Generic Bifurcations of Planar Filippov Systems via Geometric Singular Perturbations

Tiago de Carvalho Durval José Tonon

Abstract

In this paper we deal with non−smooth vector fields on the plane. We prove that the analysis of their local behavior around certain typical singularities can be treated via singular perturbation theory. In fact, after a regularization of a such system and a blow−up we are able to bring out some results that bridge the space between non−smooth dynamical systems presenting typical singularities and singularly perturbed smooth systems.

1 Introduction

This work fits within the geometric study of Singular Perturbation Problems expressed by vector fields on $\mathbb{R}^2$. We study the phase portraits of certain non−smooth planar vector fields having a curve Σ as the discontinuity set. We present some results in the framework developed by Sotomayor and Teixeira in [13] (and extended in [11]) and establish a bridge between those systems and the fundamental role played by the Geometric Singular Perturbation Theory. This transition was introduced in papers like [2] and [9], in dimensions 2 and 3 respectively. Results in this context can be found in [10]. We deal with non−smooth vector fields presenting structurally unstable configurations and we prove that these structurally unstable configurations are carried over the Geometric Singular Perturbation Problem associated. Some good surveys about Geometric Singular Perturbation Theory are [3] and [4], among others.

Received by the editors February 2011.

Communicated by P. De Maesschalk.

2000 *Mathematics Subject Classification :* Primary 34A36, 34C23, 34D15, 34D30.

Key words and phrases : geometric singular perturbation, non−smooth vector fields, bifurcation, fold−fold singularity.

Let $\mathcal{U} \subseteq \mathbb{R}^2$ be an open set and $\Sigma \subseteq \mathcal{U}$ given by $\Sigma = f^{-1}(0)$, where $f : \mathcal{U} \longrightarrow \mathbb{R}$ is a smooth function having $0 \in \mathbb{R}$ as a regular value (i.e. $\nabla f(p) \neq 0$, for any $p \in f^{-1}(0)$). Clearly Σ is the separating boundary of the regions $\Sigma_+ = \{q \in \mathcal{U} | f(q) \geq 0\}$ and $\Sigma_- = \{q \in \mathcal{U} | f(q) \leq 0\}$. We can assume that Σ is represented, locally around a point $q = (x, y)$, by the function $f(x, y) = x$.

Designate by $\mathfrak{X}^r$ the space of C^r-vector fields on a compact set $K \subset \mathcal{U}$ endowed with the C^r-topology with $r \geq 1$. Call $\widetilde{\Omega}^r = \widetilde{\Omega}^r(K, f)$ the space of vector fields $Z : K \setminus \Sigma \to \mathbb{R}^2$ such that

$$Z(x,y) = \begin{cases} X(x,y), & \text{for} \quad (x,y) \in \Sigma_+, \\ Y(x,y), & \text{for} \quad (x,y) \in \Sigma_-, \end{cases} \tag{1}$$

where $X = (h_1, g_1)$, $Y = (h_2, g_2)$ are in $\mathfrak{X}^r$. The trajectories of Z are solutions of $\dot{q} = Z(q)$, which has, in general, discontinuous right$-$hand side.

In what follows we will use the notation

$$X.\nabla f(p) = \langle \nabla f(p), X(p) \rangle \quad \text{and} \quad Y.\nabla f(p) = \langle \nabla f(p), Y(p) \rangle .$$

We distinguish the following regions on the discontinuity set Σ :

- ▶ $\Sigma_1 \subseteq \Sigma$ is the *sewing region* if $(X.\nabla f)(Y.\nabla f) > 0$ on Σ_1 .
- ▶ $\Sigma_2 \subseteq \Sigma$ is the *escaping region* if $(X.\nabla f) > 0$ and $(Y.\nabla f) < 0$ on Σ_2.
- ▶ $\Sigma_3 \subseteq \Sigma$ is the *sliding region* if $(X.\nabla f) < 0$ and $(Y.\nabla f) > 0$ on Σ_3.

