On the classification of rational homotopy types of elliptic spaces with homotopy Euler characteristic zero for dim < 8

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Abstract

We classify rational homotopy types of elliptic spaces with homotopy Euler characteristic zero for dim < 8.

1 Introduction

Throughout the paper we consider connected, simply connected spaces.

Definition 1.1. A space X is said to be *elliptic* if $\dim \pi_*(X) \otimes \mathbb{Q} < \infty$ and $\dim H^*(X;\mathbb{Q}) < \infty$.

$\chi_\pi(X) = \sum_p (-1)^p \dim \pi_p(X) \otimes \mathbb{Q}$ is called the *homotopy Euler characteristic*;

$\chi_c(X) = \sum_p (-1)^p \dim H^p(X;\mathbb{Q})$ is called the (*cohomology*) *Euler characteristic*.

Then in general there hold

$$\chi_\pi(X) \leq 0 \quad \text{and} \quad \chi_c(X) \geq 0.$$

Furthermore it is shown in [Ha, Theorem 1, p.175] that the following conditions are equivalent:

(1) $\chi_\pi(X) = 0$, (2) $\chi_c(X) > 0$, (3) $H^*(X;\mathbb{Q})$ is evenly graded,

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and that $H^*(X;\mathbb{Q})$ is a polynomial algebra truncated by a Borel ideal in this case.

The purpose of this paper is to classify the rational homotopy types of elliptic spaces with $\chi_\pi(X) = 0$ for $\dim H^*(X;\mathbb{Q}) < 8$.

By the dimension formula (2.2), the cohomology algebra of such a space is isomorphic to either $\mathbb{Q}[x_1]/(f_1)$ or $\mathbb{Q}[x_1, x_2]/(f_1, f_2)$ as a graded algebra, where (f_1, f_2) is the ideal generated by a regular sequence $\{f_1, f_2\}$, and hence the rational homotopy types of this kind are intrinsically *formal*, that is, two spaces with the isomorphic rational cohomology algebras are rationally homotopy equivalent. Thus, for our purpose, it is sufficient to classify graded algebras of the type $\mathbb{Q}[x_1, x_2]/(f_1, f_2)$.

M.R.Hilali tried in his thesis [Hi] to classify such elliptic rational homotopy types whose dimension of the cohomology algebra is not greater than 6. However his argument seems to be incorrect. Correcting it is a starting point of our work [MS]; in fact, there are infinitely many non-isomorphic $\mathbb{Q}$-algebras A such that

$$A \underset{\mathbb{Q}}{\otimes} \overline{\mathbb{Q}} \cong \overline{\mathbb{Q}}[x_1, x_2]/(x_1^2, x_2^2).$$

Let X be a graded algebra over $\mathbb{Q}$ and K a Galois extension of $\mathbb{Q}$. A graded algebra Y over $\mathbb{Q}$ is said to be a $K/\mathbb{Q}$ form if Y becomes isomorphic to X when the ground field is extended to K. The set of $\mathbb{Q}$-isomorphism classes of X forms a set $E(K/\mathbb{Q}, X)$. It is known that the set $E(K/\mathbb{Q}, X)$ corresponds bijectively to the Galois cohomology $H^1(\mathrm{Gal}(K/\mathbb{Q}), A(K))$, where $A(K)$ denotes the group of K-automorphisms of X (see [W], p.136).

Our result of classifying them is given as follows:

Theorem 1.2. *Let A be the cohomology algebra of an elliptic space with $\chi_\pi = 0$. If $\dim H^*(X;\mathbb{Q}) < 8$, then A is isomorphic to one of the following:*

dim	*isomorphic classes of graded algebras*
1	$\mathbb{Q}$
2	$\{\mathbb{Q}[x]/(x^2), \lvert x\rvert = 2n \mid n \in \mathbb{N}\}$
3	$\{\mathbb{Q}[x]/(x^3), \lvert x\rvert = 2n \mid n \in \mathbb{N}\}$
4	$\{\mathbb{Q}[x]/(x^4), \lvert x\rvert = 2n \mid n \in \mathbb{N}\}$,
	$\{\mathbb{Q}[x_1, x_2]/(x_1^2 + ax_2^2, x_1x_2), \lvert x_1\rvert = \lvert x_2\rvert = 2n \mid a \in \mathbb{Q}^\times/\mathbb{Q}^{\times 2}, n \in \mathbb{N}\}$,
	$\{\mathbb{Q}[x_1, x_2]/(x_1^2, x_2^2), \lvert x_1\rvert = 2n, \lvert x_2\rvert = 2m \mid (n, m) \in \mathbb{N}^2, n \neq m\}$
5	$\{\mathbb{Q}[x]/(x^5), \lvert x\rvert = 2n \mid n \in \mathbb{N}\}$,
	$\{\mathbb{Q}[x_1, x_2]/(x_1x_2, x_1^3 + x_2^2), \lvert x_1\rvert = 4n, \lvert x_2\rvert = 6n \mid n \in \mathbb{N}\}$
6	$\{\mathbb{Q}[x]/(x^6), \lvert x\rvert = 2n \mid n \in \mathbb{N}\}$,
	$\{\mathbb{Q}[x_1, x_2]/(x_1^2 + ax_2^2, sx_1^3 + tx_1^2x_2), \lvert x_1\rvert = \lvert x_2\rvert = 2n \mid (a, [s, t]) \in T, n \in \mathbb{N}\}$,
	$\{\mathbb{Q}[x_1, x_2]/(x_1^2, x_2^3), \lvert x_1\rvert = 2n, \lvert x_2\rvert = 2m \mid (n, m) \in \mathbb{N}, n \neq m\}$,
	$\{\mathbb{Q}[x_1, x_2]/(x_1x_2, x_2^2 + ax_1^4), \lvert x_1\rvert = 2n, \lvert x_2\rvert = 4n \mid n \in \mathbb{N}, a \in \mathbb{Q}^\times/\mathbb{Q}^{\times 2}\}$
7	$\{\mathbb{Q}[x]/(x^7), \lvert x\rvert = 2n \mid n \in \mathbb{N}\}$,
	$\{\mathbb{Q}[x_1, x_2]/(x_1^3 + x_2^2, x_1^2x_2), \lvert x_1\rvert = 4n, \lvert x_2\rvert = 6n \mid n \in \mathbb{N}\}$,
	$\{\mathbb{Q}[x_1, x_2]/(x_1x_2, x_1^5 + x_2^2), \lvert x_1\rvert = 4n, \lvert x_2\rvert = 10n \mid n \in \mathbb{N}\}$,
	$\{\mathbb{Q}[x_1, x_2]/(x_1x_2, x_1^4 + x_2^3), \lvert x_1\rvert = 6n, \lvert x_2\rvert = 8n \mid n \in \mathbb{N}\}$

