Finite Groups containing Certain Abelian TI-subgroups

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Abstract

We determine the structure of finite groups whose cyclic subgroups, elementary abelian 2-subgroups and abelian subgroups of order at most $4p$, p a prime, are TI-subgroups.

1 Introduction

A subgroup K of a finite group G is called a TI-subgroup of G if $K \cap K^g = 1$ or K for each $g \in G$. The classification of the finite groups containing certain TI-subgroups are of special interest in group theory. Walls in [4] has described the finite groups all of whose subgroups are TI-subgroups. Recently in [2], finite groups all of whose abelian subgroups are TI-subgroups are classified. In this article we classify all finite groups whose cyclic subgroups, elementary abelian 2-subgroups and abelian subgroups of order at most $4p$ are TI-subgroups, where p is a prime dividing the order of G.

Throughout this article all the groups are finite and G is a finite group whose cyclic subgroups, elementary abelian 2-subgroups and abelian subgroups of order at most $4p$ are TI-subgroups, where p is a prime dividing the order of G. We follow [1] for notation in group theory. In this paper we shall prove the following theorems.

Theorem 1.1. *Let G be a group of even order and $z \in G$ be an involution, then $C_G(z)$ is nilpotent. Furthermore, if $C_G(t)$ is not a 2-group for some involution $t \in G$, then G is solvable.*

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A finite group H is called a *CIT-group* if $C_H(t)$ is 2-group for each involution $t \in H$. Theorem 1.1 shows that if G is nonsolvable, then G is a CIT-group. The next theorem shows that there are few simple groups whose cyclic subgroups, elementary abelian 2-subgroups and abelian subgroups of order at most $4p$, p a prime, are TI-subgroups.

Theorem 1.2. *Let G be a nonsolvable group, then G is isomorphic to $L_2(4)$, $L_2(7)$ or $L_2(9)$.*

In the next theorem we assume that G is solvable.

Theorem 1.3. *Let G be a solvable group , then one of the following holds.*
i) G is nilpotent.
ii) $G \cong S_4$ or A_4.
iii) Either G is of odd order or G has a normal 2-complement and a Sylow 2-subgroup of G is cyclic or is isomorphic to Q_8.

2 Proofs of theorems 1.1 and 1.3

In this section, we assume that G is a finite group of even order all of whose cyclic subgroups, elementary abelian 2-subgroups and abelian subgroups of order at most $4p$ are TI-subgroups, where p is a prime dividing the order of G. We refer the reader to [1] for information on coprime action theorem and Frobenius' normal p-complement theorem.

Lemma 2.1. *Let K be a TI-subgroup of a group G and Let T a non-trivial normal subgroup of G contained in K. Then K is also normal in G.*

Proof: Since $T \leq K^g \cap K$ for all $g \in G$ and K is a TI-subgroup of G we obtain $K^g = K$, for each $g \in G$. This proves the lemma. ∎

Let S be a Sylow 2-subgroup of the group G and $z \in S$ be an involution. Then we put $H = C_G(z)$, the centralizer of z in G. Using this notation we prove the following lemma.

Lemma 2.2. *The subgroup H is nilpotent. Furthermore, if T is any cyclic p-subgroup of H, then $H = N_G(T)$, for an odd prime number p.*

Proof: Let p be an odd prime number, $Q \in Syl_p(H)$ and $1 \neq x \in Q$ be a p-element. Set $T = \langle x \rangle$ and $R = N_G(T)$, then $z \in R$. We have $U = \langle z, T \rangle$ is cyclic and hence it is a TI-subgroup of G. Now by Lemma 2.1, we get that U is normal in $\langle R, H \rangle$. This gives us that $\langle z \rangle$ is normal in R and T is normal in H. Hence $R = H$. Let $1 \neq t \in H$ be an involution. Then by Lemma 2.1, we have $Y = \langle z, t \rangle$ is normal in H. Therefore $O(H) \leq C_G(t)$. Now let $1 \neq r \in H$ be a 2-element. Then $\langle r \rangle^{O(H)} \cap \langle r \rangle \neq 1$. Therefore $\langle r \rangle$ is normalized by $O(H)$. By this and since each cyclic subgroup of $O(H)$ is normal in H, we get that H is nilpotent and the lemma is proved. ∎

