Semilinear hyperbolic functional differential problem on a cylindrical domain

W. Czernous

Abstract

We consider the initial boundary value problem for a semi-linear partial functional differential equation of the first order on a cylindrical domain in $n+1$ dimensions. Projection of the domain onto the n-dimensional hyperplane is a connected set with boundary satisfying certain type of cone condition. Using the method of characteristics and the Banach contraction theorem, we prove the global existence, uniqueness and continuous dependence on data of Carathéodory solutions of the problem. This approach cover equations with deviating variables as well as differential integral equations.

1 Introduction

Let the symbol $B(x, r)$ denote an open ball in the n-dimensional Euclidean space $\mathbb{R}^n$, with centre at $x \in \mathbb{R}^n$ and radius $r > 0$. For $U \subset \mathbb{R}^{1+n}$ and a normed space Y, equipped with the norm $\| \cdot \|_Y$, we define $C(U, Y)$ to be the set of all continuous functions $w : U \to Y$; this space is equipped with the usual supremum norm $\|w\|_{C(U,Y)} = \sup_{P \in U} \|w(P)\|_Y$. We write it simply $C(U)$ when no confusion can arise; following [1], we denote by $C(\overline{U})$ the set of all continuous and bounded functions on U.

Let us formulate the functional differential problem. The sets, considered here, are:

$$E_0 = [-b_0, 0] \times \overline{\Omega}, \quad E = [0, a] \times \overline{\Omega}, \quad \partial_0 E = (0, a] \times \partial\Omega,$$

Received by the editors November 2010.

Communicated by P. Godin.

2000 *Mathematics Subject Classification* : 35R10, 35A30, 35A01.

Key words and phrases : Carathéodory solutions, global existence, characteristics, uniform cone condition.

where $a > 0$, $b_0 \geq 0$, and Ω is a nonempty, open, connected subset of $\mathbb{R}^n$, satisfying a (modified) *uniform cone condition* (its original form may be found in [1]), to be specified later. Write $E^* = [-b_0, a] \times \overline{\Omega}$ and $D = [-b_0 - a, 0] \times (\overline{\Omega} - \overline{\Omega})$. For $(t, x) \in E$ define

$$\mathscr{D}[t,x] = \Big\{ (\tau, y) \in \mathbb{R}^{1+n} \, : \, \tau \leq 0 \quad \text{and} \quad (t+\tau, x+y) \in E^* \Big\}.$$

Then $\mathscr{D}[t,x] \subset D$ for $(t,x) \in E^*$. Given $z : E^* \to \mathbb{R}$ and $(t,x) \in E$, define $z_{(t,x)} \colon \mathscr{D}[t,x] \to \mathbb{R}$ by $z_{(t,x)}(\tau,y) = z(t+\tau, x+y)$, $(\tau,y) \in \mathscr{D}[t,x]$. Then $z_{(t,x)}$ is the restriction of z to $E^* \cap ([-b_0, t] \times \mathbb{R}^n)$, shifted to $\mathscr{D}[t,x]$.

Suppose that $\phi_0 \colon [0,a] \to [0,a]$, $0 \leq \phi_0(t) \leq t$ and $\phi = (\phi_1, \ldots, \phi_n) \colon E \to \Omega$ are given functions. Write $\varphi(t,x) = (\phi_0(t), \phi(t,x))$ for $(t,x) \in E$. Let $f_j \colon E \to \mathbb{R}$, $1 \leq j \leq n$, $F \colon E \times C(\overline{D}) \to \mathbb{R}$ and $\psi \colon E_0 \cup \partial_0 E \to \mathbb{R}$ be given. Put $f = (f_1, \ldots, f_n)^T$. We consider the functional differential equation

$$\partial_t z(t,x) + \partial_x z(t,x) f(t,x) = F(t,x,z_{\varphi(t,x)}), \tag{1}$$

augmented with the initial boundary condition

$$z(t,x) = \psi(t,x) \tag{2}$$

on $E_0 \cup \partial_0 E$. A function $\tilde{z} \in C(\overline{E^*})$ is called a global Carathéodory solution of (1), (2) if it is absolutely continuous, has partial derivatives almost everywhere, and satisfies (1) almost everywhere on E and (2) on $E_0 \cup \partial_0 E$.

The aim of this paper is to prove a theorem on the existence and continuous dependence of global Carathéodory solutions to (1), (2). We use the method of characteristics. The initial-boundary value problem is transformed into a functional integral equation, for which the existence of a solution is proved by means of the Banach fixed point theorem. Classical solutions of the functional integral equation lead to Carathéodory solutions to (1), (2).

We give examples of semilinear equations which can be obtained from (1) by specializing F.

Example 1.1 *Suppose that* $\tilde{F} : E \times \mathbb{R} \to \mathbb{R}$ *is a given function. Set* $F(t,x,w) = \tilde{F}(t,x,w(0,0))$. *Then* (1) *becomes the equation with deviated variables*

$$\partial_t z(t,x) + \partial_x z(t,x) f(t,x) = \tilde{F}(t,x,z(\varphi(t,x))).$$

Example 1.2 *Suppose that* $\varphi(t,x) = (t,x)$ *on* E. *For the above* $\tilde{F}$ *we put*

$$F(t,x,w) = \tilde{F}(t,x, \int_{\mathscr{D}[t,x]} w(\tau,y) \, d\tau \, dy).$$

Then (1) *is equivalent to the integro-differential equation*

$$\partial_t z(t,x) + \partial_x z(t,x) f(t,x) = \tilde{F}(t,x, \int_{\mathscr{D}[t,x]} z_{(t,x)}(\tau,y) \, d\tau \, dy).$$

It is clear that more complicated examples of differential equations with deviating variables and integro-differential equations can be obtained from (1) for suitable F and φ.

Let us give a brief review of existence results concerning initial-boundary value problems for first order functional differential equations.

Initial boundary value problems for almost linear systems for unknown functions of two independent variables were considered in [16]. A continuous function is a solution of a mixed problem if it satisfies an integral functional system by integrating along bicharacteristics. Existence theorems and differential inequalities related to almost linear functional problems can be found in [5]. Distributional solutions of almost linear problems were investigated in [17]. The method used in this paper is constructive; the existence result is based on a difference scheme.

The existence and uniqueness results for quasilinear systems with initial or initial-boundary conditions in the class of Carathéodory solutions can be found in [7],[18]. Initial problems for nonlinear equations were considered in [8].

