Conjugate convolution operators and inner amenability

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Abstract

Let G be a group and $L^\infty(G)$ be the C^*-algebra of bounded complex-valued functions on G. G is called inner amenable if there exists a positive norm 1 functional m on $L^\infty(G)$ such that $m(\rho(y)f) = m(f)$ for each $y \in G$, $f \in L^\infty(G)$ (where $\rho(y)f(x) = f(yxy^{-1})$); the functional m is called an inner invariant mean.

In this paper, among the other things, we prove a variety of characterizations of inner amenable groups. We also give sufficient conditions on an inner invariant mean to be a topologically inner invariant mean.

1 Introduction

There are a lot of results in abstract harmonic analysis on amenability of a locally compact group. A good deal of attention was paid to the study of inner amenable groups. The study of inner invariant means was initiated by Effros [5] and pursued by Akemann [1], Yuan [25] for discrete groups, Lau and Paterson [13] and Yuan [26] for locally compact groups, and Ling [15] and Mohammadzadeh and Nasr-Isfahani [18] for semigroups. Amenable locally compact groups and [IN]-groups are inner amenable. Furthermore when G is connected, then G is amenable if and only if G is inner amenable [16]. Amenability and inner amenability of Lau algebras is studied in [12] and [19]. For terminologies regarding invariant means on locally compact groups, the reader is

Received by the editors September 2010 - In revised form in March 2011.
Communicated by F. Bastin.
2000 *Mathematics Subject Classification :* Primary: 43A60; Secondary: 22D15.
Key words and phrases : Amenability, Banach algebras, conjugate convolution, inner amenability, locally compact group.

referred to [20]. Let π_∞ be the isometric representation of G on $L^\infty(G)$ given by $\pi_\infty(x)f(t) = f(x^{-1}tx)$. It is shown that $L^\infty(G)$ has an inner invariant mean if and only if the commutant $\pi_\infty(G)'$ of $\pi_\infty(G)$ contains a nonzero compact operator [14]. The literature on inner amenability has grown substantially in recent years, see [9], [11] and [17].

In this paper, we investigate inner invariant means on $L^\infty(G)$ and its closed subalgebra $U^\infty(G)$ of all $f \in L^\infty(G)$ for which the mapping $y \mapsto \rho(y)f$ is continuous [7]. We also study topologically inner invariant means on certain closed subspaces X of $U^\infty(G)$ and their relation with inner invariant means on X. We show that every topologically inner invariant mean on $L^\infty(G)$ is also inner invariant. The converse remains open. Sufficient conditions on an inner invariant mean to be a topologically inner invariant mean are given. We characterize inner amenable groups by introducing the so-called conjugate convolution operators which develop the techniques of the usual convolution operators. We give sufficient conditions and some necessary conditions for G to have an inner invariant mean.

2 Preliminaries and notations

Throughout this paper G will denote a locally compact group with left Haar measure dx, modular function Δ, and identity e. For $1 \leq p < \infty$, $L^p(G)$ is the space of complex-valued measurable functions φ on G such that $\int |\varphi(x)|^p dx < \infty$. Let $L^\infty(G)$ be the algebra of essentially bounded measurable complex-valued functions on G. For $y \in G$ and f a function on G we use the notation

$$_yf(x) = f(y^{-1}x), \;\; \rho(y)f(x) = f(yxy^{-1}) \quad (x \in G).$$

If $\varphi \in L^1(G)$, $\psi \in L^p(G)$ $(1 \leq p < \infty)$ and $f \in L^\infty(G)$, then $\varphi \circledast \psi$ as member of $L^p(G)$ is given

$$\varphi \circledast \psi(x) = \int \varphi(y)\psi(y^{-1}xy)\Delta(y)^{\frac{1}{p}} dy \quad (x \in G)$$

while $\varphi \odot f$ as member of $L^\infty(G)$ is given by

$$\varphi \odot f(x) = \int \varphi(y) f(yxy^{-1}) dy \quad (x \in G).$$

We have $\|\varphi \circledast \psi\|_p \leq \|\varphi\|_1 \|\psi\|_p$ and $\|\varphi \odot f\| \leq \|\varphi\|_1 \|f\|$. More information on this product can be found in [23] and [24]. More generally, for $1 \leq p \leq \infty$, let π_p be the isometric representation of G on $L^p(G)$ given by

$$\pi_p(y)\varphi(x) = \varphi(y^{-1}xy)\Delta(y)^{\frac{1}{p}} \qquad (x, y \in G, \; \varphi \in L^p(G)).$$

Thus for all $y \in G$, we have $\|\varphi\|_p = \|\pi_p(y)\varphi\|_p$. We denote by $P^p(G)$ the convex set of all nonnegative functions φ in $L^p(G)$ such that $\|\varphi\|_p = 1$. If A is measurable subset of G, then $|A|$ is the measure of A. For any subset A of G, 1_A denotes the characteristic function of A. If $0 < |A| < \infty$, we also consider the mapping $\xi_A(x) = \frac{1_A(x)}{|A|}$ defined on G.

