New existence results on periodic solutions of nonautonomous second order differential systems with (q, p)–Laplacian*

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Abstract

Some new existence theorems are obtained for periodic solutions of nonautonomous second-order differential systems with (q, p)–Laplacian.

1 Introduction and main results

In the last years many authors starting with Mawhin and Willem (see [1]) proved the existence of solutions for problem

$$\begin{aligned} &\ddot{u}(t) = \nabla F(t, u(t)) \text{ a.e. } t \in [0, T], \\ &u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{aligned} \tag{1}$$

under suitable conditions on the potential F (see [2]-[16]). Also in a series of papers (see [17]-[19]) we have generalized some of these results for the case when the potential F is just locally Lipschitz in the second variable x not continuously differentiable and after (see [20]-[22]) we have considered the second order inclusions systems with p–Laplacian. Very recent we have proved the existence of periodic solutions for systems with (q, p)–Laplacian (see [23]-[25]).

In [16] the authors proved the following critical point theorem (see Theorem 1.1 in [16]):

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Theorem 1. *Suppose that V_1 and V_2 are reflexive Banach spaces, $\psi \in C^1(V_1 \times V_2, \mathbb{R})$, $\psi(v_1, \cdot)$ is weakly upper semi-continuous for all $v_1 \in V_1$ and $\psi(\cdot, v_2) : V_1 \to \mathbb{R}$ is convex for all $v_2 \in V_2$, and ψ' is weakly continuous. Assume that*

$$\psi(0, v_2) \to -\infty \tag{2}$$

as $\|v_2\| \to \infty$ and, for every $M > 0$,

$$\psi(v_1, v_2) \to +\infty \tag{3}$$

as $\|v_1\| \to \infty$ uniformly for $\|v_2\| \leq M$. Then ψ has at least one critical point.

Using this theorem, in [16], the authors proved some new existence results of periodic solutions for problem (1). The aim of this paper is to show how some of these results can be generalized. More exactly our results represent the extensions to second-order differential systems with (q, p)–Laplacian.

Consider the second order system

$$\begin{cases} -\frac{d}{dt}\left(|\dot{u}_1(t)|^{q-2}\dot{u}_1(t)\right) = \nabla_{u_1}F(t, u_1(t), u_2(t)), \\ -\frac{d}{dt}\left(|\dot{u}_2(t)|^{p-2}\dot{u}_2(t)\right) = \nabla_{u_2}F(t, u_1(t), u_2(t)) \text{ a.e. } t \in [0, T], \\ u_1(0) - u_1(T) = \dot{u}_1(0) - \dot{u}_1(T) = 0, \\ u_2(0) - u_2(T) = \dot{u}_2(0) - \dot{u}_2(T) = 0, \end{cases} \tag{4}$$

where $1 < p, q < \infty$, $T > 0$, and $F : [0, T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ satisfy the following assumption (A):

- F is measurable in t for each $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$;
- F is continuously differentiable in (x_1, x_2) for a.e. $t \in [0, T]$;
- there exist $a_1, a_2 \in C(\mathbb{R}_+, \mathbb{R}_+)$ and $b \in L^1(0, T; \mathbb{R}_+)$ such that

  $$|F(t, x_1, x_2)|, \quad |\nabla_{x_1}F(t, x_1, x_2)|, \quad |\nabla_{x_2}F(t, x_1, x_2)| \leq \left[a_1(|x_1|) + a_2(|x_2|)\right]b(t)$$

  for all $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $t \in [0, T]$.

The corresponding functional associated to system (4) is $\varphi : W \to \mathbb{R}$ given by

$$\varphi(u_1, u_2) = -\frac{1}{q}\int_0^T |\dot{u}_1(t)|^q dt - \frac{1}{p}\int_0^T |\dot{u}_2(t)|^p dt + \int_0^T F(t, u_1(t), u_2(t))dt$$

where $W = W_T^{1,q} \times W_T^{1,p}$.