An essential extension of some ideas concerning classical solutions of hyperbolic functional differential problems is given in [3],[4], where the Cinquini-Cibrario solutions are considered. This class of solutions is placed between classical solutions and solutions in the Carathéodory sense.

The monograph [13] contains an exposition of existence and uniqueness of generalized and classical solutions to hyperbolic functional differential problems.

Solutions of all these evolutionary problems are shown to exist locally in time. The only global existence result [18] concerns equations with two independent variables.

We propose a result on global existence of Carathéodory solutions to a class of hyperbolic mixed problems in several independent variables. Our result cannot be obtained from the above mentioned ones. What is more, we consider an initial boundary value problem in a cylindrical domain of a very general shape. In particular, it may be $[0, a] \times \Omega$, with (bounded or not) Ω having locally Lipschitz boundary. If Ω is bounded, then the uniform cone condition, assumed here, implies the Lipschitz condition, see [11]. All known results on such problems for first order partial functional differential equations are formulated for Ω being an interval in $\mathbb{R}^n$.

First order partial differential equations with deviated variables and differential integral equations find applications in different fields of knowledge. We give a few examples.

An equation with deviated variables ([10]) describes a density of households at time t depending on their estates, in the theory of the distribution of wealth. Hyperbolic integral differential equations perturbed by a dissipative integral terms of the Volterra type are proposed [2] as simple mathematical models for the non linear phenomenon of harmonic generation of laser radiation through piezoelectric crystals for non dispersive materials and of the Maxwell - Hopkinson type. There are various problems in nonlinear optic which lead to hyperbolic integral differential problems [2]. A system of nonlinear functional differential equations, which model an age dependent epidemic of a disease with vertical transmission, is investigated in [9]. Almost linear functional differential equations may be used ([6]) to describe a model of proliferating cell population. Nonlinear equations may be used to describe the growth of a population of cells which constantly differentiate (change their properties) in time. A model for erythroid production based on a continuous maturation - proliferation scheme is developed in [14]. The paper [12] discusses optimal harvesting policies for age - structured population harvested with effort independent of age. Hyperbolic functional differential equations are considered in the nonlinear theory describing the motion of viscoelastic media [15].

For further bibliography on applications of functional partial differential equations see the monographs [13], [19].

The paper is organized as follows. In the next section, we formulate and prove new results on existence and uniqueness of characteristics. The method of characteristics is used to transform the mixed problem into a system of integral functional equations, described and analyzed in the last section.

2 Properties of characteristics

For a point $(t,x) \in E$, we consider the Cauchy problem

$$\eta'(\tau) = f(\tau, \eta(\tau)), \qquad \eta(t) = x = (x_1, \ldots, x_n)^T, \tag{3}$$

and denote by $g(\cdot, t, x)$ its Carathéodory solution. This function is the characteristic of (1). Since the problem (3) is formulated on $[0,a] \times \overline{\Omega}$, we need first to give our requirements on the domain.

2.1 A modified uniform cone condition and its consequences

To be specific, we cite here the definition from [1].

Let v be a nonzero vector in $\mathbb{R}^n$, and for each $x \neq 0$ let $\angle(x,v)$ be the angle between the position vector x and v. For such given v, $\rho > 0$, and γ satisfying $0 < \gamma \leq \pi$, the set

$$C = \{x \in \mathbb{R}^n \,:\, x = 0 \quad \text{or} \quad 0 < \|x\| \leq \rho,\ \angle(x,v) \leq \gamma/2\}$$

is called a *finite cone* of height ρ, axis direction v and aperture angle γ with vertex at the origin. Note that $x + C = \{x + y \,:\, y \in C\}$ is a finite cone with vertex at x but the same dimensions and axis direction as C and is obtained by parallel translation of C.

An open cover $\mathscr{O}$ of a set $S \subset \mathbb{R}^n$ is said to be *locally finite* if any compact set in $\mathbb{R}^n$ can intersect at most finitely many members of $\mathscr{O}$. Such locally finite collections of sets must be countable, so their elements can be listed in a sequence. If S is closed, then any open cover of S by sets with a uniform bound on their diameters possesses a locally finite subcover.

We denote by Ω_δ the set of points in Ω within distance δ of the boundary of Ω:

$$\Omega_\delta = \{x \in \Omega \,:\, \operatorname{dist}(x, \partial\Omega) < \delta\}.$$

A set Ω satisfies the *uniform cone condition* if there exists a locally finite open cover $\{U_j\}$ of the boundary of Ω and a corresponding sequence $\{C_j\}$ of finite cones, each congruent to some fixed finite cone C, such that

(i) There exists $M < \infty$ such that every U_j has diameter less than M,

(ii) $\Omega_\delta \subset \bigcup_{j=1}^{\infty} U_j$ for some $\delta > 0$.

(iii) $Q_j \equiv \bigcup_{x \in \Omega \cap U_j} (x + C_j) \subset \Omega$ for every j.

(iv) For some finite R, every collection of $R+1$ of the sets Q_j has empty intersection.

In the paper, we assume that Ω satisfies the above uniform cone condition, together with the following:

(v) For every pair of points $x, y \in \Omega_\delta$ such that $\|x - y\| < \delta$, there exists j such that

$$x, y \in V_j(\delta) = \left\{ x \in U_j \,:\, \operatorname{dist}(x, \partial U_j) > \delta \right\}.$$

(vi) There is a family $\mathscr{C}$ of finite cones, each congruent to some fixed finite cone C^*, such that for every finite set J of indices,

$$\text{if} \quad \Omega \cap \bigcap_{j \in J} U_j \neq \emptyset \quad \text{then there is } C \in \mathscr{C},\, C \subset \bigcap_{j \in J} C_j.$$

(vii) The domain Ω is Lipschitz path-connected. That is, there exists a constant L_Ω such that, for all $x, y \in \Omega$, the set Ω contains a path between x and y whose length does not exceed $L_\Omega \|x - y\|$.

(viii) All axis direction vectors v_j have unit length.