Duality between Banach spaces is denoted by $\langle\ \rangle$; thus for $f \in L^{\infty}(G)$ and $\varphi \in L^1(G)$, we have $\langle f, \varphi\rangle = \int f(x)\varphi(x)dx$. As far as possible, we follow [7] in our notation and refer to [22] for basic functional analysis and to [10] for basic harmonic analysis.

3 Main results

We start by recalling the following definition.

Definition 3.1. Let X be a subspace of $L^{\infty}(G)$ with $1_G \in X$ that is closed under complex conjugation:

(i) We say that X is *invariant* (*topologically invariant*), if $\rho(y)f \in X$ ($\varphi \odot f \in X$) whenever $y \in G$, $f \in X$ and $\varphi \in P^1(G)$;

(ii) A *mean* on X is a norm one nonnegative functional m on X such that $m(1_G) = 1$;

(iii) Let X be an invariant subspace of $L^{\infty}(G)$. A mean m on X is called *inner invariant mean* if $\langle m, \rho(y)f\rangle = \langle m, f\rangle$ for all $f \in X$ and $y \in G$;

(iv) Let X be a topologically invariant subspace of $L^{\infty}(G)$. A mean m on X is called *topologically inner invariant mean* if

$$\langle m, \varphi \odot f\rangle = \langle m, f\rangle$$

for all $\varphi \in P^1(G)$ and $f \in X$;

(v) A locally compact group G is called *inner amenable* group if it admits an inner invariant mean on $L^{\infty}(G)$.

We denote by $U^{\infty}(G)$ the Banach space consisting of the complex-valued functions f in $L^{\infty}(G)$ that are uniformly continuous, that is, the mapping $y \mapsto \rho(y)f$ from G into $L^{\infty}(G)$ is continuous [7]. The present author has proved that $U^{\infty}(G)$ is a Banach algebra and $\varphi \odot f \in U^{\infty}(G)$ for every $\varphi \in L^1(G)$ and $f \in L^{\infty}(G)$ (see Lemma 2.3 in [7]). Clearly $U^{\infty}(G)$ is an invariant subspace of $L^{\infty}(G)$.

Lemma 3.2. Let G be a locally compact group. Then the following statements hold:

(i) Let X be a closed subspace of $U^{\infty}(G)$. Then X is invariant if and only if it is topologically invariant;

(ii) Let X be a closed subspace of $U^{\infty}(G)$ with $1_G \in X$ that is closed under complex conjugation and topologically invariant. A mean m on X is inner invariant if and only if it is topologically inner invariant.

Proof. (i): By the same argument as used at the proof of Lemma 2.5 in [7], we see that X is invariant if and only if it is topologically invariant.

(ii): Let m be an inner invariant mean on X, and let $f \in X$ and $\varphi \in P^1(G)$. Since the measures in $P^1(G)$ with compact supports are norm dense in $P^1(G)$, without loss of generality we may assume that φ has a compact support. By Theorem 3.27 in [22],

$$\langle m, \varphi \odot f\rangle = \int \langle m, \rho(y)f\rangle \varphi(y)dy = \int \langle m, f\rangle \varphi(y)dy = \langle m, f\rangle.$$

This shows that m is topologically inner invariant mean.

To prove the converse, let m be a topologically inner invariant mean on X and fix $\varphi \in P^1(G)$. For $f \in X$ and $y \in G$,

$$\langle m, \rho(y)f\rangle = \langle m, \varphi \odot \rho(y)f\rangle = \langle m, {}_y\varphi \odot f\rangle = \langle m, f\rangle.$$

Thus, m is an inner invariant mean on X. ∎

Let G be a locally compact group. For $\varphi, \psi \in L^1(G)$, $f \in L^\infty(G)$ and $m, n \in L^\infty(G)^*$, the elements $f.\varphi$ and $n.f$ of $L^\infty(G)$ and $m.n \in L^\infty(G)^*$ are defined by

$$\langle f.\varphi, \psi\rangle = \langle f, \varphi \circledast \psi\rangle, \ \ \langle n.f, \varphi\rangle = \langle n, f.\varphi\rangle, \ \ \langle m.n, f\rangle = \langle m, n.f\rangle,$$

respectively. Clearly $\|f.\varphi\| \leq \|f\|\|\varphi\|_1$, $\|n.f\| \leq \|n\|\|f\|$ and $\|m.n\| \leq \|m\|\|n\|$. Elementary calculations shows that $\varphi \odot f = f.\varphi$ for every $f \in L^\infty(G)$ and $\varphi \in L^1(G)$.