Definition 1. *The sliding vector field associated to $Z \in \widetilde{\Omega}^r$ is the vector field Z^s tangent to Σ_3 and defined at $q \in \Sigma_3$ by $Z^s(q) = m - q$ with m being the point of the segment joining $q + X(q)$ and $q + Y(q)$ such that $m - q$ is tangent to Σ_3 (see Figure 1).*

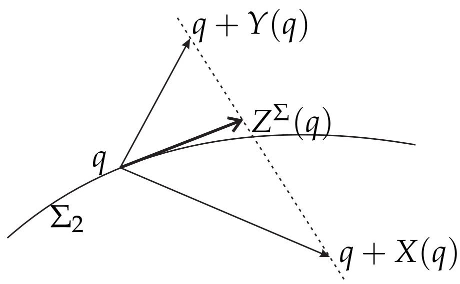

Figure 1: Filippov's convention.

Observe that if $q \in \Sigma_2$ for $Z \in \widetilde{\Omega}^r$ then $q \in \Sigma_3$ for $-Z$. Therefore we can define the *escaping vector field* on Σ_2 associated to Z by $Z^e = -(-Z)^s$. The *sewing vector field* associated to Z is the vector field Z^w defined in $q \in \Sigma_1$ as an arbitrary convex combination of $X(q)$ and $Y(q)$, i.e., $Z^w(q) = \lambda X(q) + (1 - \lambda) Y(q)$ where $\lambda \in [0, 1]$. In what follows we use the notation Z^Σ for all these cases.

Let $\Omega^r = \Omega^r(K, f)$ be the space of vector fields $Z : K \to \mathbb{R}^2$ such that $Z \in \widetilde{\Omega}^r$ and $Z(q) = Z^\Sigma(q)$ for all $q \in \Sigma$. We write $Z = (X, Y)$, which we will accept to be multivalued in points of Σ. The basic results of differential equations, in this context, were stated by Filippov in [5]. Related theories can be found in [7, 13, 15].

An approximation of the non−smooth vector field $Z = (X, Y)$ by a 1−parameter family Z_ϵ of smooth vector fields is called an ϵ−regularization of Z. We give the details about this process in Section 3. A transition function is used to average X and Y in order to get a family of smooth vector fields that approximates Z. The main goal of this process is to deduce certain dynamical properties of the non−smooth dynamical system from the regularized system. The regularization process developed by Sotomayor and Teixeira produces a singular problem for which the discontinuous set is a center manifold. Via a blow up we establish a bridge between non−smooth dynamical systems and the geometric singular perturbation theory. This paper deals almost exclusively with the critical (or singular) dynamics, namely the limit $r = 0$ in a singular perturbation of the form $r\dot{x} = a(x, y, r), \dot{y} = b(x, y, r)$, except for giving regularized vector fields in a form that allows them to be analyzed in the nonsingular limit.

Roughly speaking, the main results of this paper are the following:

Theorem 1. *Consider $Z(x, y) = Z_\lambda(x, y) = (X(x, y), Y_\lambda(x, y)) \in \Omega^r$ a non−smooth planar vector field, where $\lambda \in (-1, 1) \subset \mathbb{R}$, and Σ identified with the y−axis. Let the trajectories of X be transverse to Σ and Z_0 presenting either a hyperbolic saddle of Y_0 or a hyperbolic focus of Y_0 or a Σ−cusp point of Y_0 at $q \in \Sigma$. Then there exists a singular perturbation problem*

$$\theta' = \alpha(r, \theta, y, \lambda)\,,\; y' = r\beta(r, \theta, y, \lambda)\,, \tag{2}$$

with $r \geq 0$, $\theta \in (\pi/4, 3\pi/4)$, $y \in \Sigma$ and α and β of class C^r such that the unfolding of (2) *produces the same topological behaviors as the unfolding of the corresponding topological normal forms of Z_λ presented in Subsection 5.1.*