Remark 2.1 *If Ω is bounded, then (v) follows from (ii) by the Lebesgue number lemma.*

Remark 2.2 *For the points x, y, $\|x - y\| < \delta$, in the closure of Ω_δ, there is j such that $x, y \in V_j(\delta/2)$.*

Remark 2.3 *It is easy to see that the uniform cone condition implies*

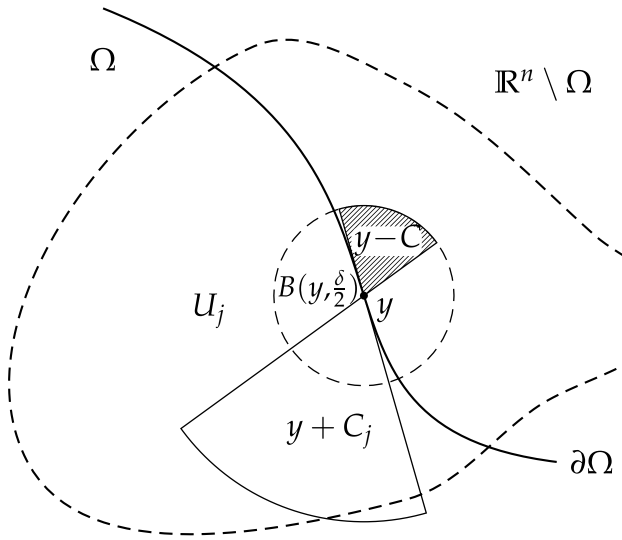
$$x + C_j \subset \overline{\Omega} \quad \textit{if} \quad x \in \overline{\Omega} \cap U_j.$$

Indeed, if $x \in \partial\Omega \cap U_j$, then there is $x_n \to x$, $\{x_n\} \subset \Omega \cap U_j$. Every $y \in x + C_j$ belongs to $\overline{\Omega}$, because $y_n := x_n + (y - x) \in x_n + C_j \subset \Omega$ tends to y.

Lemma 2.4 (on an outward cone) *Let $y \in \partial\Omega$ and let j be such that $B(y, \delta/2) \subset U_j$ (by the Remark 2.2, we may always find such j). Then, for $C = C_j \cap \overline{B(0, \delta/2)}$,*

$$(y - \operatorname{Int} C) \cap \overline{\Omega} = \emptyset.$$

Figure 1: Outward open cone at the boundary of Ω.



Proof. We first show that $y - \operatorname{Int} C$ misses Ω. Suppose that $z \in (y - \operatorname{Int} C) \cap \Omega$. Then $\|z-y\| < \delta/2$ and so $z \in U_j$ by our assumption. By the uniform cone condition, $z + C_j \subset \Omega$. Obviously, for any finite cone C,

$$z \in y - C \iff y \in z + C.$$

So we obtain

$$z \in y - C \subset y - C_j, \quad \text{and thus} \quad y \in z + C_j \subset \Omega,$$

contradicting the fact that $y \in \partial\Omega$.

Suppose now that $\bar{z} \in (y - \operatorname{Int} C) \cap \partial\Omega$. But a point from the boundary of Ω may be approached from its interior, and $y - \operatorname{Int} C$ is an open set, hence there is $z \in (y - \operatorname{Int} C) \cap \Omega$, which we have proved impossible. ∎

2.2 Regularity of characteristics

Assumption H [f]. *Let the function $f\colon E \to \mathbb{R}^n$, in variables (t,x), satisfy Carathéodory conditions on E, with usual integrable functions $\tilde{\alpha}$, $\tilde{\beta}\colon [0,a] \to \mathbb{R}_+$, playing the roles of a bound and of a generalized Lipschitz constant w.r.t. x, respectively. Denote by v_j the axis direction of cone C_j. We require from f to be well-separated from zero on a neighbourhood of $[0,a] \times \partial\Omega$: there is $\kappa > 0$, such that, for almost every $t \in [0,a]$ and for all j,*

$$v_j \circ f(t,x) \geq \kappa \quad \textit{if} \quad x \in \Omega \cap U_j. \tag{4}$$

Moreover, there exists a family of finite cones $\{C'_j\}_{j=1}^{\infty}$, with each C'_j congruent to some fixed finite cone C', and having the same axis direction and height as C_j, but a smaller aperture angle, such that for almost all $t \in [0,a]$ and for all j,

$$f(t,x) \in \bigcup_{\lambda>0} \lambda C'_j, \quad \textit{if} \quad x \in \Omega \cap U_j. \tag{5}$$

Let $\zeta(t,x)$ be the left end of the maximal interval on which the characteristic $g(\cdot,t,x)$ is defined.

Fact 2.5 *Let $f : I \to \mathbb{R}^n$, be an integrable function on some measurable set $I \subset \mathbb{R}$, and let C be an* infinite cone, *that is, $C = \{x \in \mathbb{R}^n \,:\, \angle(x,v) \le \gamma/2\}$ for some angle γ and axis v. If*

$$f(t) \in C \quad \textit{almost everywhere on } \; I,$$

then

$$\int_I f(t)\, dt \in C.$$

Proof. Intuitively, the above assertion follows from the property $C + C = C$, true for any infinite cone C. Nevertheless, a proof using a basic property of the Lebesgue integral (integral of a nonnegative-valued function is nonnegative) may be facilitated by the fact, that an infinite cone C is the intersection of all *half-spaces* containing it, $C = \bigcap \{H \,:\, H \supset C\}$, where by a half space H we mean

$$H = \{x \in \mathbb{R}^n \,:\, v \circ x \ge 0\} \quad \text{for some} \quad v \in \mathbb{R}^n, v \ne 0. \qquad \blacksquare$$

Corollary 2.6 *Take $x \in \Omega \cap U_j$ and $t \in [0,a]$. Suppose that the characteristic $g(\cdot,t,x)$ is defined on $[t,T]$ for some $T \in \mathbb{R}$, $t < T \le a$. By the condition* (5) *of the Assumption $H[\,f\,]$, if*

$$\{g(\tau,t,x) \,:\, \tau \in [t,T]\} \subset \Omega \cap U_j$$

then

$$\int_t^T f(\tau, g(\tau,t,x))\, d\tau \in \bigcup_{\lambda>0} \lambda C'_j. \tag{6}$$

Remark 2.7 *Note that, since $\Omega \cap U_j$ is an open set, we may always find $T > t$, mentioned above, if only $g(\cdot,t,x)$ exists on a right-hand neighbourhood of t.*

Corollary 2.8 *Suppose that $x \in \overline{\Omega_\delta}$, $t \in [0,a]$, and that the characteristic $g(\cdot,t,x)$ exists on $[t,T]$ such that $\int_t^T \tilde{\alpha}(\tau)\, d\tau \le \delta/2$. Then, by the Remark 2.2, there is j such that* (6) *holds.*

Lemma 2.9 *Suppose that Assumption $H[\,f\,]$ is satisfied. Then*

$$\zeta(t,x) > 0 \quad \textit{implies} \quad g(\zeta(t,x),t,x) \in \partial\Omega, \tag{7}$$

the characteristic $g(\cdot,t,x)$ is defined on $[\zeta(t,x),a]$, and there is a constant L_g, uniform for (s,y), $(t,x) \in E$, such that

$$\|g(\tau,s,y) - g(\tau,t,x)\| \le L_g\Big(\Big| \int_t^s \tilde{\alpha}(\tau)\, d\tau \Big| + \|y - x\| \Big), \tag{8}$$

for $\tau \in [\max\{\zeta(s,y),\zeta(t,x)\}, \min\{s,t\}]$.