The set T in the table is defined as follows. Let

$$P^1(\mathbb{Q}) = \mathbb{Q} \times \mathbb{Q} - \{(0,0)\}/\sim,$$

where $(t_1, s_1) \sim (t_2, s_2)$ if and only if there is an element $r \in \mathbb{Q}^\times$ such that $rt_1 = t_2$ and $rs_1 = s_2$. Set $M_1 = \mathbb{Q}^\times \times P^1(\mathbb{Q})$ and $M_2 = \mathbb{Q}^{\times 2} \times P^1(\mathbb{Q})$. We define an equivalence relation $\sim$ on $M_1 \backslash M_2$ as follows: $(\alpha_1, [s_1, t_1]) \sim (\alpha_2, [s_2, t_2])$ if and only if the following (1) and (2) are satisfied:

1. $\alpha_1 \cdot \alpha_2 \in \mathbb{Q}^{\times 2}$; (then the quadratic extensions $\mathbb{Q}(\sqrt{\alpha_1})$ and $\mathbb{Q}(\sqrt{\alpha_2})$ coincide, which we denote by $\mathbb{K}$.)
2. $$\frac{t_2 - s_2\sqrt{\alpha_2}}{t_2 + s_2\sqrt{\alpha_2}} \cdot \frac{t_1 + s_1\sqrt{\alpha_1}}{t_1 - s_1\sqrt{\alpha_1}} \in {\mathbb{K}_1}^{\times 3},$$
   where $\mathbb{K}_1$ consists of elements of $\mathbb{K}$ whose norms are 1.

Let $\tilde{M}_2 = \{(r^2, [s,t]) \in M_2 \mid t \pm sr \neq 0\}$, and on $\tilde{M}_2$ we define an equivalence relation $\sim$ as follows:

$$(r_1^2, [s_1, t_1]) \sim (r_2^2, [s_2, t_2]) \iff \frac{t_2 - s_2 r_2}{t_2 + s_2 r_2} \cdot \frac{t_1 + s_1 r_1}{t_1 - s_1 r_1} \in \mathbb{Q}^{\times 3}.$$

We set

$$T = (M_1 \backslash M_2)/\sim \cup \tilde{M}_2/\sim.$$

Then an element $(\alpha, [s,t]) \in T$ corresponds to the isomorphism classes of the algebras

$$\mathbb{Q}[x_1, x_2]/(x_1^2 - \alpha x_2^2, sx_1^3 + tx_1^2 x_2)$$

of regular type. (See the last paragraph of Section 5 for details.)

We denote by B and C the family given in the second line of dim 4 and 6 respectively:

$$B = \{\mathbb{Q}[x_1, x_2]/(x_1^2 + ax_2^2, x_1x_2), |x_1| = |x_2| = 2n \mid a \in \mathbb{Q}^\times/\mathbb{Q}^{\times 2}, n \in \mathbb{N}\},$$

$$C = \{\mathbb{Q}[x_1, x_2]/(x_1^2 + ax_2^2, sx_1^3 + tx_1^2 x_2), |x_1| = |x_2| = 2n \mid (a, [s,t]) \in T, n \in \mathbb{N}\}.$$

All the elements of the family in B (resp. C) are isomorphic as $\overline{\mathbb{Q}}$-algebra after tensoring $\overline{\mathbb{Q}}$ over $\mathbb{Q}$. However they give us a family of infinitely many non isomorphic $\mathbb{Q}$-algebras in dimensions 4 and 6 even when ignoring the gradings.

The spaces representing the algebras in the table above can be constructed as follows:

(1) The space X such that $H^*(X;\mathbb{Q}) \cong \mathbb{Q}[x]/(x^k)$; Let $\varphi : K(\mathbb{Q}, |x|) \to K(\mathbb{Q}, k|x|)$ be a map representing the element

$$x^k \in \mathbb{Q}[x] \cong H^*(K(\mathbb{Q}, |x|); \mathbb{Q}).$$

Then X is given as the homotopy fibre of φ.

(2) The space X such that $H^*(X;\mathbb{Q}) \cong \mathbb{Q}[x_1, x_2]/(f_1, f_2)$, where (f_1, f_2) is the ideal generated by elements $f_i \in \mathbb{Q}[x_1, x_2]$; Let $\varphi_i : K(\mathbb{Q}, |x_1|) \times K(\mathbb{Q}, |x_2|) \to$

$K(\mathbb{Q}, |f_i|)$ be a map representing the element $f_i \in \mathbb{Q}[x_1, x_2] \cong H^*(K(\mathbb{Q}, |x_1|) \times K(\mathbb{Q}, |x_2|); \mathbb{Q})$ for $i = 1, 2$ and let F be the homotopy fibre of φ_1. Then X is given as the homotopy fibre of the composite map

$$\varphi_2 \circ i : F \to K(\mathbb{Q}, |x_1|) \times K(\mathbb{Q}, |x_2|) \to K(\mathbb{Q}, |f_2|),$$

where i is the inclusion of the fibre.

Our method to classify the algebras is based on the dimension formula (2.2) for $n = 2$:

$$\dim_{\mathbb{Q}} \mathbb{Q}[x_1, x_2]/(f_1, f_2) = |f_1| \cdot |f_2| / |x_1| \cdot |x_2|$$

due to Koszul, where $|x_i|$ and $|f_i|$ denote the degree of x_i and f_i respectively.

The present work is the revised version of [MS]. However there are no alterations in the results but some minor modifications in the expressions. During these past years, following our method in [MS], Kono-Tamamura obtain in [KT1] and [KT2] similar results in dimensions 10, 11, 13; their arguments are entirely the same as ours given in [MS].

The paper is organized as follows. In Section 2 we consider the case of dimensions $1, 2, 3$; in Section 3 the case of dimension 4; in Section 4 the case of dimension 5; in Section 5 the case of dimension 6; in Section 6 the case of dimension 7.

