

$$\| \sum_{k=1}^{p} g^{(2)}_{p-k+1}\|_2 = \| \sum_{k=1}^{p} G^{(2)}_{p-k+1}\|_2.$$

From [20, Theorem 3] the proof follows. ■

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