Proof. The existence, up to the boundary of E, and uniqueness of the solution of (3), follow from classical theorems. Hence (7). To see that $g(\cdot,t,x)$ is defined on

$[\zeta(t,x),a]$, let us assume that for some $(t,x) \in E$ and for some $\tilde{a}$, $\zeta(t,x) < \tilde{a} < a$, we have

$$g(\tau,t,x) \in \Omega \quad \text{for} \quad \tau \in (\zeta(t,x),\tilde{a}) \quad \text{and} \quad g(\tilde{a},t,x) \in \partial\Omega.$$

Set $y = g(\tilde{a},t,x)$. Choose j such that $y \in U_j$; there is $r > 0$ such that $B(y,r) \subset U_j$. There exists $\varepsilon \in (0, \tilde{a} - \zeta(t,x))$ with the property, that the integral of $\tilde{\alpha}$, over any closed interval $I \subset [0,a]$ of length ε, does not exceed $1/2 \min\{r,h\}$, where h is the height of the cone C_j. By the definition of $\tilde{a}$,

$$g(\tilde{a}-\varepsilon,t,x) \in \Omega;$$

additionally, we have chosen ε so that

$$\|g(\tilde{a}-\varepsilon,t,x) - y\| = \left\| \int_{\tilde{a}-\varepsilon}^{\tilde{a}} f(\tau, g(\tau,t,x))\, d\tau \right\| \le \int_{\tilde{a}-\varepsilon}^{\tilde{a}} \tilde{\alpha}(\tau)\, d\tau \le r/2,$$

implying

$$g(\tilde{a}-\varepsilon,t,x) \in B(y,r) \subset U_j.$$

Hence $g(\tilde{a}-\varepsilon,t,x) \in \Omega \cap U_j$, and similarly

$$g(\tau,t,x) \in \Omega \cap U_j \quad \text{for} \quad \tau \in I = (\tilde{a}-\varepsilon, \tilde{a}). \tag{9}$$

By the Corollary 2.6,

$$\int_{\tilde{a}-\varepsilon}^{\tilde{a}} f(\tau, g(\tau,t,x))\, d\tau \in C$$

From the definition of ε, it is clear that the norm of the vector $\int_{\tilde{a}-\varepsilon}^{\tilde{a}} f(\tau, g(\tau,t,x))\, d\tau$ does not exceed the height h of C_j. All these sum up to

$$g(\tilde{a},t,x) = g(\tilde{a}-\varepsilon,t,x) + \int_{\tilde{a}-\varepsilon}^{\tilde{a}} f(\tau, g(\tau,t,x))\, d\tau \in (\Omega \cap U_j) + C_j \subset \Omega,$$

where the last inclusion follows from the uniform cone condition, contradicting $g(\tilde{a},t,x) \in \partial\Omega$.

The estimate (8) follows by use of the Gronwall's lemma for integrable data. ∎

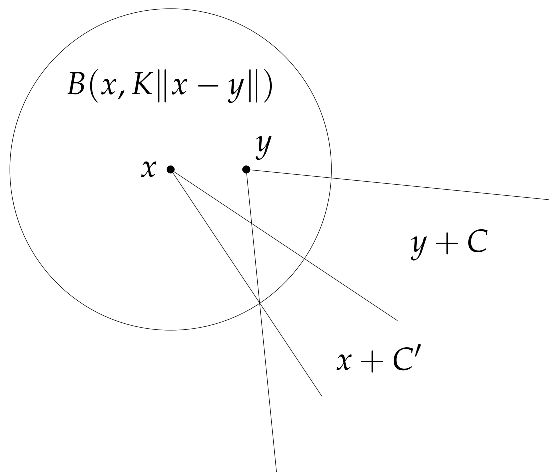
Lemma 2.10 *Suppose that Assumption H [f] is satisfied. Then there is a constant L_ζ, uniform for (s,y), $(t,x) \in E$, such that*

$$|\zeta(s,y) - \zeta(t,x)| \le L_\zeta \Big(\left| \int_t^s \tilde{\alpha}(\tau)\, d\tau \right| + L_\Omega \|y - x\| \Big), \tag{10}$$

Proof. The proof goes in several steps.

(i) Suppose that two infinite cones C, C' have the same axis direction and different aperture angles: $\gamma_{C'} < \gamma_C$, so that C' is more sloppy than C. If we take some $x \notin y + C$, then $x + C'$ crosses $y + C$ in a neighbourhood of x with radius

Figure 2: Two infinite cones with the same axis direction.



depending only on the difference $\gamma_C - \gamma_{C'}$ and on the distance $\|x-y\|$ between the vertices of the cones. More precisely, there is $K = K(\gamma_C - \gamma_{C'})$ such that

$$(x+C') \setminus \overline{B(x, K\|x-y\|)} \quad \subset \quad y + \operatorname{Int} C, \tag{11}$$

as illustrated by Figure 2. This is easy to see for cones in $\mathbb{R}^2$ (one may calculate an exact formula for $K(\cdot)$), and for $n > 2$ may be proved by taking projections onto the axis direction vector.