For each $\varphi \in L^1(G)$, define a seminorm ρ_φ on the linear space $L^\infty(G)$ by $\rho_\varphi(f) = \|f.\varphi\|$, $f \in L^\infty(G)$. Note that $\mathcal{P} = \{\rho_\varphi;\ \varphi \in L^1(G)\}$ separates the points of $L^\infty(G)$. The locally convex topology on $L^\infty(G)$ determined by these seminorms is denoted by τ_c. We first remark that the τ_c-topology may be characterized in another manner. Indeed, it is a standard device to embed $L^\infty(G)$ into $\mathcal{B}(L^1(G), L^\infty(G))$ by an operator T so that $T(f)(\varphi) = f.\varphi$, $f \in L^\infty(G)$, $\varphi \in L^1(G)$. Then T is one-to-one and linear. On the other hand, $\mathcal{B}(L^1(G), L^\infty(G))$ naturally carries the strong operator topology. So T allows us to consider the induced topology on $L^\infty(G)$ which is the same as the τ_c-topology. In [8] the author studied the τ_c-topology on the dual $M_a(S)^*$ of the semigroup algebra $M_a(S)$ of a locally compact foundation semigroup S. From these observations we immediately deduce the following Lemma.

Lemma 3.3. Let G be a locally compact group. For each $\varphi \in L^1(G)$, the mapping $f \mapsto \varphi \odot f$ from $(L^\infty(G), \tau_c)$ into $(L^\infty(G), \|.\|)$ is continuous.

We are now in a position to establish one of the main results of this section.

Theorem 3.4. Let G be a locally compact group, X a subspace of $L^\infty(G)$ with $1_G \in X$ that is closed under complex conjugation, invariant and topologically invariant. Then the following properties hold:

(i) Every topologically inner invariant mean m on X is τ_c-continuous;

(ii) An inner invariant mean on X is topologically inner invariant mean if and only if it is τ_c-continuous;

(iii) Let m be an inner invariant mean on X. Suppose there is some $\varphi_0 \in P^1(G)$ such that $\langle m, \varphi_0 \odot f\rangle = \langle m, f\rangle$ for all $f \in X$. Then m is topologically inner invariant mean.

Note that an analogue of statement (ii) for topological left invariant means has proved by Crombez, see Lemma 2.1 in [3]. Also, there is an argument similar to statement (iii) for topological left invariant means, see Proposition 22.2 in [21].

Proof. (i): Let m be a topologically inner invariant mean on X, and let $f_\alpha \to f$ in the τ_c-topology of X. By Lemma 3.3, for $\varphi \in P^1(G)$, $f_\alpha.\varphi \to f.\varphi$ in the norm topology. We conclude that

$$\begin{aligned}\lim_\alpha \langle m, f_\alpha\rangle &= \lim_\alpha \langle m, \varphi \odot f_\alpha\rangle = \lim_\alpha \langle m, f_\alpha.\varphi\rangle = \langle m, f.\varphi\rangle \\ &= \langle m, \varphi \odot f\rangle = \langle m, f\rangle.\end{aligned}$$

This shows that m is τ_c-continuous.

(ii): Let m be an inner invariant mean on X. If m is topologically inner invariant, then m is τ_c-continuous; see (i).

To prove the converse, let m be an inner invariant mean on X. Let $f \in X$, $\varphi \in P^1(G)$ and $\epsilon > 0$ be given. We further assume that φ has a compact support, say K. If $\|f\| = 0$, we have trivially $\langle m, \varphi \odot f\rangle = \langle m, f\rangle$. We now consider the case $\|f\| > 0$. The sets

$$V(\varphi \odot f, \varphi_1, ..., \varphi_n, \delta) = \{h \in X;\ \|h.\varphi_i - (\varphi \odot f).\varphi_i\| < \delta,\ i = 1, ..., n\}$$