Theorem 2. *Suppose that assumption (A) holds and $F(t, x_1, x_2)$ is convex in (x_1, x_2) for a.e. $t \in [0, T]$. Assume that the following conditions are satisfied:*

(A_1) There exist $\alpha_1, \alpha_2 \in L^1(0, T; \mathbb{R}_+)$ with $\int_0^T \alpha_1(t)dt < T^{-\frac{q}{q'}}$, $\int_0^T \alpha_2(t)dt < T^{-\frac{p}{p'}}$ where $\frac{1}{q} + \frac{1}{q'} = 1$, $\frac{1}{p} + \frac{1}{p'} = 1$, and $\gamma \in L^1(0, T; \mathbb{R}_+)$ such that

$$F(t, x_1, x_2) \leq \frac{1}{q}\alpha_1(t)|x_1|^q + \frac{1}{p}\alpha_2(t)|x_2|^p + \gamma(t)$$

for all $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $t \in [0, T]$.

(A_2)

$$\int_0^T F(t, x_1, x_2)dt \to +\infty \quad as \ |(x_1, x_2)| = \sqrt{|x_1|^2 + |x_2|^2} \to \infty,$$

$(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$.

Then problem (4) has at least one solution in W.

Theorem 3. *Suppose that assumptions (A) and (A_1) holds and there exist $\mu_1, \mu_2 \in L^1(0, T; \mathbb{R})$ with $\int_0^T \mu_i(t)dt > 0, i = 1, 2$ such that $F(t, x_1, x_2) - \frac{1}{q}\mu_1(t)|x_1|^q - \frac{1}{p}\mu_2(t)|x_2|^p$ is convex in (x_1, x_2) for a.e. $t \in [0, T]$. Then problem (4) has at least one solution in W.*

Remark 1. *Theorems 2 and 3 generalizes Theorems 3.3 and 1.3 of Tang and Wu [16]. In fact, it follows from our results by letting $p = q = 2$ and $F(t, x_1, x_2) = F_1(t, x_1)$.*

Remark 2. *Unfortunately a similar result with Theorem 1.4 from [16], when we suppose that there exist $k_1, k_2 \in L^1(0, T; \mathbb{R}_+)$ satisfying some conditions such that $-F(t, x_1, x_2) + \frac{1}{q}k_1(t)|x_1|^q + \frac{1}{p}k_2(t)|x_2|^p$ is convex in (x_1, x_2) for a.e. $t \in [0, T]$, cannot be obtain using the same technique.*

Remark 3. *There are functions F satisfying our Theorem 2 and not satisfying the results from [23] - [25]. For example, let*

$$F(t, x_1, x_2) = \frac{1}{q}\beta_1(t)|x_1|^q + \frac{1}{p}\beta_2(t)|x_2|^p + \beta_3(t)(|x_1|^2 + |x_2|^2)^{\frac{r}{2}} + (l_1(t), x_1) + (l_2(t), x_2),$$

where $1 < r < \min\{q, p\}$, $\beta_1, \beta_2, \beta_3 \in L^1(0, T; \mathbb{R}_+)$ with $0 < \int_0^T \beta_1(t)dt < T^{-\frac{q}{q'}}$, $0 < \int_0^T \beta_2(t)dt < T^{-\frac{p}{p'}}$, $\frac{1}{q} + \frac{1}{q'} = 1$, $\frac{1}{p} + \frac{1}{p'} = 1$ and $l_1 \in L^{q'}(0, T; \mathbb{R}^N)$, $l_2 \in L^{p'}(0, T; \mathbb{R}^N)$, respectively. Then the function F satisfies our Theorem 2. But the function F does not satisfy Theorems 1 and 3 in [25] and Theorem 2 in [23] because that F is neither sublinear nor subquadratic. Moreover the function F does not satisfy the results in [24] because the corresponding energy functional is unbounded either below or above.