Thus, for any $\delta > 0$, we get for $\|x-y\| < \delta' = \delta/(4K)$

$$(x+C') \setminus \overline{B(x, \delta/4)} \quad \subset \quad \operatorname{Int}(y+C). \tag{12}$$

(ii) We show that there is $\delta' > 0$ such that for $x \in \Omega_{\delta'}$, $y \in \partial\Omega$, $\|x-y\| < \delta'$, and for j, such that Remark 2.2 applies to x and y,

$$g(\tau, t, x) \in U_j \quad \text{for all } \tau \in [\zeta(t,x), t]. \tag{13}$$

Let K, $K > 1$, from (i), be chosen so that it fits for

$$C = \bigcup_{\lambda > 0} \lambda C_j, \qquad C' = \bigcup_{\lambda > 0} \lambda C_j',$$

and let $\delta' = \delta/(4K) < \delta/4$. We get, by the relation (12),

$$B(x, \delta/2 - \delta') \cap (x - C') \setminus \overline{B(x, \delta/4)} \quad \subset \quad B(y, \delta/2) \cap \operatorname{Int}(y - C). \tag{14}$$

Suppose that for some $\bar{t} < t$ we have $\|g(\bar{t}, t, x) - x\| > \delta/4$. Then, by the continuity of characteristics and by the Darboux property,

$$g(\tau, t, x) \in B(x, \delta/2 - \delta') \setminus \overline{B(x, \delta/4)} \tag{15}$$

for some $\tau \in [\bar{t}, t)$. This τ may be even chosen in such a way, that

$$\{g(s,t,x) \;:\; s \in [\tau, t]\} \subset B(x, \delta/2).$$

Moreover, recall that j is such that $B(x,\delta/2) \subset U_j$. This enables us to use the Corollary 2.6, and to get (note that $x = g(t,t,x)$)

$$g(\tau,t,x) = x - \int_{\tau}^{t} f(s,g(s,t,x))\,ds \in x - C'. \tag{16}$$

From the relations (14)-(16) we get $g(\tau,t,x) \in B(y,\delta/2) \cap \mathrm{Int}(y - C)$. By the Lemma on an outward cone, this yields $g(\tau,t,x) \notin \overline{\Omega}$, which is not possible. This shows that $g(\bar{t},t,x) \in \overline{B(x,\delta/4)}$ for $\bar{t} < t$, and, taking into account $B(x,\delta/2) \subset U_j$, we get (13).

(iii) We show a local version of the generalized Lipschitz condition, with the uniform constant, implying the global property (10).

Since (10) is obvious in the case $\zeta(s,y) = \zeta(t,x) = 0$, let us take $0 \le \zeta(s,y) < \zeta(t,x)$, so that $g(\zeta(t,x),t,x) \in \partial\Omega$. By (8), for (t,x) and (s,y) satisfying

$$\left| \int_t^s \tilde{\alpha}(\tau)\,d\tau \right| + \|y - x\| \le \delta'/(2L_g), \tag{17}$$

we get

$$\|g(\zeta(t,x),s,y) - g(\zeta(t,x),t,x)\| \le L_g \Big(\left| \int_t^s \tilde{\alpha}(\tau)\,d\tau \right| + \|y - x\| \Big) \le \delta'/2 < \delta'.$$

As we have proved in (ii), there is j such that

$$g(\tau,s,y) \in U_j \quad \text{for} \quad \tau \in [\zeta(s,y),\zeta(t,x)].$$

Thus

$$v_j \circ f(\tau,g(\tau,s,y)) \ge \kappa \quad \text{for} \quad \tau \in (\zeta(s,y),\zeta(t,x)]$$

and, recalling that condition (viii) on the page 5 sets all v_j to have unit length,

$$\begin{aligned}
\kappa\big(\zeta(t,x) - \zeta(s,y)\big) &\le \int_{\zeta(s,y)}^{\zeta(t,x)} v_j \circ f(\tau,g(\tau,s,y))\,d\tau \\
&= v_j \circ \big(g(\zeta(t,x),s,y) - g(\zeta(s,y),s,y)\big) \\
&\le \|v_j\|\;\|g(\zeta(t,x),s,y) - g(\zeta(s,y),s,y)\| \\
&\le \|g(\zeta(t,x),s,y) - g(\zeta(t,x),t,x)\| \\
&\quad + \|g(\zeta(t,x),t,x) - g(\zeta(s,y),s,y)\|.
\end{aligned}$$

Let us analyse the number $\|g(\zeta(t,x),t,x) - g(\zeta(s,y),s,y)\|$. Since, by the Corollary 2.6,

$$g(\zeta(s,y),s,y) \in g(\zeta(t,x),s,y) - C',$$

and this cone crosses the cone $g(\zeta(t,x),t,x) - C$ in a ball of radius $K\|g(\zeta(t,x),s,y) - g(\zeta(t,x),t,x)\|$, the Lemma on an outward cone shows (similarly to the proof of (13)) that

$$g(\zeta(s,y),s,y) \in \overline{B\Big(g(\zeta(t,x),t,x),r\Big)}, \qquad r = K\|g(\zeta(t,x),s,y) - g(\zeta(t,x),t,x)\|,$$

and

$$\kappa\big(\zeta(t,x)-\zeta(s,y)\big) \le (1+K)\|g(\zeta(t,x),s,y)-g(\zeta(t,x),t,x)\|$$
$$\le (1+K)L_g\Big(\Big|\int_t^s \tilde{\alpha}(\tau)\,d\tau\Big| + \|y-x\|\Big),$$

yielding

$$|\zeta(s,y)-\zeta(t,x)| \le L_\zeta\Big(\Big|\int_t^s \tilde{\alpha}(\tau)\,d\tau\Big| + \|y-x\|\Big),$$

for (t,x), (s,y) satisfying (17). Since the Lipschitz constant, appearing above, is independent of (t,x), (s,y), the global assertion (10) follows from the convexity of $[0,a]$ and from the Lipschitz path-connectedness of Ω. ■

We have proposed above a new type of domains, suitable for formulation of hyperbolic functional differential problems. Note that our conditions may be easily reformulated, so to allow for an initial problem instead of initial-boundary. This is why we have assumed the original Carathéodory conditions on f. Nevertheless, due to the necessity of Lipschitz continuity in x of $\psi(S(t,x))$ appearing in the sequel, we need f to be essentially bounded, in the sense that

$$\tilde{\alpha} \in L^\infty([0,a]) \quad \text{and} \quad \operatorname{ess\,sup} \tilde{\alpha} = B_f \quad \text{for some } B_f \in \mathbb{R}_+. \tag{18}$$

3 Functional integral system

When it does not lead to misunderstanding, we write $U_t = U \cap ([-\infty,t] \times \mathbb{R}^n)$ for $U \subset \mathbb{R}^{1+n}$ and $t \in [0,a]$. We choose the norm in $\mathbb{R}^k$ to be the ∞-norm: $\|y\| = \|y\|_\infty = \max_{1\le i\le k}|y_i|$. For the last argument w of the function F, we will use the standard supremum norm in $C(\overline{D})$, $\|w\| = \|w\|_{C(\overline{D})}$.