Lemma 2.3. *Assume that $N_G(P)$ has no normal 2-complement, for some 2-subgroup $1 \neq P$ of G. Then $N_G(P)$ is isomorphic to A_4 or S_4.*

Proof: Set $K = N_G(P)$ and let V be a minimal normal 2-subgroup of K. Then V is elementary abelian and we may assume that $z \in V$. By Lemma 2.2, and as K has no normal 2-complement, we conclude that V is not cyclic. Let $1 \neq C \leq V$ be a subgroup of index 2 in V. Then as V is minimal normal in K and C is a TI-group, we get that $C^g \cap C = 1$ for some $1 \neq g \in K$. This tells us that V is of order 4. By Lemma 2.2, we obtain that $C_K(V)$ is nilpotent. Using this and since K has no normal 2-complement, we conclude that $K/C_K(V)$ is of order 3 or isomorphic to S_3. Assume that $Q \in Syl_p(C_K(V))$, p an odd prime, and $x \in Z(Q)$ be of order p. Then by Lemma 2.1, $\langle V, x\rangle$ is normal in K. This gives that $\langle x\rangle$ is normal in K. Now by Lemma 2.1, we get that $\langle x, z\rangle$ is normal in K. But this gives that $K \leq H$ and then Lemma 2.2 implies that K has a normal 2-complement, which is a contradiction. Therefore $O(C_K(V)) = 1$ and hence $C_K(V) = O_2(K)$. We have $V \leq Z(O_2(K))$, as V is a minimal normal subgroup of K. Assume that $O_2(K) \neq V$, ten there is an abelian subgroup $W \leq O_2(K)$ of order 8 containing V. By Lemma 2.1, W is normal in K. Let $\langle s\rangle \in Syl_3(K)$, then W/V is $\langle s\rangle$-invariant. This tells us that $C_W(s)$ is of order two. Lemma 2.1 implies that $\langle z, C_W(s)\rangle$ is normal in $\langle W, s\rangle$. But this gives that $s \in H$ and then $s \in C_K(V)$, a contradiction. Hence $V = O_2(K)$ and the lemma is proved. ∎

Lemma 2.4. *Assume that H is not a 2-group. Then G is solvable and one of the following holds.*

i) G is nilpotent.

ii) S is cyclic or $S \cong Q_8$, $G = [O(G), z]H$, $(|[O(G), z]|, |H|) = 1$, $[O(G), z]$ is abelian and $C_{[O(G),z]}(x) = 1$, for each element $1 \neq x \in H$.

Proof: By the assumption $O(H) \neq 1$. Let $1 \neq y \in Z(S)$ be an involution. Set $K = C_G(y)$ and assume further that $G \neq K$. By Lemma 2.2, we have $O(H) = O(K)$. Assume that each 2-local subgroup of G has a normal 2-complement. Then Frobenius' normal p-complement theorem gives us that $G = O(G)K$. Assume that S contains an elementary abelian subgroup of order 4 containing y. Then by coprime action theorem and Lemma 2.2, we have $O(G) = O(K)$ and hence $G = K$, a contradiction. Therefore y is the unique involution in S and S is cyclic or $S \cong Q_8$. Now we have $y = z$. By coprime action theorem we have $G = [O(G), z]H$ and since z acts fixed point freely on $[O(G), z]$, we get that $[O(G), z]$ is abelian. By Lemma 2.2, we obtain that $C_{[O(G),z]}(x) = 1$, for each element $1 \neq x \in H$ and hence $(|[O(G), z]|, |H|) = 1$.