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2 The case of dimensions 1, 2, 3

Let $\{f_1, \cdots, f_n\}$ be a regular sequence of graded elements in a polynomial ring $\mathbb{Q}[x_1, \cdots, x_n]$. We can assume that each f_i $(i = 1, \cdots, n)$ has no constant or linear terms and that

(2.1) $$|x_1| \leq \cdots \leq |x_n|, \quad |f_1| \leq \cdots \leq |f_n|.$$

Put $A = \mathbb{Q}[x_1, \cdots, x_n]/(f_1, \cdots, f_n)$. Then by the *dimension formula* (see [FHT; (32.14), p.446]), we have

(2.2) $$\dim_{\mathbb{Q}} A = |f_1| \cdots |f_n| / |x_1| \cdots |x_n|.$$

Lemma 2.1. $2|x_i| \leq |f_i|$ *for* $i = 1, \cdots, n$.

Proof. We prove by induction on i. Since f_1 has no linear terms, we have $|f_1| \geq 2|x_1|$. As inductive hypothesis we assume that $2|x_i| \leq |f_i|$ for $i = 1, \cdots, k$. If $|x_k| = |x_{k+1}|$, then $|f_{k+1}| \geq |f_k| \geq 2|x_k| = 2|x_{k+1}|$. Let $|x_{k+1}| > |x_k|$ and suppose $|f_{k+1}| < 2|x_{k+1}|$. Then f_{k+1} is contained in the ideal ($x_{k+1}x_i$ for $i \leq k$, x_ix_j for $i, j \leq k$), and hence we see that $f_{k+1} \in (x_1, \cdots, x_k)$, the ideal generated by $\{x_1, \cdots, x_k\}$. Thus $f_1, \cdots, f_{k+1}$ are all contained in the ideal $(x_1, \cdots, x_k)$, that is, $(f_1, \cdots, f_{k+1}) \subset (x_1, \cdots, x_k)$. Then, for (any irreducible component of) varieties of $\overline{\mathbb{Q}}$-points, we have

$$V(f_1, \cdots, f_{k+1}) \supset V(x_1, \cdots, x_k),$$

where

$$\begin{aligned} V(f_1, \cdots, f_{k+1}) &= \{\mathbf{x} \in \overline{\mathbb{Q}}^n \mid f_i(\mathbf{x}) = 0, \quad 1 \le i \le k+1\}, \\ V(x_1, \cdots, x_k) &= \{\mathbf{x} \in \overline{\mathbb{Q}}^n \mid x_i = 0, \quad 1 \le i \le k\}. \end{aligned}$$

Hence we have

$$\dim V(f_1, \cdots, f_{k+1}) \ge \dim V(x_1, \cdots, x_k) = n - k,$$

which contradicts the fact that $\{f_1, \cdots, f_{k+1}\}$ is a regular sequence. ∎

Combining (2) and Lemma 2.1, we have

$$\dim_{\mathbb{Q}} A \ge 2^n. \tag{2.3}$$

If $\dim_{\mathbb{Q}} A = 1$, then $n = 0$ and $A \cong \mathbb{Q}$. If $\dim_{\mathbb{Q}} A = 2$, then $n = 1$ and $A \cong \mathbb{Q}[x]/(x^2)$. If $\dim_{\mathbb{Q}} A = 3$, then $n = 1$ and $A \cong \mathbb{Q}[x]/(x^3)$.

3 The case of dimension 4

Let A be the cohomology algebra of an elliptic space with $\chi_\pi = 0$ such that $\dim_{\mathbb{Q}} A = 4$. Then $n = 1$ or 2 in (2). If $n = 1$, then $A \cong \mathbb{Q}[x]/(x^4)$. If $n = 2$, then it follows from Lemma 2.1 and (2.2) that

$$|f_1| = 2|x_1|, \quad |f_2| = 2|x_2|.$$

If $|x_1| < |x_2|$, then $(f_1) = (x_1^2)$, and f_2 is of the following form:

$$f_2 = ax_2^2 + bx_1^{k_1}x_2 + cx_1^{k_2}$$

with $a \neq 0$, where $k_2 > k_1 \ge 2$. Hence we obtain that

$$(f_1, f_2) = (x_1^2, x_2^2).$$

If $|x_1| = |x_2|$, then we may set

$$f_1 = ax_1^2 + bx_1x_2 + cx_2^2, \quad f_2 = dx_1^2 + ex_1x_2 + fx_2^2 \qquad (a, b, c, d, e, f \in \mathbb{Q}).$$

If $a = c = 0$, then $(f_1, f_2) = (x_1x_2, x_1^2 + \alpha x_2^2)$, where $\alpha = \dfrac{f}{d} \in \mathbb{Q}^\times$. If $a \neq 0$, by setting $a\left(x_1 + \dfrac{b}{2a}x_2\right) = u_1$, we have

$$f_1 = u_1^2 + \alpha x_2^2, \quad \alpha = \frac{4ac - b^2}{4a}.$$

By using f_1, we obtain the form $(f_1, f_2) = (u_1^2 + \alpha x_2^2, gu_1x_2 + hx_2^2)$. If $g = 0$, then we have $(f_1, f_2) = (u_1^2, x_2^2)$. If $g \neq 0$, we set $v_1 = gu_1 + hx_2$. Then $f_2 = v_1x_2$; using f_2 we have $(f_1, f_2) = (v_1^2 + \beta x_2^2, v_1x_2)$ *for some* $\beta \in \mathbb{Q}^\times$. The case $c \neq 0$ is similar. Thus we have shown the following

Lemma 3.1. *Let f_1 and f_2 be homogeneous polynomials of degree 2. Then $\mathbb{Q}[x_1, x_2]/(f_1, f_2)$ is isomorphic to $\mathbb{Q}[x_1, x_2]/(x_1^2 + \alpha x_2^2, x_1x_2)$ for some $\alpha \in \mathbb{Q}^\times$.*

Remark. $\mathbb{Q}[x_1, x_2]/(x_1^2, x_2^2)$ is isomorphic to $\mathbb{Q}[x_1, x_2]/(x_1^2 + x_2^2, x_1x_2)$ as $\mathbb{Q}$- algebras.

Notation. $A_\gamma = \mathbb{Q}[x_1, x_2]/(x_1^2 + \gamma x_2^2, x_1x_2)$ for $\gamma \in \mathbb{Q}^\times$.