2 Preliminaries

We introduce some functional spaces. Let $T > 0$ be a positive number, $1 < q, p < \infty$ and $1 < q', p' < \infty$ such that $\frac{1}{q} + \frac{1}{q'} = 1$, $\frac{1}{p} + \frac{1}{p'} = 1$. We use $|\cdot|$ to denote the Euclidean norm in $\mathbb{R}^N$. We denote by $W_T^{1,p}$ the Sobolev space of functions $u \in L^p(0, T; \mathbb{R}^N)$ having a weak derivative $\dot{u} \in L^p(0, T; \mathbb{R}^N)$. The norm in $W_T^{1,p}$ is defined by

$$\|u\|_{W_T^{1,p}} = \Big(\int_0^T \big(|u(t)|^p + |\dot{u}(t)|^p\big)dt\Big)^{\frac{1}{p}}.$$

Moreover, we use the space W defined by

$$W = W_T^{1,q} \times W_T^{1,p}$$

with the norm $\|(u_1,u_2)\|_W = \|u_1\|_{W_T^{1,q}} + \|u_2\|_{W_T^{1,p}}$. It is clear that W is a reflexive Banach space.
We recall that

$$\|u\|_p = \Big(\int_0^T |u(t)|^p dt \Big)^{\frac{1}{p}} \text{ and } \|u\|_\infty = \max_{t\in[0,T]} |u(t)|.$$

For our aims it is necessary to recall some very well know results (for proof and details see [1]).

Proposition 4. *Each $u_1 \in W_T^{1,q}$ and each $u_2 \in W_T^{1,p}$ can be written as $u_i(t) = \bar{u}_i + \tilde{u}_i(t)$, $i = 1,2$ with*

$$\bar{u}_i = \frac{1}{T}\int_0^T u_i(t)dt, \qquad \int_0^T \tilde{u}_i(t)dt = 0.$$

We have the Sobolev's inequality

$$\|\tilde{u}_1\|_\infty \le T^{\frac{1}{q'}} \|\dot{\tilde{u}}_1\|_q,\ \|\tilde{u}_2\|_\infty \le T^{\frac{1}{p'}} \|\dot{\tilde{u}}_2\|_p \qquad \text{for each } u_1 \in W_T^{1,q},\ u_2 \in W_T^{1,p}.$$

In [15] the authors have proved the following result (see Lemma 3.1) which generalize a very well known result proved by Jean Mawhin and Michel Willem (see Theorem 1.4 in [1]):

Lemma 5. *Let $L : [0,T]\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N \to \mathbb{R}$, $(t,x_1,x_2,y_1,y_2) \to L(t,x_1,x_2,y_1,y_2)$ be measurable in t for each (x_1,x_2,y_1,y_2), and continuously differentiable in (x_1,x_2,y_1,y_2) for a.e. $t \in [0,T]$. If there exist $a_i \in C(\mathbb{R}_+,\mathbb{R}_+)$, $b \in L^1(0,T;\mathbb{R}_+)$, and $c_1 \in L^{q'}(0,T;\mathbb{R}_+)$, $c_2 \in L^{p'}(0,T;\mathbb{R}_+)$, $1 < p,q < \infty$, $\frac{1}{q}+\frac{1}{q'} = 1$, $\frac{1}{p}+\frac{1}{p'} = 1$ such that for a.e. $t\in[0,T]$ and every $(x_1,x_2,y_1,y_2) \in \mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N\times\mathbb{R}^N$, one has*

$$\begin{aligned}
|L(t,x_1,x_2,y_1,y_2)| &\le \big[a_1(|x_1|)+a_2(|x_2|)\big]\big[b(t)+|y_1|^q+|y_2|^p\big],\\
|D_{x_1}L(t,x_1,x_2,y_1,y_2)| &\le \big[a_1(|x_1|)+a_2(|x_2|)\big]\big[b(t)+|y_2|^p\big],\\
|D_{x_2}L(t,x_1,x_2,y_1,y_2)| &\le \big[a_1(|x_1|)+a_2(|x_2|)\big]\big[b(t)+|y_1|^q\big],\\
|D_{y_1}L(t,x_1,x_2,y_1,y_2)| &\le \big[a_1(|x_1|)+a_2(|x_2|)\big]\big[c_1(t)+|y_1|^{q-1}\big],\\
|D_{y_2}L(t,x_1,x_2,y_1,y_2)| &\le \big[a_1(|x_1|)+a_2(|x_2|)\big]\big[c_2(t)+|y_2|^{p-1}\big],
\end{aligned}$$

then the function $\varphi : W_T^{1,q}\times W_T^{1,p} \to \mathbb{R}$ defined by