Now let $1 \neq P \leq S$ be a subgroup of S and assume that $N = N_G(P)$ has no normal 2-complement. Lemma 2.3, implies that N is isomorphic to A_4 or S_4. This gives us that P is of order 4 and $C_N(P) = P$. On the other hand, Lemma 2.2 implies that $1 \neq O(H) \leq C_N(P)$, which is a contradiction. This contradiction proves the lemma. ∎

Lemma 2.5. *Let G be solvable and a CIT-group, then one of the following holds.*

i) $G = S$.

ii) $G \cong S_4$ or A_4.

iii) $G = O(G)S$ and S is either cyclic or isomorphic to Q_8.

Proof: Assume that $O(G) = 1$. Assume further that the normalizer of each 2-subgroup of G has a normal 2-complement. Then by Frobenius' normal p-complement theorem we get that $G = O(G)S$ and hence $G = S$. Now assume that $1 \neq V$ is a subgroup of S such that $K = N_G(V)$ has no normal 2-complement. Then by Lemma 2.3, we get that V is an elementary abelian group of order 4 and $K \cong A_4$ or S_4. Since G is solvable, $O(G) = 1$ and G is CIT-group, we obtain $O_2(G) = V$ and hence $K = G$.

Assume that $O(G) \neq 1$. As G is a CIT-group, we conclude that $O_2(G) = 1$. Since G is a CIT-group and $O(G) \neq 1$, the coprime action theorem implies that S is cyclic or $S \cong Q_8$. This and Lemma 2.2 tell us that the normalizer of each 2-subgroup of G has a normal 2-complement in G and hence $G = O(G)S$, by Frobenius normal p-complement theorem. Thus the lemma is proved. ∎

Proof of Theorem 1.1. The proof of the theorem follows from Lemmas 2.2 and 2.4. ∎

Proof of Theorem 1.3. Theorem 1.3 follows from Lemmas 2.2, 2.4 and 2.5. ∎

3 Proof of Theorem 1.2

In this section G is a finite nonsolvable group whose cyclic subgroups, elementary abelian 2-subgroups and abelian subgroups of order at most $4p$ are TI-subgroups, where p is a prime dividing the order of G. Since G is nonsolvable, it is of even order. We keep the notations H, z and S from Section 2. We make use of the following lemma.

Lemma 3.1. *Let A be a simple group and $T \in Syl_2(A)$. Then*

i) if $T \cong D_8$ and $C_A(t) \cong T$ for each involution $t \in T$, then $A \cong A_6 \cong L_2(9)$ or $A \cong L_3(2) \cong L_2(7)$;

ii) if T is elementary abelian of order 4 and $C_A(t) \cong T$ for each involution $t \in T$, then $A \cong A_5 \cong L_2(4)$.

Proof: i) this part is an elementary exercise in Character Theory, see for example [3, Theorem 7.10]. ii) It can be proved using a similar method or by counting argument. ∎

Lemma 2.4 implies that $C_G(x)$ is 2-group, for each involution $1 \neq x \in G$ and hence G is a CIT-group. By this and coprime action theorem we obtain $O(G) = 1$. Since G is nonsolvable, by Frobenius' normal p-complement theorem we get that $N_G(V)$ is not 2-group for some 2-subgroup $1 \neq V$ of G. Set $K = N_G(V)$, then lemma 2.3 implies that V is an elementary abelian group of order 4 and $K \cong A_4$ or S_4.

Let $T \in Syl_2(K)$ and we may assume that $T \leq S$. Assume that $V^g \leq T$, for some $g \in G$. As K/V is a subgroup of S_3 and V^g is elementary abelian of order 4, we have $V^g \cap V \neq 1$. Note that V is a TI-subgroup of G and hence $V = V^g$. This shows that we may assume that $T = S$. Since K/V is a subgroup of S_3, one obtain that either $V = S$ or $S \cong D_8$.

Lemma 3.2. *i) If $V = S$, then $G \cong A_5$.*
ii) If $S \cong D_8$, then $G \cong A_6$ or $L_2(7)$.

Proof: Assume that $S = V$. Since $O(G) = 1$, we may assume that G is simple. Now i) follows from Lemma 3.1 (ii). Assume that $S \cong D_8$. Then ii) follows from Lemma 3.1(i) and hence the lemma holds. ∎

Now Theorem 1.2 follows from Lemma 3.2.

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