Proposition 3.2. *The algebras A_α and A_β $(\alpha, \beta \in \mathbb{Q}^\times)$ are isomorphic if and only if $\alpha \cdot \beta^{-1} \in \mathbb{Q}^{\times 2}$.*

Proof. Suppose that there is an isomorphism $\varphi : A_\alpha \to A_\beta$. Then we can set

$$\varphi(x_1) = p_1x_1 + q_1x_2, \quad \varphi(x_2) = p_2x_1 + q_2x_2 \qquad (p_i, q_i \in \mathbb{Q}).$$

Then we have

$$\begin{aligned}\varphi(x_1^2 + \alpha x_2^2) &= (p_1^2 + \alpha p_2^2)x_1^2 + 2(p_1q_1 + \alpha p_2q_2)x_1x_2 + (q_1^2 + \alpha q_2^2)x_2^2,\\ \varphi(x_1x_2) &= p_1p_2x_1^2 + (p_1q_2 + p_2q_1)x_1x_2 + q_1q_2x_2^2.\end{aligned}$$

Since these elements are zero in A_β, we have $(p_1^2 + \alpha p_2^2)\beta = q_1^2 + \alpha q_2^2$ and $p_1p_2\beta = q_1q_2$. Thus we have

$$\alpha\beta^{-1} = (p_1/q_2)^2 \in \mathbb{Q}^{\times 2}.$$

Conversely, if $\alpha\beta^{-1} \in \mathbb{Q}^{\times 2}$, the map $\varphi : A_\alpha \to A_\beta$ defined by

$$\varphi(x_1) = x_1, \quad \varphi(x_2) = rx_2$$

gives an isomorphism φ, where r is an element of $\mathbb{Q}^\times$ such that $r^2 = \alpha^{-1}\beta$. ■

4 The case of dimension 5

Let A be the cohomology algebra of an elliptic space with $\chi_\pi = 0$ such that $\dim_{\mathbb{Q}} A = 5$. Then $n = 1$ or 2 in (2.2). If $n = 1$, then $A \cong \mathbb{Q}[x]/(x^5)$. If $n = 2$, then we have $|f_1| \cdot |f_2| = 5|x_1| \cdot |x_2|$ in (2.2).

(a) Assume that $|f_1|$ is an integer multiple of $|x_1|$, that is, $|f_1| = k|x_1|$ for some integer $k \geq 2$. By Lemma 2.1 we have

$$2|x_2| \leq |f_2| = \frac{5}{k}|x_2|.$$

Hence we have $k = 2$. Then f_2 is contained in the ideal generated by x_1. By regularity f_1 is not contained in the ideal (x_1). Then $|f_1| = \ell|x_2|$ for some integer $\ell \geq 2$. Then we have

$$2|x_2| \leq |f_1| = 2|x_1|.$$

Hence we have $|x_1| = |x_2|$. But this contradicts that $|f_2| = \frac{5}{2}|x_2|$.

(b) Assume that $|f_1|$ is an integer multiple of $|x_2|$, that is, $|f_1| = k|x_2|$ for some integer $k \geq 1$. Then by Lemma 2.1 we have

$$2|x_2| \leq |f_2| = \frac{5}{k}|x_1| \leq \frac{5}{k}|x_2|.$$

Thus we have $k = 1$ or 2.

If $k = 1$, then f_1 is a polynomial of x_1 since f_1 has no linear terms. But then $|f_1|$ is an integer multiple of $|x_1|$, which is impossible by (a).

If $k = 2$, then f_2 is contained in the ideal (x_2), since $|f_2| = \frac{5}{2}|x_1|$. By regularity f_1 is not contained in the ideal (x_2). This implies that $|f_1|$ is an integer multiple of $|x_1|$, which is impossible by (a).

(c) Thus $|f_1|$ is neither integer multiple of $|x_1|$ nor of $|x_2|$, that is, f_1 is contained in both (x_1) and (x_2). Hence f_2 is an integer multiple of both $|x_1|$ and $|x_2|$, that is, $|f_2| = k_1|x_1| = k_2|x_2|$ for some integers $k_1, k_2 \geq 2$. Then from the inequality

$$2|x_1| \leq |f_1| = \frac{5}{k_2}|x_1| \leq \frac{5}{k_2}|x_2|,$$

we deduce $k_2 = 2$. If $k_1 = 2$, then $|x_1| = |x_2|$, and so $|f_1|$ is an integer multiple of $|x_1|$. This contradicts the assumptions. Thus $k_1 \geq 3$. Then we have

$$\frac{5}{2}|x_1| = |f_1| \geq |x_1| + |x_2| = |x_1| + \frac{k_1}{2}|x_1|,$$

which implies that $k_1 = 3$. Then we have

$$|f_1| = |x_1| + |x_2|, \quad |f_2| = 2|x_2|, \quad 3|x_1| = 2|x_2|.$$

Thus the only possibility is that

$$(f_1, f_2) = (x_1x_2, x_1^3 + \alpha x_2^2), \qquad \alpha \in \mathbb{Q}^\times.$$

Proposition 4.1. *For any $\alpha, \beta \in \mathbb{Q}^\times$, there is a graded algebra isomorphism*

$$\varphi : \frac{\mathbb{Q}[x_1, x_2]}{(x_1x_2, x_1^3 + \alpha x_2^2)} \longrightarrow \frac{\mathbb{Q}[x_1, x_2]}{(x_1x_2, x_1^3 + \beta x_2^2)}.$$

Proof. Since $|x_1| < |x_2|$, the graded map is of the following form:

$$\varphi(x_1) = p_1x_1, \quad \varphi(x_2) = q_2x_2$$

for some $p_1, q_2 \in \mathbb{Q}^\times$. This correspondence φ defines an isomorphism if and only if $p_1^3\beta = \alpha q_2^2$. Hence by setting $p_1 = q_2 = \alpha\beta^{-1} \in \mathbb{Q}^\times$, we obtain the desired isomorphism. ∎

5 The case of dimension 6

Let A be the cohomology algebra of an elliptic space with $\chi_\pi = 0$ such that $\dim_{\mathbb{Q}} A = 6$. Then $n = 1$ or 2 in (2). If $n = 1$, then $A \cong \mathbb{Q}[x]/(x^6)$. So we let $n = 2$ for rest of the section.

First we consider the case $|x_1| < |x_2|$.