$$\varphi(u_1,u_2) = \int_0^T L(t,u_1(t),u_2(t),\dot{u}_1(t),\dot{u}_2(t))dt$$

is continuously differentiable on $W_T^{1,q}\times W_T^{1,p}$ and

$$\begin{aligned}
\langle\varphi'(u_1,u_2),(v_1,v_2)\rangle = \int_0^T &\big[(D_{x_1}L(t,u_1(t),u_2(t),\dot{u}_1(t),\dot{u}_2(t)),v_1(t))\\
&+(D_{y_1}L(t,u_1(t),u_2(t),\dot{u}_1(t),\dot{u}_2(t)),\dot{v}_1(t))\\
&+(D_{x_2}L(t,u_1(t),u_2(t),\dot{u}_1(t),\dot{u}_2(t)),v_2(t))\\
&+(D_{y_2}L(t,u_1(t),u_2(t),\dot{u}_1(t),\dot{u}_2(t)),\dot{v}_2(t))\big]dt.
\end{aligned}$$

Corollary 6. *Let $L : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be defined by*

$$L(t, x_1, x_2, y_1, y_2) = -\frac{1}{q}|y_1|^q - \frac{1}{p}|y_2|^p + F(t, x_1, x_2)$$

where $F : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ satisfy condition (A). If $(u_1, u_2) \in W_T^{1,q} \times W_T^{1,p}$ is a solution of the corresponding Euler equation $\varphi'(u_1, u_2) = 0$, then (u_1, u_2) is a solution of (4).

3 The proofs of the theorems

We can apply Theorem 1 with the following cast of characters:

- Let $V_1 = \mathbb{R}^N \times \mathbb{R}^N$, $V_2 = \tilde{W} = \tilde{W}_T^{1,q} \times \tilde{W}_T^{1,p}$, where $\tilde{W}_T^{1,q} = \{x \in W_T^{1,q} \mid \int_0^T x(t)dt = 0\}$ and $\tilde{W}_T^{1,p} = \{x \in W_T^{1,p} \mid \int_0^T x(t)dt = 0\}$; V_1 and V_2 are reflexive Banach spaces;
- Let $\psi : V_1 \times V_2 \to \mathbb{R}$ be given by $\psi(v_1, v_2) = \varphi(v_1 + v_2) = \varphi((\bar{u}_1, \bar{u}_2) + (\tilde{u}_1, \tilde{u}_2))$ where $v_1 = (\bar{u}_1, \bar{u}_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and $v_2 = (\tilde{u}_1, \tilde{u}_2) \in \tilde{W}_T^{1,q} \times \tilde{W}_T^{1,p}$;
- By assumption (A) it is obviously that $\psi \in C^1(V_1 \times V_2)$, $\psi(v_1, \cdot)$ is weakly upper semi-continuous for all $v_1 \in V_1$ and ψ' is weakly continuous.

To get our results remains to show that $\varphi((\bar{u}_1, \bar{u}_2) + (\tilde{u}_1, \tilde{u}_2))$ is convex in $(\bar{u}_1, \bar{u}_2)$ and to prove the corresponding conditions (2) and (3) for our situation:

$$\varphi((0,0) + (\tilde{u}_1, \tilde{u}_2)) \to -\infty \tag{5}$$

as $\|(\tilde{u}_1, \tilde{u}_2)\| \to \infty$ and, for every $M > 0$,

$$\varphi((\bar{u}_1, \bar{u}_2) + (\tilde{u}_1, \tilde{u}_2)) \to +\infty \tag{6}$$

as $\|(\bar{u}_1, \bar{u}_2)\| \to \infty$ uniformly for $\|(\tilde{u}_1, \tilde{u}_2)\| \le M$.