where $\delta > 0$ and $\{\varphi_1, ..., \varphi_n\}$ is a finite subset of $L^1(G)$, form a basis of open neighborhoods of $\varphi \odot f$ in the τ_c-topology of X. Now, we choose a neighborhood $V(\varphi \odot f, \varphi_1, ..., \varphi_n, \delta)$ of $\varphi \odot f$ in X such that $|\langle m, h\rangle - \langle m, \varphi \odot f\rangle| < \epsilon$ whenever $h \in V(\varphi \odot f, \varphi_1, ..., \varphi_n, \delta)$. Since the mapping $y \mapsto {}_y\varphi_i$ is continuous [6], for every $y \in K$, there exists a relatively compact neighbourhood U_y of y in G such that $\|{}_y\varphi_i - {}_x\varphi_i\|_1 < \frac{\delta}{\|f\|}$ whenever $x \in U_y$ and $i \in \{1, ..., n\}$. Now cover K by $\{U_y;\ y \in K\}$. By compactness we may extract a finite subcover $U_{y_1}, ..., U_{y_l}$ of K. We can find l Borel subsets $A_1, ..., A_l$ of K such that

$$K = \bigcup_{j=1}^{l} A_j,\ \ A_j \cap A_r = \emptyset\ (j \neq r),\ \ \|{}_y\varphi_i - {}_{y_j}\varphi_i\|_1 < \frac{\delta}{\|f\|}$$

whenever $y \in A_j$ and $i \in \{1, ..., n\}$. If $j \in \{1, ..., l\}$, we also put $\alpha_j = \int_{A_j} \varphi(y)dy$. Then $\sum_{j=1}^{l} \alpha_j = 1$. For every $i \in \{1, ..., n\}$,

$$\begin{aligned}\Big|\sum_{j=1}^{l} \alpha_j \rho(y_j) f.\varphi_i - (\varphi \odot f).\varphi_i\Big| &= \Big|\sum_{j=1}^{l} \alpha_j {}_{y_j}\varphi_i \odot f - \varphi_i \odot (\varphi \odot f)\Big| \\ &\leq \sum_{j=1}^{l} \int_{A_j} \varphi(z) |{}_{y_j}\varphi_i \odot f - {}_z\varphi_i \odot f| dz \\ &\leq \sum_{j=1}^{l} \int_{A_j} \varphi(z) \|{}_{y_j}\varphi_i - {}_z\varphi_i\|_1 \|f\| dz < \delta.\end{aligned}$$

This shows that $\sum_{j=1}^{l} \alpha_j \rho(y_j) f \in V(\varphi \odot f, \varphi_1, ..., \varphi_n, \delta)$, and so

$$|\langle m, f\rangle - \langle m, \varphi \odot f\rangle| = \left|\left\langle m, \sum_{j=1}^{l} \alpha_j \rho(y_j) f \right\rangle - \langle m, \varphi \odot f\rangle\right| < \epsilon.$$

As $\epsilon > 0$ may be chosen arbitrarily, we have $\langle m, f\rangle = \langle m, \varphi \odot f\rangle$. Finally, if φ is any element in $P^1(G)$, let $\{\varphi_n\} \subseteq P^1(G)$ be a sequence of elements with compact support such that $\varphi_n \to \varphi$. Then from the above special case, we conclude that $\langle m, \varphi \odot f\rangle = \langle m, f\rangle$.

(iii): Let m be an inner invariant mean and $\langle m, \varphi_0 \odot f\rangle = \langle m, f\rangle$ for all $f \in X$. To show that m is topologically inner invariant mean, it is sufficient to prove that m is τ_c-continuous. But suppose $f_\alpha \to f$ in the τ_c-topology. Since $\varphi_0 \odot f_\alpha = f_\alpha.\varphi_0 \to f.\varphi_0 = \varphi_0 \odot f$ in the norm topology, we see that

$$\lim_\alpha \langle m, f_\alpha\rangle = \lim_\alpha \langle m, \varphi_0 \odot f_\alpha\rangle = \langle m, \varphi_0 \odot f\rangle = \langle m, f\rangle.$$

Hence m is topologically inner invariant mean. ∎

Let G be a compact nondiscrete abelian group. By Proposition 22.3 in [21], there exists a left invariant mean m on $L^\infty(G)$ such that $\langle m, \varphi * f\rangle \neq \langle m, f\rangle$ for some $f \in L^\infty(G)$ and $\varphi \in P^1(G)$. This shows that m can not be a topologically left invariant mean. It is easy to see that every topologically inner invariant mean on $L^\infty(G)$ is inner invariant mean on $L^\infty(G)$. We do not know whether or not the converse holds. The next theorem of this section exhibits a number of assertions which are equivalent to inner amenability of a locally compact group G.