Condition 3.1 (V) *Suppose that $F\colon E \times C(\overline{D}) \to \mathbb{R}$ is a given function of the variables (t,x,w). We will say that F satisfies condition (V) if for each $(t,x) \in E$ and for w, $\bar{w} \in C(\overline{D})$ such that $w(\tau,y) = \bar{w}(\tau,y)$ for $(\tau,y) \in \mathscr{D}[\varphi(t,x)]$ we have $F(t,x,w) = F(t,x,\bar{w})$.*

Note that condition (V) means that the value of F at the point (t,x,w) depends on (t,x) and on the restriction of w to the set $\mathscr{D}[\varphi(t,x)]$ only.

Assumption H [φ]. *The deviating function ϕ is Lipschitz continuous in x with constant L_ϕ.*

Assumption H [ψ]. *The function ψ is bounded and Lipschitz continuous, with the bound s_0 and the Lipschitz constant s_1.*

Denote by $C^{0.1}[s](\overline{E_0 \cup \partial_0 E})$ the class of all functions ψ satisfying Assumption H [ψ]. For a such ψ fixed, write $C_\psi = \{z \in C(E^*) : z(t,x) = \psi(t,x) \text{ on } E_0 \cup \partial_0 E\}$.

Assumption H [F]. *The function $F\colon E \times C(\overline{D})$ of the variables (t,x,w) is measurable in t, continuous in x,w,*

$$F(\,\cdot\,,x,w) \in L([0,a]) \quad \textit{for} \quad (x,w) \in \overline{\Omega} \times C(\overline{D}),$$

and one of the following two conditions is satisfied: either

(i) F is bounded by B_F and $|F(t,x,w) - F(t,\bar{x},\bar{w})| \le \gamma(t,p)[\|x-\bar{x}\| + \|w-\bar{w}\|]$ for $\|w\|, \|\bar{w}\| \le p$, where $\gamma(\cdot,p)$ is integrable on $[0,a]$ for $p \in \mathbb{R}_+$,

or

(ii) $|F(t,x,0)|$ is bounded by B_F and $|F(t,x,w) - F(t,x,\bar{w})| \le \lambda(t)\|w-\bar{w}\|$ with $\lambda \in L([0,a])$ and $|F(t,x,w) - F(t,\bar{x},w)| \le \gamma(t,\|w\|)\|x-\bar{x}\|$ with $\gamma(\cdot,p)$ integrable on $[0,a]$ for $p \in \mathbb{R}_+$.

Put

$$Q[z](\tau,t,x) = (\tau, g(\tau,t,x), z_{\varphi(\tau,g(\tau,t,x))}), \qquad S(t,x) = (\zeta(t,x), g(\zeta(t,x),t,x))$$

and

$$\mathbb{F}[z](t,x) = \begin{cases} \psi(S(t,x)) + \int_{\zeta(t,x)}^{t} F(Q[z](\tau,t,x))\,d\tau & \text{on } E, \\ \psi(t,x) & \text{on } E_0 \cup \partial_0 E. \end{cases}$$

We consider the functional integral problem

$$z = \mathbb{F}[z]. \tag{19}$$

An easy estimate (possibly using Gronwall lemma) of solutions to (19) follows.

Lemma 3.2 *Suppose that Assumptions $H[\,f\,]$, $H[\,\varphi\,]$, $H[\,\psi\,]$, $H[\,F\,]$ are satisfied, the relation* (18) *holds, and*

1) *the data $\psi \in C^{0.1}[s](\overline{E_0 \cup \partial_0 E})$,*

2) *the function $\bar{z} \in C_\psi$ satisfies* (19)*,*

3) *$\bar{z}$ is bounded and absolutely continuous on E^*.*

Then

$$\|\bar{z}\|_{C(E_t^*)} \le \mu(t),$$

where $\mu(t) = s_0 + tB_F$, in the case (i) of Assumption $H[\,F\,]$, and

$$\mu(t) = s_0 + tB_F + \int_0^t (s_0 + \tau B_F)\lambda(\tau) \exp\Big\{ \int_\tau^t \lambda(s)\,ds \Big\}\,d\tau \tag{20}$$

in the second case.

Denote, for $z \in C(E^*)$, $\alpha \in \mathbb{R}_+$, $\beta \in L([0,a])$,

$$|z|_{\alpha,\beta} = \sup_{\substack{t\in[0,a]\\ x\neq\bar{x}}} \Big\{ \frac{|z(t,x) - z(t,\bar{x})|}{\|x-\bar{x}\|} \exp\Big(-\alpha \int_0^t \beta(\tau)\,d\tau \Big) \Big\}.$$

Let the class $C_\psi[d;\alpha,\beta;\mu] \subset C_\psi$ of candidates for a solution of our problem be the space of all continuous extensions of ψ, satisfying $|z|_{\alpha,\beta} \le d$, and such that $\|z\|_{C(E_t^*)} \le \mu(t)$, $t \in [0,a]$.

Corollary 3.3 *Suppose that Assumptions H [f], H [φ], H [ψ], H [F] are satisfied, the relation* (18) *holds, and the data $\psi \in C^{0.1}[s](\overline{E_0 \cup \partial_0 E})$. Then for $z, \bar{z} \in C_\psi[d;\alpha,\beta;\mu]$,*

$$|F(t,x,z_{\varphi(t,x)})| \leq B(t), \tag{21}$$

$$|F(t,x,z_{\varphi(t,x)}) - F(t,\bar{x},z_{\varphi(t,\bar{x})})| \leq L(t)\|x-\bar{x}\|, \tag{22}$$

$$|F(t,x,z_{\varphi(t,x)}) - F(t,x,\bar{z}_{\varphi(t,x)})| \leq Z(t)\|z-\bar{z}\|_{C(E_t^*)} \quad \text{if} \quad \|\bar{z}\|_{C(E^*)} \leq \mu(a), \tag{23}$$

where, in the case (i) of Assumption H [F],

$$B(t) = B_F, \quad Z(t) = \gamma(t,\mu(a))$$

or, in the second case of that Assumption,

$$B(t) = B_F + \lambda(t)\mu(a), \quad Z(t) = \lambda(t),$$

and

$$L(t) = \gamma(t,\mu(a)) + Z(t)\exp\{\alpha \textstyle\int_0^t \beta(\tau)\,d\tau\}|z|_{\alpha,\beta}L_\phi.$$

Proof. The relations (21), (23) are clear; functional arguments in (22) have different domains, so we have to treat it more carefully. There is a continuous extension $\tilde{z}\colon \varphi(E)+D \to \mathbb{R}$ of z, Lipschitz continuous in x with the same constant. Define $w,\bar{w}\colon D \to \mathbb{R}$ by

$$\begin{aligned} w(s,y) &= \tilde{z}_{\varphi(t,x)}(s,y) = \tilde{z}(\varphi(t,x)+(s,y)), \\ \bar{w}(s,y) &= \tilde{z}_{\varphi(t,\bar{x})}(s,y) = \tilde{z}(\varphi(t,\bar{x})+(s,y)). \end{aligned}$$