Proof of the Theorem 2. Since $F(t, x_1, x_2)$ is convex in (x_1, x_2) for a.e. $t \in [0,T]$ it is obvious that $F(t, (\bar{u}_1, \bar{u}_2) + (\tilde{u}_1(t), \tilde{u}_2(t)))$ is convex in $(\bar{u}_1, \bar{u}_2) \in \mathbb{R}^N \times \mathbb{R}^N$, so is $\int_0^T F(t, (\bar{u}_1, \bar{u}_2) + (\tilde{u}_1(t), \tilde{u}_2(t)))dt$. Hence for every $(\tilde{u}_1(t), \tilde{u}_2(t))) \in \tilde{W} = \tilde{W}_T^{1,q} \times \tilde{W}_T^{1,p}$,

$$\varphi((\bar{u}_1, \bar{u}_2) + (\tilde{u}_1, \tilde{u}_2)) = -\frac{1}{q}\|\dot{\tilde{u}}_1\|_q^q - \frac{1}{p}\|\dot{\tilde{u}}_2\|_p^p + \int_0^T F(t, (\bar{u}_1, \bar{u}_2) + (\tilde{u}_1(t), \tilde{u}_2(t)))dt$$

is convex in $(\bar{u}_1, \bar{u}_2) \in \mathbb{R}^N \times \mathbb{R}^N$.

By the convexity of $F(t, (\cdot, \cdot))$, assumption (A) and Sobolev's inequality, we have

$$\int_0^T F(t, (\bar{u}_1, \bar{u}_2) + (\tilde{u}_1(t), \tilde{u}_2(t)))dt \ge$$

$$\ge 2\int_0^T F\Big(t, \frac{1}{2}(\bar{u}_1, \bar{u}_2)\Big)dt - \int_0^T F(t, -\tilde{u}_1(t), -\tilde{u}_2(t))dt \ge$$

$$\geq 2\int_0^T F\Big(t,\frac{1}{2}(\bar{u}_1,\bar{u}_2)\Big)dt - \int_0^T \Big[a_1\big(|\tilde{u}_1(t)|\big) + a_2\big(|\tilde{u}_2(t)|\big)\Big]b(t)dt \geq$$

$$\geq 2\int_0^T F\Big(t,\frac{1}{2}(\bar{u}_1,\bar{u}_2)\Big)dt - \Big[\max_{0\leq s\leq \|\tilde{u}_1\|_\infty} a_1(s) + \max_{0\leq s\leq \|\tilde{u}_2\|_\infty} a_2(s)\Big]\int_0^T b(t)dt \geq$$

$$\geq 2\int_0^T F\Big(t,\frac{1}{2}(\bar{u}_1,\bar{u}_2)\Big)dt - \Big[\max_{0\leq s\leq T^{\frac{1}{q'}}M} a_1(s) + \max_{0\leq s\leq T^{\frac{1}{p'}}M} a_2(s)\Big]\int_0^T b(t)dt$$

for all $(\bar{u}_1,\bar{u}_2)\in\mathbb{R}^N\times\mathbb{R}^N$ and $(\tilde{u}_1,\tilde{u}_2)\in\tilde{W}$ with $\|(\tilde{u}_1,\tilde{u}_2)\|\leq M$, which implies that

$$\varphi((\bar{u}_1,\bar{u}_2)+(\tilde{u}_1,\tilde{u}_2)) \geq -\frac{1}{q}\|\dot{\tilde{u}}_1\|_q^q - \frac{1}{p}\|\dot{\tilde{u}}_2\|_p^p + 2\int_0^T F\Big(t,\frac{1}{2}(\bar{u}_1,\bar{u}_2)\Big)dt -$$

$$-\Big[\max_{0\leq s\leq T^{\frac{1}{q'}}M} a_1(s) + \max_{0\leq s\leq T^{\frac{1}{p'}}M} a_2(s)\Big]\int_0^T b(t)dt \geq$$

$$\geq -\frac{1}{q}M^q - \frac{1}{p}M^p + 2\int_0^T F\Big(t,\frac{1}{2}(\bar{u}_1,\bar{u}_2)\Big)dt -$$

$$-\Big[\max_{0\leq s\leq T^{\frac{1}{q'}}M} a_1(s) + \max_{0\leq s\leq T^{\frac{1}{p'}}M} a_2(s)\Big]\int_0^T b(t)dt$$

for all $(\bar{u}_1,\bar{u}_2)\in\mathbb{R}^N\times\mathbb{R}^N$ and $(\tilde{u}_1,\tilde{u}_2)\in\tilde{W}$ with $\|(\tilde{u}_1,\tilde{u}_2)\|\leq M$. Now, from (A_2) we get that $\varphi((\bar{u}_1,\bar{u}_2)+(\tilde{u}_1,\tilde{u}_2))\to+\infty$ as $|(\bar{u}_1,\bar{u}_2)|\to\infty$, $(\bar{u}_1,\bar{u}_2)\in\mathbb{R}^N\times\mathbb{R}^N$, uniformly for $(\tilde{u}_1,\tilde{u}_2)\in\tilde{W}$ with $\|(\tilde{u}_1,\tilde{u}_2)\|\leq M$.