(a) Assume that $|f_1|$ is an integer multiple of $|x_2|$, that is, $|f_1| = k|x_2|$ for some integer $k \geq 1$. Then we have

$$2|x_2| \leq |f_2| = \frac{6}{k}|x_1| < \frac{6}{k}|x_2|,$$

which implies that $k = 1$ or 2.

If $k = 1$, then $f_1 = x_1^m$ and $|x_2| = m|x_1|$ with $m \geq 2$. By the dimension formula (2.2) for $n = 2$ we have

$$|f_2| = \frac{6}{m}|x_2|.$$

As f_2 is not contained in the ideal (x_1), we deduce that $|f_2|$ is an integer multiple of $|x_2|$. Hence $m = 2$ or 3. If $m = 2$, then $(f_1, f_2) = (x_1^2, x_2^3)$ with $|x_2| = 2|x_1|$. If $m = 3$, then $(f_1, f_2) = (x_1^3, x_2^2)$.

If $k = 2$, then $|f_1| = 2|x_2|$ and $|f_2| = 3|x_1|$. Hence we have $|x_1| < |x_2| \leq \frac{3}{2}|x_1|$. Suppose $|x_1| < |x_2| < \frac{3}{2}|x_1|$. Then, since we have $|x_1| + |x_2| < 2|x_2| = |f_1| < 3|x_1| = |f_2| < 2|x_1| + |x_2|$, we can deduce

$$(f_1, f_2) = (x_2^2, x_1^3).$$

Suppose $|x_2| = \frac{3}{2}|x_1|$. Then we have

$$f_1 = ax_1^3 + bx_2^2, \quad f_2 = cx_1^3 + dx_2^2$$

for some $a, b, c, d \in \mathbb{Q}$ satisfying $ad - bc \neq 0$. Hence $(f_1, f_2) = (x_1^3, x_2^2)$.

(b) Assume that $|f_1|$ is an integer multiple of $|x_1|$ and not of $|x_2|$, that is, $|f_1| = k|x_1|$ for some integer $k \geq 2$. If $k \geq 4$, then $|f_2| \leq \frac{3}{2}|x_2|$ and $|f_1|$ is an integer multiple of $|x_2|$, which is not allowed. Hence $k = 2$ or 3.

If $k = 2$, then $|f_1| = 2|x_1|$ and $|f_2| = 3|x_2|$. Thus we have

$$(f_1, f_2) = (x_1^2, x_2^3).$$

If $k = 3$, then $|f_1| = 3|x_1|$ and $|f_2| = 2|x_2|$. If $|x_2| \neq 2|x_1|$, we see $(f_1, f_2) = (x_1^3, x_2^2)$.

If $|x_2| = 2|x_1|$, then we have

$$(f_1, f_2) = (ax_1^3 + bx_1x_2, cx_2^2 + dx_1^4)$$

for some $a, b, c, d \in \mathbb{Q}$ such that $a^2c + b^2d \neq 0$ and $c \neq 0$.

Proposition 5.1. *The graded algebras* $\mathbb{Q}[x_1, x_2]/(ax_1^3 + bx_1x_2, cx_2^2 + dx_1^4)$*, where* $a, b, c, d \in \mathbb{Q}$*, such that* $a^2c + b^2d \neq 0$ *and that* $c \neq 0$ *are isomorphic to one of the following*

$$\mathbb{Q}[x_1, x_2]/(x_1x_2, x_2^2 + \alpha x_1^4) \text{ with } \alpha \in \mathbb{Q}^\times, \quad \mathbb{Q}[x_1, x_2]/(x_1^3, x_2^2).$$

Moreover $\mathbb{Q}[x_1, x_2]/(x_1x_2, x_2^2 + \alpha x_1^4)$ *and* $\mathbb{Q}[x_1, x_2]/(x_1x_2, x_2^2 + \beta x_1^4)$ *are isomorphic if and only if* $\alpha^{-1} \cdot \beta \in \mathbb{Q}^{\times 2}$*.*

Proof. If $b \neq 0$, we set $ax_1^2 + bx_2 = X_2$. Then

$$\begin{aligned} (f_1, f_2) &= (x_1X_2, \frac{c}{b^2}X_2^2 + (\frac{a^2c}{b^2} + d)x_1^4) \\ &= (x_1X_2, X_2^2 + \alpha x_1^4), \text{ where } \alpha = \frac{a^2c + b^2d}{c} \in \mathbb{Q}^\times. \end{aligned}$$

The second part of the proposition follows from an easy calculation.

If $b = 0$, then they are isomorphic to $\mathbb{Q}[x_1, x_2]/(x_1^3, x_2^2)$.

(c) If $|f_1|$ is not an integer multiple of $|x_1|$ and not of $|x_2|$, then by the regularity $|f_2|$ is an integer multiple of $|x_2|$. Let $|f_2| = k|x_2|$ for some integer $k \geq 2$. Then $|f_1| = \dfrac{6}{k}|x_1|$ and $k \leq 3$. Hence $k = 2$ or 3, and so we have $|f_1| = 2|x_1|$ or $3|x_1|$, which is not allowed. ∎

The case $n = 2$ and $|x_1| < |x_2|$ can be summarized as follows.

Proposition 5.2. *The set of isomorphism classes of graded algebras of dimension 6 with* $n = 2$ *satisfying the condition* $|x_1| \neq |x_2|$ *are*

$$\{\mathbb{Q}[x_1, x_2]/(x_1^2, x_2^3), |x_1| = 2n, |x_2| = 2m \mid (n, m) \in \mathbb{N}^2, n \neq m\},$$

$$\{\mathbb{Q}[x_1, x_2]/(x_1x_2, x_2^2 + \alpha x_1^4), |x_1| = 2n, |x_2| = 4n \mid n \in \mathbb{N}, \alpha \in \mathbb{Q}^\times/\mathbb{Q}^{\times 2}\}.$$

We consider the case $|x_1| = |x_2|$. Then f_1 and f_2 are homogeneous polynomials of degree 2 and 3 respectively. As in Lemma 3.1, we may set

$$f_1 = x_1^2 - \alpha x_2^2, \qquad \alpha \in \mathbb{Q}.$$

By the same way as in Proposition 3.2, we have the following: If there is an isomorphism

$$\mathbb{Q}[x_1, x_2]/(x_1^2 - \alpha_1 x_2^2, f_2) \longrightarrow \mathbb{Q}[x_1, x_2]/(x_1^2 - \alpha_2 x_2^2, f_2'),$$

then we have

(1) $\alpha_1 = \alpha_2 = 0$ or (2) $\alpha_1 \cdot \alpha_2 \in \mathbb{Q}^{\times 2}$.