Theorem 3.5. A locally compact group G is inner amenable if and only if there exists a net $\{\varphi_\alpha\}$ in $P^1(G)$ satisfying any one of the following conditions:

(i) For every $\varphi, \psi \in P^1(G)$, $\lim_\alpha \|\psi \circledast (\varphi \circledast \varphi_\alpha) - \psi \circledast \varphi_\alpha\|_1 = 0$;

(ii) For every $\varphi \in P^1(G)$ and $f \in U^\infty(G)$, $\lim_\alpha \langle f, \varphi \circledast \varphi_\alpha - \varphi_\alpha\rangle = 0$;

(iii) For every compact subset K of G and every $f \in U^\infty(G)$,

$$\lim \sup\{|\langle f, \pi_1(y)\varphi_\alpha - \varphi_\alpha\rangle|;\ y \in K\} = 0.$$

Proof. Let G be inner amenable. By Theorem 2 in [24], there exists a net $\{\varphi_\alpha\}$ in $P^1(G)$ such that $\lim_\alpha \|\varphi \circledast \varphi_\alpha - \varphi_\alpha\|_1 = 0$ for every $\varphi \in P^1(G)$. For every $\varphi, \psi \in P^1(G)$,

$$\lim_\alpha \|\psi \circledast (\varphi \circledast \varphi_\alpha) - \psi \circledast \varphi_\alpha\|_1 \leq \lim_\alpha \|\varphi \circledast \varphi_\alpha - \varphi_\alpha\|_1 = 0.$$

(i) implies (ii): Let $f \in U^\infty(G)$ and $\varphi \in P^1(G)$. By Cohen's factorization theorem, $U^\infty(G) = L^1(G) \odot L^\infty(G)$ [23]. Therefore f is of the form $f = \psi_0 \odot f_0$ for some $\psi_0 \in L^1(G)$ and $f_0 \in L^\infty(G)$. By considering Jordan decomposition, it is clear that statement (i) holds for any $\psi \in L^1(G)$. Hence

$$\lim_\alpha \langle f, \varphi \circledast \varphi_\alpha - \varphi_\alpha\rangle = \lim_\alpha \langle f_0, \psi_0 \circledast (\varphi \circledast \varphi_\alpha) - \psi_0 \circledast \varphi_\alpha\rangle = 0.$$

(ii) implies G is inner amenable: It suffices to show that $U^\infty(G)$ has a topologically inner invariant mean. By Proposition 3.3 in [21], the net $\{\varphi_\alpha\}$ admits a subnet $\{\phi_\beta\}$ converging to a mean m in the weak* topology of $L^\infty(G)$. For all $f \in U^\infty(G)$ and $\varphi \in P^1(G)$,

$$\langle m, \varphi \odot f - f \rangle = \lim_\beta \langle f, \varphi \circledast \varphi_\beta - \varphi_\beta \rangle = 0.$$

(iii) implies G is inner amenable: This is similar to the last implication. Let $\{\varphi_\alpha\}$ be as in statement (iii) and define m as above. Then for $f \in U^\infty(G)$ and $x \in G$,

$$\langle m, \rho(x) f - f \rangle = \lim_\beta \langle f, \pi_1(x)\varphi_\beta - \varphi_\beta \rangle = 0.$$

Inner amenable implies (iii): This is an immediate consequence of Theorem 1 of [24]. ■

Theorem 3.6. Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. A locally compact group G is inner amenable if and only if

$$\inf\{\sup\{\inf\{\langle \pi_p(y)\varphi + \varphi, \psi \rangle; y \in K\}; \varphi \in P^p(G), \psi \in P^q(G)\}; K \in \mathcal{K}\} = 2,$$

where $\mathcal{K}$ is the family of compact subsets of G.

Proof. Suppose that G is inner amenable. Let K be a compact subset of G and $\epsilon > 0$. By Theorem 1 in [26], there exists $\phi \in P^1(G)$ such that, for every $y \in K$, $\|\pi_1(y)\phi - \phi\|_1 < \epsilon^p$. For $a \geq 0$, the map $x \mapsto x^p - a^p - (x-a)^p$ is increasing from $\mathbb{R}^+$ into $\mathbb{R}$. So that $(b-a)^p \leq b^p - a^p$ for all $b \geq a$. Let $\varphi = \phi^{\frac{1}{p}}$. For every $y \in K$, we obtain

$$\begin{aligned} \|\pi_p(y)\varphi - \varphi\|_p^p &= \int \left| \phi^{\frac{1}{p}}(y^{-1}xy)\Delta(y)^{\frac{1}{p}} - \phi^{\frac{1}{p}}(x) \right|^p dx \\ &\leq \int |\phi(y^{-1}xy)\Delta(y) - \phi(x)| dx \\ &\leq \|\pi_1(y)\phi - \phi\|_1 < \epsilon^p. \end{aligned}$$