By (A_1) and Sobolev's inequality, we have

$$\varphi(\tilde{u}_1,\tilde{u}_2) = -\frac{1}{q}\|\dot{\tilde{u}}_1\|_q^q - \frac{1}{p}\|\dot{\tilde{u}}_2\|_p^p + \int_0^T F(t,\tilde{u}_1(t),\tilde{u}_2(t))dt \leq$$

$$\leq -\frac{1}{q}\|\dot{\tilde{u}}_1\|_q^q - \frac{1}{p}\|\dot{\tilde{u}}_2\|_p^p + \frac{1}{q}\int_0^T \alpha_1(t)|\tilde{u}_1(t)|^q dt + \frac{1}{p}\int_0^T \alpha_2(t)|\tilde{u}_2(t)|^p dt + \int_0^T \gamma(t)dt \leq$$

$$\leq -\frac{1}{q}\|\dot{\tilde{u}}_1\|_q^q - \frac{1}{p}\|\dot{\tilde{u}}_2\|_p^p + \frac{1}{q}\int_0^T \alpha_1(t)dt\ \|\tilde{u}_1\|_\infty^q + \frac{1}{p}\int_0^T \alpha_2(t)dt\ \|\tilde{u}_2\|_\infty^p + \|\gamma\|_1 \leq$$

$$\leq -\frac{1}{q}\|\dot{\tilde{u}}_1\|_q^q + \frac{1}{q}\int_0^T \alpha_1(t)dt\cdot T^{\frac{q}{q'}}\ \|\dot{\tilde{u}}_1\|_q^q - \frac{1}{p}\|\dot{\tilde{u}}_2\|_p^p + \frac{1}{p}\int_0^T \alpha_2(t)dt\cdot T^{\frac{p}{p'}}\ \|\dot{\tilde{u}}_2\|_p^p + \|\gamma\|_1 \leq$$

$$\leq -\frac{1}{q}\Big(1 - T^{\frac{q}{q'}}\int_0^T \alpha_1(t)dt\Big)\|\dot{\tilde{u}}_1\|_q^q - \frac{1}{p}\Big(1 - T^{\frac{p}{p'}}\int_0^T \alpha_2(t)dt\Big)\|\dot{\tilde{u}}_2\|_p^p + \|\gamma\|_1$$

which implies that $\varphi(\tilde{u}_1,\tilde{u}_2)\to-\infty$ as $\|(\tilde{u}_1,\tilde{u}_2)\|\to\infty$, $(\tilde{u}_1,\tilde{u}_2)\in\tilde{W}$.

Proof of the Theorem 3. Let $G(t,x_1,x_2) = F(t,x_1,x_2) - \frac{1}{q}\mu_1(t)|x_1|^q - \frac{1}{p}\mu_2(t)|x_2|^p$ for all $(x_1,x_2)\in\mathbb{R}^N\times\mathbb{R}^N$ and a.e. $t\in[0,T]$. Then $G(t,(\bar{u}_1,\bar{u}_2)+(\tilde{u}_1(t),\tilde{u}_2(t)))$ is convex in $(\bar{u}_1,\bar{u}_2)\in\mathbb{R}^N\times\mathbb{R}^N$ so is

$\int_0^T G(t,(\bar{u}_1,\bar{u}_2)+(\tilde{u}_1(t),\tilde{u}_2(t)))dt$. Hence for every $(\tilde{u}_1(t),\tilde{u}_2(t)) \in \tilde{W}$

$$\varphi((\bar{u}_1,\bar{u}_2)+(\tilde{u}_1,\tilde{u}_2)) = -\frac{1}{q}\|\dot{\tilde{u}}_1\|_q^q - \frac{1}{p}\|\dot{\tilde{u}}_2\|_p^p + \frac{1}{q}\int_0^T \mu_1(t)|\bar{u}_1+\tilde{u}_1(t)|^q dt +$$

$$+\frac{1}{p}\int_0^T \mu_2(t)|\bar{u}_2+\tilde{u}_2(t)|^p dt + \int_0^T G(t,(\bar{u}_1,\bar{u}_2)+(\tilde{u}_1(t),\tilde{u}_2(t)))dt$$

is convex in $(\bar{u}_1,\bar{u}_2)$ as a sum of three convex functions: one is convex in $\bar{u}_1$, the second one is convex in $\bar{u}_2$ and the last one is convex in $(\bar{u}_1,\bar{u}_2)$.