For the case (1), we have isomorphisms

$$\mathbb{Q}[x_1, x_2](x_1^2, f_2) \cong \mathbb{Q}[x_1, x_2](x_1^2, x_2^3 + ax_1x_2^2) \cong \mathbb{Q}[x_1, x_2]/(x_1^2, x_2^3).$$

Next we consider the case (2). Assume that $\alpha_1 \in \mathbb{Q}^\times$ and $\alpha_1 \notin \mathbb{Q}^{\times 2}$ and that there is an isomorphism

$$\varphi : \mathbb{Q}[x_1, x_2]/(x_1^2 - \alpha_1 x_2^2, f_1) \longrightarrow \mathbb{Q}[x_1, x_2]/(x_1^2 - \alpha_2 x_2^2, f_2)$$

defined by
$$\varphi(x_1) = px_1 + qx_2, \quad \varphi(x_2) = rx_1 + sx_2$$
with $p, q, r, s \in \mathbb{Q}^\times$. Then $pq = \alpha_1 rs$ and
$$-\alpha_2 = \frac{q^2 - s^2\alpha_1}{p^2 - r^2\alpha_1} = -\frac{qs}{rp}, \text{ so } \alpha_1 \cdot \alpha_2 = \left(\frac{q}{r}\right)^2 \in \mathbb{Q}^{\times 2}.$$
The case that one of p, q, r, s is zero is similar.

So we set $\alpha_2 = r^2\alpha_1$ for some $r \in \mathbb{Q}^\times$. The polynomials f_2, f_2' can be chosen as
$$f_2 = s_1x_1^3 + t_1x_1^2x_2, \;\; f_2' = s_2x_1^3 + t_2x_1^2x_2$$
with some $s_i, t_i \in \mathbb{Q}$ $(i = 1, 2)$. Set

(5.3) $$X_1 = x_1 + \sqrt{\alpha_1}x_2, \quad X_2 = x_1 - \sqrt{\alpha_1}x_2.$$

Let $\mathbb{K} = \mathbb{Q}(\sqrt{\alpha_1})$ be the quadratic field. Then we have an isomorphism
$$\mathbb{Q}[x_1, x_2]/(x_1^2 - \alpha_1x_2^2, f_1) \underset{\mathbb{Q}}{\otimes} \mathbb{K} \cong \mathbb{K}[X_1, X_2]/(X_1X_2, \bar{f}_1),$$
where $\bar{f}_1 = (t_1 + s_1\sqrt{\alpha_1})X_1^3 + (-t_1 + s_1\sqrt{\alpha_1})X_2^3$. Hence φ induces an isomorphism
$$\overline{\varphi} : \mathbb{K}[X_1, X_2]/(X_1X_2, X_1^3 + a_1X_2^3) \longrightarrow \mathbb{K}[X_1, X_2]/(X_1X_2, X_1^3 + a_2X_2^3),$$
where $a_1 = \dfrac{-t_1 + s_1\sqrt{\alpha_1}}{t_1 + s_1\sqrt{\alpha_1}}$ and $a_2 = \dfrac{-t_2 + s_2r\sqrt{\alpha_1}}{t_2 + s_2r\sqrt{\alpha_1}}$. Remark here that $a_1a_2 \neq 0$ by the regularity of the ideals appearing in the above.

Let
$$\overline{\varphi}(X_i) = p_iX_1 + q_iX_2, \;\; p_i, q_i \in \mathbb{K}$$
for $i = 1, 2$. We have $p_1p_2 = 0$ and $q_1q_2 = 0$, since $\overline{\varphi}(X_1X_2) \in (X_1X_2)$. Thus $p_2 = q_1 = 0$ or $p_1 = q_2 = 0$.

First, we consider the case $p_2 = q_1 = 0$. Then we have $p_1q_2 \neq 0$ and that

(5.4) $$a_2a_1^{-1} = (q_2p_1^{-1})^3.$$

It follows from (3) that

(5.5) $$\begin{aligned} \overline{\varphi}(x_1) &= \frac{1}{2}\{(p_1 + q_2)x_1 + \sqrt{\alpha_1}(p_1 - q_2)x_2\}, \\ \overline{\varphi}(x_2) &= \frac{1}{2\sqrt{\alpha_1}}\{(p_1 - q_2)x_2 + \sqrt{\alpha_1}(p_1 + q_2)x_2\}. \end{aligned}$$

Since $\overline{\varphi}$ is defined over $\mathbb{Q}$, we have
$$p_1 + q_2 \in \mathbb{Q} \quad \text{and} \quad (p_1 - q_2)\sqrt{\alpha_1} \in \mathbb{Q},$$
which implies that p_1 and q_2 are conjugate elements over $\mathbb{Q}$ by the equalities (5.5). Then $q_2p_1^{-1}$ are of the form $u^\sigma u^{-1}$, where u^σ is the conjugate of $u \in \mathbb{K}^\times$

if we take $u = p_1$ and $q_2 = u^\sigma$. By Hilbert's Theorem 90 (see [M; p.93]), the set $\{u^\sigma u^{-1} | u \in \mathbb{K}^\times\}$ coincides with the set $\mathbb{K}_1^\times = \{\gamma \in \mathbb{K}^\times | N_{\mathbb{K}}(\gamma) = 1\}$, where $N_{\mathbb{K}}(\gamma)$ is the norm $c^2 - \alpha_1 d^2$ for the element $\gamma = c + d\sqrt{\alpha_1}$. It follows from the condition $a_2 a_1^{-1} \in \mathbb{K}_1^{\times 3}$ that

$$\frac{t_2 - s_2 r\sqrt{\alpha_1}}{t_2 + s_2 r\sqrt{\alpha_1}} \cdot \frac{t_1 - s_1\sqrt{\alpha_1}}{t_1 + s_1\sqrt{\alpha_1}} \in \mathbb{K}_1^{\times 3}. \tag{5.6}$$

For the case $p_1 = q_2 = 0$, quite similarly to the above we have $p_2 q_1 \neq 0$ and that

$$a_2 a_1^{-1} = (q_1 p_2^{-1})^3.$$

For the same reasons as the above, p_2 and q_1 are conjugate over $\mathbb{Q}$. Hence we also have $a_2 a_1^{-1} \in \mathbb{K}_1^{\times 3}$.