Now let $\psi = \varphi^{\frac{p}{q}}$. For every $y \in K$,

$$\langle \pi_p(y)\varphi + \varphi, \psi \rangle = \langle \pi_p(y)\varphi - \varphi, \psi \rangle + 2\langle \varphi, \psi \rangle > 2 - \epsilon.$$

As $\epsilon > 0$ and $K \in \mathcal{K}$ are arbitrary, we have

$$\inf\{\sup\{\inf\{\langle \pi_p(y)\varphi + \varphi, \psi \rangle; y \in K\}; \varphi \in P^p(G), \psi \in P^q(G)\}; K \in \mathcal{K}\} = 2.$$

Conversely if the condition holds, let K be a compact subset of G and $\epsilon > 0$. Then there exist $\varphi \in P^p(G)$ and $\psi \in P^q(G)$ such that $\langle \pi_p(y)\varphi + \varphi, \psi \rangle > 2 - \epsilon$ for every $y \in K$. It follows that $\|\pi_p(y)\varphi + \varphi\|_p > 2 - \epsilon$ for every $y \in K$. For every $y \in K$, by the Clarkson's inequalities, we obtain

$$\|\pi_p(y)\varphi + \varphi\|_p^p + \|\pi_p(y)\varphi - \varphi\|_p^p \leq 2^{p-1}(\|\pi_p(y)\varphi\|_p^p + \|\varphi\|_p^p) = 2^p$$

in case $p \geq 2$, and so $\|\pi_p(y)\varphi - \varphi\|_p^p < 2^p - (2-\epsilon)^p$. We have

$$\|\pi_p(y)\varphi + \varphi\|_p^q + \|\pi_p(y)\varphi - \varphi\|_p^q \leq 2^{q+1-p}(\|\pi_p(y)\varphi\|_p^p + \|\varphi\|_p^p)^{p-1} = 2^q$$

in case $1 < p < 2$, and so $\|\pi_p(y)\varphi - \varphi\|_p^q < 2^q - (2-\epsilon)^q$. Since this holds for all $y \in K$, we conclude that G is inner amenable [24]. ■

Corollary 3.7. Let $1 < p, q < \infty$ and $\frac{1}{p} + \frac{1}{q} = 1$. The following conditions are equivalent:

(i) G is inner amenable;

(ii) $\inf\{\sup\{\langle \phi \circledast \varphi + \varphi, \psi\rangle;\ \varphi \in P^p(G),\ \psi \in P^q(G)\},\ \phi \in P^1(G)\} = 2.$

Proof. (i) implies (ii): Let $\phi \in P^1(G)$ and $\epsilon \in (0,1)$. Choose $\phi_1 \in C_c(G)^+$ with compact support K such that $\|\phi - \phi_1\|_1 < \epsilon$, hence $\|\phi_1\|_1 > 1 - \epsilon$ [10]. By Theorem 3.6, we may determine $\varphi \in P^p(G)$ and $\psi \in P^q(G)$ such that $\langle \pi_p(y)\varphi + \varphi, \psi\rangle > 2 - \epsilon$ for all $y \in K$. By integration, we obtain $\langle \phi_1 \circledast \varphi + \varphi, \psi\rangle \geq (2-\epsilon)\|\phi_1\|_1 > (2-\epsilon)(1-\epsilon)$. We have

$$\langle \phi \circledast \varphi + \varphi, \psi\rangle + \epsilon \geq \langle \phi_1 \circledast \varphi + \varphi, \psi\rangle \geq (2-\epsilon)(1-\epsilon).$$

This shows that

$$\inf\{\sup\{\langle \phi \circledast \varphi + \varphi, \psi\rangle;\ \varphi \in P^p(G),\ \psi \in P^q(G)\},\ \phi \in P^1(G)\} = 2.$$