By the definition of subdifferential of convex function, we have

$$F(t,x_1,x_2) - \frac{1}{q}\mu_1(t)|x_1|^q - \frac{1}{p}\mu_2(t)|x_2|^p = G(t,x_1,x_2) \geq$$

$$\geq G(t,0,0) + (\nabla_{x_1}G(t,0,0),x_1) + (\nabla_{x_2}G(t,0,0),x_2) =$$

$$= F(t,0,0) + (\nabla_{x_1}F(t,0,0),x_1) + (\nabla_{x_2}F(t,0,0),x_2) \geq$$

$$\geq -\left[a_1(0)+a_2(0)\right]b(t)(1+|x_1|+|x_2|)$$

for all $(x_1,x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $t \in [0,T]$. It follows from assumption (A) and Sobolev's inequality that

$$\varphi((\bar{u}_1,\bar{u}_2)+(\tilde{u}_1,\tilde{u}_2)) \geq -\frac{1}{q}\|\dot{\tilde{u}}_1\|_q^q - \frac{1}{p}\|\dot{\tilde{u}}_2\|_p^p + \frac{1}{q}\int_0^T \mu_1(t)|\bar{u}_1+\tilde{u}_1(t)|^q dt +$$

$$+\frac{1}{p}\int_0^T \mu_2(t)|\bar{u}_2+\tilde{u}_2(t)|^p dt - [a_1(0)+a_2(0)]\int_0^T b(t)\big(1+|\bar{u}_1+\tilde{u}_1(t)|+|\bar{u}_2+\tilde{u}_2(t)|\big)dt \geq$$

$$\geq -\frac{1}{q}\|\dot{\tilde{u}}_1\|_q^q + \frac{1}{q2^q}|\bar{u}_1|^q\int_0^T \mu_1(t)dt - \frac{1}{q}\|\mu_1\|_1\|\tilde{u}_1\|_\infty^q -$$

$$-\frac{1}{p}\|\dot{\tilde{u}}_2\|_p^p + \frac{1}{p2^p}|\bar{u}_2|^p\int_0^T \mu_2(t)dt - \frac{1}{p}\|\mu_2\|_1\|\tilde{u}_2\|_\infty^p -$$

$$-\left[a_1(0)+a_2(0)\right]\left(1+|\bar{u}_1|+|\bar{u}_2|+\|\tilde{u}_1\|_\infty+\|\tilde{u}_2\|_\infty\right)\int_0^T b(t)dt \geq$$

$$\geq -\frac{1}{q}M^q + \frac{1}{q2^q}|\bar{u}_1|^q\int_0^T \mu_1(t)dt - \frac{1}{q}\|\mu_1\|_1 T^{\frac{q}{q'}}M -$$

$$-\frac{1}{p}M^p + \frac{1}{p2^p}|\bar{u}_2|^p\int_0^T \mu_2(t)dt - \frac{1}{p}\|\mu_2\|_1 T^{\frac{p}{p'}}M -$$

$$-\left[a_1(0)+a_2(0)\right]\left(1+|\bar{u}_1|+|\bar{u}_2|+(T^{\frac{1}{q'}}+T^{\frac{1}{p'}})M\right)\int_0^T b(t)dt$$

for all $(\bar{u}_1,\bar{u}_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and $(\tilde{u}_1,\tilde{u}_2) \in \tilde{W}$ with $\|(\tilde{u}_1,\tilde{u}_2)\| \leq M$. Now condition (6) follows from $\int_0^T \mu_i(t)dt > 0, i=1,2$. Condition (5) follows like in the proof of Theorem 2.

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