Conversely, we have

Proposition 5.3. *Let $(\alpha_i, [s_i, t_i])$ be elements of $M_1 \backslash M_2$ $(i = 1, 2)$. If $(\alpha_1, [s_1, t_1])$ and $(\alpha_2, [s_2, t_2])$ are equivalent, then there is a graded algebra isomorphism*

$$\overline{\varphi} : \mathbb{Q}[x_1, x_2]/(x_1^2 - \alpha_1 x_2^2, s_1 x_1^3 + t_1 x_1^2 x_2) \to \mathbb{Q}[x_1, x_2]/(x_1^2 - \alpha_2 x_2^2, s_2 x_1^3 + t_2 x_1^2 x_2^3).$$

(See the statement below Theorem 1.2 in Section 1 for the definitions of M_1, M_2 and the equivalence relation.)

Proof. Since $(\alpha_1, [s_1, t_1])$ and $(\alpha_2, [s_2, t_2])$ are equivalent, there is $r \in \mathbb{Q}^\times$ so that $\alpha_2 = r^2 \alpha_1$, and we may set

$$\frac{t_1 + s_1\sqrt{\alpha_1}}{t_1 - s_1\sqrt{\alpha_1}} \cdot \frac{t_2 - s_2 r\sqrt{\alpha_1}}{t_2 + s_2 r\sqrt{\alpha_1}} = t^3, \ t \in \mathbb{K}^\times.$$

Then $t \in \mathbb{K}_1$. Again by Hilbert's Theorem 90, we may write

$$t = \frac{a + b\sqrt{\alpha_1}}{a - b\sqrt{\alpha_1}}, \ a, b \in \mathbb{Q}.$$

Let X_1 and X_2 be as in (5.3). We can define a $\mathbb{K}$-graded algebra map

$$\psi : \mathbb{K}[X_1, X_2]/(X_1 X_2, X_1^3 - \frac{t_1 - s_1\sqrt{\alpha_1}}{t_1 + s_1\sqrt{\alpha_1}} X_2^3) \longrightarrow \mathbb{K}[X_1, X_2]/(X_1 X_2, X_1^3 - \frac{t_2 - s_2 r\sqrt{\alpha_2}}{t_2 + s_2 r\sqrt{\alpha_1}} X_2^3)$$

by

$$\psi(X_1) = (a + b\sqrt{\alpha_1})X_1, \ \psi(X_2) = (a - b\sqrt{\alpha_1})X_2$$

for some $a, b \in \mathbb{Q}$. Then we have

$$\begin{aligned} \psi(x_1) = \psi\left(\tfrac{X_1 + X_2}{2}\right) &= \tfrac{1}{2}\{(a + b\sqrt{\alpha_1})X_1 + (a - b\sqrt{\alpha_1})X_2\} \\ &= \tfrac{1}{2}\{(a + b\sqrt{\alpha_1})(x_1 + \sqrt{\alpha_1}x_2)) \\ &\qquad + (a - b\sqrt{\alpha_1})(x_1 - \sqrt{\alpha_1}x_2)\} \\ &= a x_1 + b\alpha_1 x_2, \end{aligned}$$

$$\psi(x_2) = \psi\left(\frac{X_1 - X_2}{2\sqrt{\alpha_1}}\right) = \frac{1}{2\sqrt{\alpha_1}}\{(a + b\sqrt{\alpha_1})(x_1 + \sqrt{\alpha_1}x_2) - (a - b\sqrt{\alpha_1})(x_1 - \sqrt{\alpha_1}x_2)\} = bx_1 + ax_2.$$

Hence ψ is defined over $\mathbb{Q}$. Thus we have a graded $\mathbb{Q}$-algebra isomorphism

$$\overline{\psi} : \mathbb{Q}[x_1, x_2]/(x_1^2 - \alpha_1 x_2^2, s_1 x_1^3 + t_1 x_1^2 x_2) \longrightarrow \mathbb{Q}[x_1, x_2]/(x_1^2 - \alpha_2 x_2^2, s_2 x_1^3 + t_2 x_1^2 x_2).$$

Next we consider the case that $(\alpha_i, [s_i, t_i]) \in \tilde{M}_2$ $(i = 1, 2)$. (For the definition of $\tilde{M}_2$ see Section 1.)

Proposition 5.4. *The two graded algebras*

$$\mathbb{Q}[x_1, x_2]/(x_1^2 - \gamma_1^2 x_2^2, s_1 x_1^3 + t_1 x_1^2 x_2) \text{ and } \mathbb{Q}[x_1, x_2]/(x_1^2 - \gamma_2^2 x_2^2, s_2 x_1^3 + t_2 x_1^3 x_2),$$

where $(\gamma_i^2, [s_i, t_i]) \in \tilde{M}_2$ $(i = 1, 2)$*, are isomorphic if and only if* $(\alpha_1, [s_1, t_1])$ *and* $(\alpha_2, [s_2, t_2])$ *are equivalent, that is,*

$$\frac{t_2 - s_2 r_2}{t_2 + s_2 r_2} \cdot \frac{t_1 + s_1 r_1}{t_1 - s_1 r_1} \in \mathbb{Q}^{\times 3}.$$

Proof. By setting

$$y_1 = x_1 + r_1 x_2, \quad y_2 = x_1 - r_1 x_2,$$

the graded algebra over $\mathbb{Q}$

$$\mathbb{Q}[x_1, x_2]/(x_1^2 - \gamma_1^2 x_2^2, s_1 x_1^3 + t_1 x_1^2 x_2)$$

is isomorphic to

$$\mathbb{Q}[y_1, y_2]/(y_1 y_2, (t_1 + s_1 r_1) y_1^3 + (-t_1 + s_1 r_1) y_2^3).$$

Observe that there is an isomorphism

$$\varphi : \mathbb{Q}[y_1, y_2]/(y_1 y_2, (t_1 + s_1 r_1) y_1^3 + (-t_1 + s_1 r_1) y_2^3) \to \mathbb{Q}[y_1, y_2]/(y_1 y_2, (t_2 + s_2 r_2) y_1^3 + (-t_2 + s_2 r_2) y_2^3)$$

if and only if

$$\frac{t_2 - s_2 r_2}{t_2 + s_2 r_2} \cdot \frac{t_1 + s_1 r_1}{t_1 - s_1 r_1} \in \mathbb{Q}^{\times 3}.$$