(ii) implies (i): Let $\phi \in P^1(G)$. By assumption, given $\epsilon \in (0,1)$, there exist $\varphi \in P^p(G)$ and $\psi \in P^q(G)$ such that $\langle \phi \circledast \varphi + \varphi, \psi\rangle > 2 - \epsilon$. It follows that $\langle \phi \circledast \varphi, \psi\rangle > 1 - \epsilon$. We consider $L_\phi : L^p(G) \to L^p(G)$ by $L_\phi(\varphi) = \phi \circledast \varphi$. Clearly $\|L_\phi\| > 1 - \epsilon$, and so $\|L_\phi\| = 1$. Since this holds for all $\phi \in P^1(G)$, by a form of the Riesz-Thorin Convexity Theorem ([4], VI.10.11), $L_\phi : L^2(G) \to L^2(G)$ has norm 1. Define $\omega_1 : \{L_\phi;\ \phi \in L^1(G)\} \to \mathbb{C}$ by $\omega_1(L_\phi) = \int \phi(x)dx$. By the Hahn Banach theorem for states (see Proposition 2.3.24 in [2]), we can extend ω_1 to a state ω on the algebra $\mathcal{B}(L^2(G))$ of bounded operators on $L^2(G)$. Therefore G is inner amenable by Theorem 2 in [26]. ■

Lau and Paterson [13] gave a necessary condition on a locally compact group G to have an inner invariant mean m such that $\langle m, 1_V\rangle = 0$ for some compact neighborhood V of G invariant under the inner automorphisms. Let A be a Borel subset of G. In the following theorem, we provide a necessary and sufficient condition for G to have an inner invariant mean m with $\langle m, 1_A\rangle = 1$.

Theorem 3.8. Let G be an inner amenable group and let A be a Borel subset of G. Then the following statements are equivalent:

(i) There is a topologically inner invariant mean on $L^\infty(G)$ such that $\langle m, 1_A\rangle = 1$;

(ii) $\inf\{\sup\{\inf\{\langle \pi_1(y)\varphi, 1_A\rangle;\ y \in K\};\ \varphi \in P^1(G)\};\ K \in \mathcal{K}\} = 1.$

Proof. (i) implies (ii): Assume that there is a topologically inner invariant mean m on $L^\infty(G)$ such that $\langle m, 1_A\rangle = 1$. As $P^1(G)$ is weak* dense in the convex set of all means on $L^\infty(G)$ (see Proposition 3.3 in [21]), there exists a net $\{\varphi_\alpha\}$ in $P^1(G)$ such that, for every $\varphi \in P^1(G)$, $\{\varphi \circledast \varphi_\alpha - \varphi_\alpha\}$ converges to 0 in the weak topology of $L^1(G)$. Let $\varphi_0 \in P^1(G)$ be fixed and put $\psi_\alpha = \varphi_0 \circledast \varphi_\alpha$. It is easy to see that $\{\psi_\alpha\}$ converging to $\varphi_0.m$ in the weak* topology of $L^\infty(G)$, and also $\langle \varphi_0.m, 1_A\rangle = 1$. Let $\epsilon > 0$ and $K \subseteq G$ compact be given. As $\varphi_0 \in L^1(G)$, the mapping $y \mapsto {}_y\varphi_0$ is continuous [10], so there exists an open neighbourhood V of e in G such that, for all $y \in V$, $\|{}_y\varphi_0 - \varphi_0\|_1 < \frac{\epsilon}{2}$ [6]. We may determine a subset $\{y_1, ..., y_n\}$ in K such that $K \subseteq \bigcup_{i=1}^n y_i V$ and $\|{}_y\varphi_0 - {}_{y_i}\varphi_0\|_1 < \frac{\epsilon}{2}$ whenever $y \in y_i V \cap K$ and $i \in \{1, ..., n\}$. There exists $\alpha_0 \in I$ such that, for every $\alpha \in I$ with $\alpha \succeq \alpha_0$ and every $i \in \{1, ..., n\}$

$$\begin{aligned} |\langle \pi_1(y_i)\psi_\alpha - \psi_\alpha, 1_A\rangle| &\leq |\langle \pi_1(y_i)\psi_\alpha - \varphi_\alpha, 1_A\rangle| + |\langle \psi_\alpha - \varphi_\alpha, 1_A\rangle| \\ &\leq |\langle {}_{y_i}\varphi_0 \circledast \varphi_\alpha - \varphi_\alpha, 1_A\rangle| + |\langle \varphi_0 \circledast \varphi_\alpha - \varphi_\alpha, 1_A\rangle| \\ &< \frac{\epsilon}{2}. \end{aligned}$$