Note that $z_{\varphi(t,x)}$ and $\tilde{z}_{\varphi(t,x)}$ are equivalent after the restriction to $\mathscr{D}[\varphi(t,x)]$. Hence, by Condition (V),

$$\begin{aligned} |F(t,x,z_{\varphi(t,x)}) - F(t,\bar{x},z_{\varphi(t,\bar{x})})| &= |F(t,x,\tilde{z}_{\varphi(t,x)}) - F(t,\bar{x},\tilde{z}_{\varphi(t,\bar{x})})| \\ &\leq \gamma(t,\mu(a))\|x-\bar{x}\| + Z(t)\|w-\bar{w}\| \\ &\leq \gamma(t,\mu(a))\|x-\bar{x}\| + Z(t)\exp\{\alpha \textstyle\int_0^t \beta(\tau)\,d\tau\}|z|_{\alpha,\beta}L_\phi\|x-\bar{x}\|. \quad \blacksquare \end{aligned}$$

Lemma 3.4 *Suppose that Assumptions H [f], H [φ], H [ψ], H [F] are satisfied, the relation* (18) *holds, and the data $\psi \in C^{0.1}[s](\overline{E_0 \cup \partial_0 E})$. Then*

$$\mathbb{F}\colon C_\psi[d;\alpha,\beta;\mu] \to C_\psi[d;\alpha,\beta;\mu].$$

with $\alpha = 2L_g L_\phi$, $\beta = Z$ and some $d \in \mathbb{R}_+$.

Proof. In the case (i) of the Assumption H [F], relation $|F[z](t,x)| \leq \mu(t)$ is clear. Consider the case (ii). Note that the function μ, given by (20), fulfils the integral inequality

$$\mu(t) \geq s_0 + \int_{\zeta(t,x)}^t \big(B_F + \lambda(\tau)\mu(\tau)\big)\,d\tau,$$

for any $(t,x) \in E$, whence the asserted boundedness of $F[z](t,x)$.

Let us prove the Lipschitz continuity of $F[z](t,\cdot)$. We have

$$\begin{aligned}&|\psi(S(t,x))-\psi(S(t,\bar{x}))|\\&\qquad=|\psi(\zeta(t,x),g(\zeta(t,x),t,x))-\psi(\zeta(t,\bar{x}),g(\zeta(t,\bar{x}),t,\bar{x}))|\le A_1\|x-\bar{x}\|\end{aligned}$$

with $A_1=s_1((1+B_f)L_\zeta L_\Omega+L_g)$, and, by (22),

$$|F(Q[z](\tau,t,x))-F(Q[z](\tau,t,\bar{x}))|\le L(\tau)L_g\|x-\bar{x}\|.$$

Using (21) and (22), we thus get

$$\begin{aligned}&|\mathbb{F}[z](t,x)-\mathbb{F}[z](t,\bar{x})|\\&\le A_1\|x-\bar{x}\|+\left|\int_{\zeta(t,x)}^{\zeta(t,\bar{x})}F(Q[z](\tau,t,\bar{x}))\,d\tau\right|+L_g\|x-\bar{x}\|\int_{\zeta(t,x)}^{t}L(\tau)\,d\tau\\&\le\Big(A_1+BL_\zeta L_\Omega+L_g\int_0^t L(\tau)d\tau\Big)\|x-\bar{x}\|\\&\le\Big(A_1+BL_\zeta L_\Omega+L_g\int_0^t\gamma(\tau,\mu(a))d\tau\\&\qquad\qquad+L_g\int_0^t\beta(\tau)\exp\{\alpha\textstyle\int_0^\tau\beta(s)ds\}|z|_{\alpha,\beta}L_\phi d\tau\Big)\|x-\bar{x}\|.\end{aligned}$$

Let $A=A_1+BL_\zeta L_\Omega+L_g\int_0^a\gamma(\tau,\mu(a))d\tau$. Then, for $x\neq\bar{x}$, quotient $\frac{|\mathbb{F}[z](t,x)-\mathbb{F}[z](t,\bar{x})|}{\|x-\bar{x}\|}$ is uniformly bounded by

$$\begin{aligned}&A\exp\{\alpha\textstyle\int_0^t\beta(\tau)d\tau\}+L_gL_\phi|z|_{\alpha,\beta}\displaystyle\int_0^t\beta(\tau)\exp\{\alpha\textstyle\int_0^\tau\beta(s)ds\}d\tau\\&\qquad=A\exp\{\alpha\textstyle\int_0^t\beta(\tau)d\tau\}+\dfrac{L_gL_\phi}{\alpha}|z|_{\alpha,\beta}\Big(\exp\{\alpha\textstyle\int_0^t\beta(\tau)d\tau\}-1\Big)\\&\qquad\qquad\le A\exp\{\alpha\textstyle\int_0^t\beta(\tau)d\tau\}+\dfrac{L_gL_\phi}{\alpha}|z|_{\alpha,\beta}\exp\{\alpha\textstyle\int_0^t\beta(\tau)d\tau\}\end{aligned}$$

and hence, taking $\alpha=2L_gL_\phi$ and $d=2A$,

$$\big|F[z]\big|_{\alpha,\beta}\le A+\frac{1}{2}|z|_{\alpha,\beta}\le 2A.$$

Finally, it is easy to see that $F[z]$ is continuous in t. ∎

Theorem 3.5 *Under the assumptions of Lemma 3.2, there is exactly one solution of* (1), (2) *in the space* $C_\psi[d;\alpha,\beta;\mu]$ *with* α, β *and* d *taken from the preceding Lemma.*

Moreover, taken any other $\bar{\psi}$, *satisfying Assumption H [* ψ *] with the same constants as* ψ, *and the respective solution* $\bar{z}\in C_{\bar{\psi}}[d;\alpha,\beta;\mu]$, *we have*

$$\|z-\bar{z}\|_{C(E_t^*)}\le L\int_0^t\|\psi-\bar{\psi}\|_{C(E_0\cup\partial_0E_t)}\,d\tau,\qquad t\in[0,a],\tag{24}$$

with the constant L *independent of* ψ, $\bar{\psi}$.