In fact, if we set $\varphi(y_i) = p_i y_1 + q_i y_2$ for $p_i, q_i \in \mathbb{Q}$ $(i = 1, 2)$, then $p_1 p_2 = 0$ and $q_1 q_2 = 0$. The condition $t \pm sr \neq 0$ in M_2 is equivalent to the one that the sequence $\{x_1^2 - r^2 x_2^2, s x_1^3 + t x_1^2 x_2\}$ is regular. ∎

By Propositions 5.3 and 5.4 we have

Proposition 5.5. *The set of isomorphism classes of graded algebras over* $\mathbb{Q}$

$$\mathbb{Q}[x_1, x_2]/(x_1^2 + \alpha x_2^2, s x_1^2 + t x_1^2 x_2)$$

corresponds bijectively to the set

$$T = (M_1 \backslash M_2)/\sim \cup \tilde{M}_2/\sim$$

In the case $\{0\} \times P^1(\mathbb{Q})$ it corresponds to the algebra

$$\mathbb{Q}[x_1, x_2]/(x_1^2, s x_2^3 + t x_2^2 x_1) \cong \mathbb{Q}[x_1, x_2]/(x_1^2, x_2^3).$$ ∎

6 The case of dimension 7

Let A be the cohomology algebra of an elliptic space with $\chi_\pi = 0$ such that $\dim_{\mathbb{Q}} A = 7$. Then $n = 1$ or 2 in (2). If $n = 1$, then $A \cong \mathbb{Q}[x]/(x^7)$. If $n = 2$, then $|f_1| \cdot |f_2| = 7|x_1| \cdot |x_2|$.

(a) Assume that $|f_1|$ is an integer multiple of $|x_1|$, that is, $|f_1| = k|x_1|$ for some integer $k \geq 2$. Then $k = 2$ or 3.

If $k = 2$, then $|f_2| = \dfrac{7}{2}|x_2|$, which implies $f_2 \in (x_1)$. By regularity f_1 contains the term cx_2^2, and hence $|x_1| = |x_2|$. This implies that $|f_2|$ is an integer multiple of $|x_2|$. This is a contradiction.

If $k = 3$, then $|f_2| = \dfrac{7}{3}|x_2|$ and $f_2 \in (x_1)$. Thus we have that $|f_1| = 2|x_2|$ and $|f_2| = \dfrac{7}{2}|x_1|$, which implies that $(f_1, f_2) = (x_1^3 + ax_2^2, x_1^2 x_2)$, where $a \in \mathbb{Q}^\times$.

(b) Assume that $|f_1|$ is an integer multiple of $|x_2|$, that is, $|f_1| = k|x_2|$ for some integer $k \geq 1$. Then $|f_2| = \frac{7}{k}|x_1|$ and so $f_2 \in (x_2)$. This implies that $|f_1|$ is an integer multiple of $|x_1|$, and so we are reduced to the case (a).

(c) Assume that $|f_1|$ is neither an integer multiple of $|x_1|$ nor of $|x_2|$. Then $f_1 \in (x_1) \cap (x_2)$, and hence f_2 contains a non zero multiple of $x_1^{k_1}$ and $x_2^{k_2}$ for some integers k_1, k_2. Then

$$|f_2| = k_1|x_1| = k_2|x_2|$$

and $k_1 > k_2 \geq 2$.

If $k_2 \geq 4$, then $|f_1| \leq \dfrac{7}{4}|x_1|$, which is impossible by Lemma 2.1.

Thus we can deduce that $k_2 = 2$ or 3.

(1) Let $k_2 = 2$. If $k_1 \geq 6$, then

$$|f_1| \geq |x_1| + |x_2| \geq \left(1 + \frac{k_1}{2}\right)|x_1| \geq 4|x_1|,$$

and hence we have by (2.2) for $n = 2$ that

$$|f_2| \leq \frac{7}{4}|x_2|,$$

which contradicts Lemma 2.1. Thus $k_1 = 3$ or 4 or 5.

If $k_1 = 3$, then

$$|f_1| = \frac{7}{3}|x_2| = \frac{7}{2}|x_1| > 3|x_1| = |f_2|.$$

This contradicts the assumption.

If $k_1 = 4$, then $|x_2| = 2|x_1|$ and $|f_1|$ is an integer multiple of $|x_1|$. This contradicts the assumption.

If $k_1 = 5$, then $|f_1| = \dfrac{7}{5}|x_2| = \dfrac{7}{2}|x_1| = |x_1| + |x_2|$. Then we have

$$(f_1, f_2) = (x_1 x_2, x_1^5 + ax_2^2), \qquad a \in \mathbb{Q}^\times.$$

(2) Let $k_2 = 3$. Then we have that

$$\frac{7}{3}|x_1| = |f_1| \geq |x_1| + |x_2| = \left(1 + \frac{k_1}{3}\right)|x_1|.$$

Since $k_1 > k_2$, we see that $k_1 = 4$ and $|f_1| = |x_1| + |x_2|$, which implies that

$$(f_1, f_2) = (x_1 x_2, x_1^4 + a x_2^3), \qquad a \in \mathbb{Q}^\times,$$

where $4|x_1| = 3|x_2|$.

Proposition 6.1. *The isomorphism classes of the algebras*

$\mathbb{Q}[x_1, x_2]/(x_1^3 + a x_2^2, x_1^2 x_2)$, $\mathbb{Q}[x_1, x_2]/(x_1 x_2, x_1^5 + a x_2^2)$, $\mathbb{Q}[x_1, x_2]/(x_1 x_2, x_1^4 + a x_2^3)$

do not depend on the choice of $a \in \mathbb{Q}^\times$.

Proof. The correspondence

$$\varphi(x_1) = p x_1, \quad \varphi(x_2) = q x_2 \qquad (p, q \in \mathbb{Q}^\times)$$

defines an isomorphism

$$\mathbb{Q}[x_1, x_2]/(x_1^3 + a x_2^2, x_1^2 x_2) \longrightarrow \mathbb{Q}[x_1, x_2](x_1^3 + b x_2^2, x_1^2 x_2)$$

if and only if $p^3 b = q^2 a$. Hence, if we take $p = q = ab^{-1}$, we obtain the desired isomorphism. The cases of the other algebras can be similarly proved. ■

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