For any $y \in K$, there exist $i \in \{1, ..., n\}$ and $v \in V$ such that $y = y_i v$. Then we have

$$\begin{aligned} |\langle \pi_1(y)\psi_\alpha - \psi_\alpha, 1_A\rangle| &= |\langle \pi_1(y)\psi_\alpha - \pi_1(y_i)\psi_\alpha + \pi_1(y_i)\psi_\alpha - \psi_\alpha, 1_A\rangle| \\ &\leq |\langle {}_y\varphi_0 \circledast \varphi_\alpha - {}_{y_i}\varphi_0 \circledast \varphi_\alpha, 1_A\rangle + \frac{\epsilon}{2} < \epsilon. \end{aligned}$$

for every $\alpha \succeq \alpha_0$. This shows that $\lim_\alpha \langle \pi_1(y)\psi_\alpha - \psi_\alpha, 1_A\rangle = 0$ uniformly on compacta.

Now let K be a compact subset of G and $\epsilon > 0$. Then there is some $\alpha_0 \in I$ such that

$$|1 - \langle \psi_{\alpha_0}, 1_A\rangle| = |\langle \varphi_0.m, 1_A\rangle - \langle \psi_{\alpha_0}, 1_A\rangle| < \frac{\epsilon}{2}$$

and $|\langle \pi_1(y)\psi_{\alpha_0} - \psi_{\alpha_0}, 1_A\rangle| < \frac{\epsilon}{2}$ for all $y \in K$. Clearly $\langle \pi_1(y)\psi_{\alpha_0}, 1_A\rangle > 1 - \epsilon$ for all $y \in K$. We conclude that

$$\inf\{\sup\{\inf\{\langle \pi_1(y)\varphi, 1_A\rangle;\ y \in K\};\ \varphi \in P^1(G)\};\ K \in \mathcal{K}\} = 1.$$

(ii) implies (i): We consider the directed set $I = \mathcal{K} \times (0, 1)$ where, for $\alpha = (K, \epsilon) \in I$, $\alpha' = (K', \epsilon') \in I$, $\alpha' \succeq \alpha$ in case $K \subseteq K'$ and $\epsilon' \leq \epsilon$. By assumption, given $\alpha = (K, \epsilon)$, there exist $\varphi_\alpha \in P^1(G)$ such that $\langle \pi_1(y)\varphi_\alpha, 1_A\rangle > 1 - \epsilon$ for all $y \in K$. Let $\varphi \in P^1(G)$ be such that φ is supported on K. We have

$$\langle \varphi \circledast \varphi_\alpha, 1_A\rangle = \int \langle \pi_1(y)\varphi_\alpha, 1_A\rangle \varphi(y)dy \geq 1 - \epsilon.$$

Since the measures in $P^1(G)$ with compact supports are norm dense in $P^1(G)$ [10], it follows that $\lim_\alpha \langle \varphi \circledast \varphi_\alpha, 1_A\rangle = 1$ for all $\varphi \in P^1(G)$. By Proposition 3.3 in [21], the net $\{\varphi_\alpha\}$ admits a subnet $\{\phi_\beta\}$ converging to a mean n in the weak* topology of $L^\infty(G)$. It follows that $\langle n, 1_A.\varphi\rangle = 1$ for all $\varphi \in P^1(G)$. Since G is inner amenable, let m_1 be a topologically inner invariant mean on $L^\infty(G)$. Indeed, if m_0 is an inner invariant mean on $L^\infty(G)$, then $m_0|_{U^\infty(G)}$ is an inner invariant mean on $U^\infty(G)$. By Lemma 3.2, $m_0|_{U^\infty(G)}$ is a topologically inner invariant mean. On the other hand, any topologically inner invariant mean on $U^\infty(G)$ may be extended to a topologically inner invariant mean on $L^\infty(G)$. Thus we can find a topologically inner invariant mean m_1 on $L^\infty(G)$. Clearly $m = m_1.n$ is a mean on $L^\infty(G)$. Let $\{\psi_\gamma\}$ be a net in $P^1(G)$ converging to m_1 in the weak* topology of $L^\infty(G)$. We have

$$\begin{aligned} |\langle m, 1_A\rangle| &= |\langle m_1.n, 1_A\rangle| = |\langle m_1, n.1_A\rangle| = \lim_\gamma |\langle \psi_\gamma, n.1_A\rangle| \\ &= \lim_\gamma |\langle n, 1_A.\psi_\gamma\rangle| = 1. \end{aligned}$$

It is straightforward to verify that m is a topologically inner invariant mean (since m_1 is) on $L^\infty(G)$. This completes our proof. ∎

Acknowledgements I would like to thank the referee for his/her careful reading of my paper and many valuable suggestions.

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