Proof. Define a norm in $C(E^*)$, equivalent to the supremum norm, by

$$[|u|] = \max\left\{ \|u\|_{C(E_t^*)} \exp\left\{ -2\int_0^t Z(\tau)d\tau \right\} \,:\, t \in [0,a] \right\}.$$

Thanks to (23), it is easy to see that

$$\begin{aligned} |(F[z]-F[\bar{z}])(t,x)| &\le \int_0^t Z(\tau)\|z-\bar{z}\|_{C(E_\tau^*)}\, d\tau \\ &\le [|z-\bar{z}|] \int_0^t Z(\tau) \exp\left\{2\int_0^\tau Z(s)ds\right\} d\tau \\ &= \frac{1}{2}[|z-\bar{z}|] \left[\exp\left\{2\int_0^t Z(\tau)\, d\tau\right\} - 1\right] \\ &< \frac{1}{2}[|z-\bar{z}|] \exp\left\{2\int_0^t Z(\tau)\, d\tau\right\}, \end{aligned}$$

whence $[|F[z]-F[\bar{z}]|] \le \frac{1}{2}[|z-\bar{z}|]$. The space $C_\psi[d;\alpha,\beta;\mu]$ is a Banach space as a closed subset of $(C(\overline{E^*}),[|\cdot|])$, so the Banach theorem yields exactly one fixed point $\tilde{z}$ of F in $C_\psi[d;\alpha,\beta;\mu]$.

Note that, by (18) and by Lemmata 2.9, 2.10, ζ is Lipschitz continuous in t, and so is the function $(t,x)\mapsto \psi(S(t,x))$. Consequently, $F[\tilde{z}]$ is absolutely continuous in t. This allows the application of the chain rule to $\tilde{z}(t,g(t,s,y))$ in what follows.

It is easily seen that the relation

$$\tilde{z}(t,x) = F[\tilde{z}](t,x) \quad \text{on } E$$

is equivalent to

$$\tilde{z}(t,g(t,s,y)) = \psi(s,y) + \int_s^t F(\tau,g(\tau,s,y),\tilde{z}_{\varphi(\tau,g(\tau,s,y))})\, d\tau,$$
$$(s,y)\in E\cap(E_0\cup\partial_0 E), \quad t\in[s,a].$$

By differentiating the above equality with respect to t and by putting $x=g(t,s,y)$ we obtain that $\tilde{z}$ is a Carathéodory solution to (1), (2).

We prove the Lipschitz dependence (24). In a way analogous to that leading to (23), we get

$$|(z-\bar{z})(t,x)| \le \|\psi-\bar{\psi}\|_{C(E_0\cup\partial_0 E_t)} + \int_0^t Z(\tau)\|z-\bar{z}\|_{C(E_\tau^*)}\, d\tau,$$

Taking maximum over E_t and applying the Gronwall inequality, we obtain the assertion with $L = 1+Me^M$, $M=\int_0^a Z(\tau)\, d\tau$. This completes the proof. ∎

References

[1] R. A. Adams, J. J. F. Fournier, *Sobolev Spaces*, Pure and Applied Mathematics 140, Academic Press, Amsterdam, 2003

[2] P. Bassanini and M. C. Galaverni, Un problema ai limiti per sistemi integrodifferenziali nonlinear di tipo iperbolico. *Boll. Un. Mat. Ital.* (1981) **7**, No.18-B, 785–798.

[3] P. Brandi, R. Ceppitelli, Existence, uniqueness and continuous dependence for a hereditary nonlinear functional partial differential equations, Ann. Polon. Math. 47, 1986, 121-136.

[4] P. Brandi, Z. Kamont, A. Salvadori, Existence of generalized solutions of hyperbolic functional differential equations, Nonl. Anal., T.M.A. 50, 2002, 919-940.

[5] R. Ceppitelli, Z. Kamont, Extremal solutions for semilinear differential functional systems in two independent variables, Atti Sem. Mat. Fis. Univ. Modena 42, 1994, 329-341.

[6] C. J. Chyan and G. F. Webb, A model of proliferating cell populations with correlation of mother - daughter mitotic times. *Ann. Mat. pura ed appl.* (1991) **157**, 1–11.

[7] T. Człapiński, On the Cauchy problem for quasilinear hyperbolic systems of partial differential functional equations of the first order, Zeit. Anal. Anwend. 10, 1991, 169-182.

[8] T. Człapiński, On the existence of generalized solutions of nonlinear first order partial differential functional equations in two independent variables, Czechosl. Math. Journ. 41, 1992, 490-506.

[9] M. El-Doma, Analysis of nonlinear integro differential equations arising in age dependent epidemic models. *Nonl. Anal. TMA* (1978) **11**, 913–937.

[10] W. Eichhorn and W. Gleissner, On a functional differential equation arising in the theory of distribution of wealth. *Aequat. Math.* (1985) **28**, 190–198.

[11] P. Grisvard, *Elliptic Problems in Nonsmooth Domains*. Pitman Advanced Publishing Program, Boston, 1985.

[12] M. E. Gurtin and L. F. Murphy, On the optimal harvesting of persistent age - structured population. *J. Math. Biol.* (1981) **13**, 131–148.

[13] Z. Kamont, Hyperbolic Functional Differential Inequalities and Applications, Kluwer Acad. Publ., Dordrecht, 1999.

[14] A. Lasota, M. C. Mackey, and M. Ważewska - Czyżewska, Minimizing therapeutically induced anemia. *J. Math. Biol.* (1981) **13**, 149–158.

[15] A. Lorenzi, An identification problem related to a nonlinear hyperbolic integro differential equation. *Nonl. Anal. TMA* (1994) **22**, 21–44.

[16] A. D. Myshkis, A. S. Slopak, A mixed problem for systems of differential functional equations with partial derivatives and with operators of the Volterra type (Russian), Mat. Sb. (N.S.) 41, 1957, 239-256.

[17] K. Topolski, Generalized solutions of first order partial differential functional equations in two independent variables, Atti Sem. Mat. Fis. Univ. Modena 39, 1991, 669-684.

[18] J. Turo, Global solvability of the mixed problem for first order functional partial differential equations, Ann. Polon. Math. 52, 1991, 231-238.

[19] J. Wu, *Theory and Applications of Partial Functional Differential Equations*. Springer, Berlin, 1996.

Institute of Mathematics, University of Gdańsk,
Wit Stwosz Street 57, 80-952 Gdańsk, Poland
email: czernous@mat.ug.edu.pl,
phone: +48 58 523 25 05, fax: +48 58 341 49 14