Theorem 2. *Consider $Z(x, y) = Z_\mu(x, y) = (X_\mu(x, y), Y_\mu(x, y)) \in \Omega^r$ a non−smooth planar vector field, where either $\mu = \lambda \in \mathbb{R}$ or $\mu = (\lambda, \varepsilon) \in \mathbb{R}^2$, and Σ identified with the y−axis. Consider that $q = (x_q, y_q) \in \Sigma$ is a Σ−fold point of both X_μ and Y_μ when $\mu = 0$ or $\mu = (0, 0)$. Then there exists a singular perturbation problem*

$$\theta' = \alpha(r, \theta, y, \mu)\,,\; y' = r\beta(r, \theta, y, \mu)\,, \tag{3}$$

with $r \geq 0$, $\theta \in (\pi/4, 3\pi/4)$, $y \in \Sigma$, α and β of class C^r such that the following statements holds:

(a) For all small neighborhood U of q in Σ the region $(\Sigma_2 \cup \Sigma_3) \cap (U - \{y_q\})$ is homeomorphic to the slow critical manifold $\alpha(0, \theta, y, \mu) = 0$ of (3) *where $y \in (U - \{y_q\})$.*

(b) The vector field Z^Σ, on $(\Sigma_2 \cup \Sigma_3) \cap (U - \{y_q\})$, and the reduced problem of (3)*, with $y \in (U - \{y_q\})$, are topologically equivalent.*

(c) *The slow critical manifold* $\alpha(0,\theta,y,0) = 0$ *of* (3)*, where* $y = y_q$*, has just an horizontal component, i.e.,* $\alpha(0,\theta,y_q,0) = 0$ *can be identified with* $\{(\theta,y)\,|\,\theta \in (\pi/4, 3\pi/4)\,,\, y = y_q\}$*. Moreover, this configuration is structurally unstable.*

The unfolding of (3) *produces the same topological behaviors as the unfolding of the corresponding normal forms of* Z_λ *presented in Table 1 and in Equation* (20)*.*

For a precise definition of $\Sigma-$cusp and $\Sigma-$fold points see Section 2. For a precise definition of slow critical manifold see Section 4.

Observe that Theorem 2 generalizes Theorem 1.1 of [10], because here we allow that $X.\nabla f(q) = Y.\nabla f(q) = 0$. In fact, if for any $q \in \Sigma$ we have that $X.\nabla f(q) \neq 0$ or $Y.\nabla f(q) \neq 0$ then Theorem 1.1 of [10] says that there exists a singular perturbation problem such that the sliding region is homeomorphic to the slow critical manifold and the sliding vector field is topologically equivalent to the reduced problem.

Remark 1. *In [2] the blow up parameter* θ *belongs to the interval* $(0,\pi)$*, however the transition function* φ *(see Section 3) is constant when* $\theta \in (0,\pi/4) \cup (3\pi/4,\pi)$*. So the restriction of* φ *to the interval* $(\pi/4, 3\pi/4)$ *is enough to describe the dynamic.*

The paper is organized as follows: in Section 2 we give the basic theory about Non$-$Smooth Vector Fields on the plane, in Section 3 we give the theory about the regularization process, in Section 4 we present a few relevant methods of Geometric Singular Perturbation Theory, in Section 5 we present the singularities treated in Theorem 1, give its topological normal forms and prove Theorem 1, in Section 6 we present the singularities treated in Theorem 2, give its topological normal forms and prove Theorem 2.

2 Preliminaries

We say that $q \in \Sigma$ is a $\Sigma-$*regular point* if

(i) $X.\nabla f(q) Y.\nabla f(q) > 0$ or

(ii) $X.\nabla f(q) Y.\nabla f(q) < 0$ and $Z^\Sigma(q) \neq 0$ (that is $q \in \Sigma_2 \cup \Sigma_3$ and it is not a singular point of Z^Σ).

The points of Σ which are not $\Sigma-$regular are called $\Sigma-$*singular*. We distinguish two subsets in the set of $\Sigma-$singular points: Σ^t and Σ^p. Any $q \in \Sigma^p$ is called a *pseudo equilibrium of* Z and it is characterized by $Z^\Sigma(q) = 0$. Any $q \in \Sigma^t$ is called a *tangential singularity* and is characterized by $Z^\Sigma(q) \neq 0$ and $X.\nabla f(q) Y.\nabla f(q) = 0$ (q is a contact point of Z^Σ).

A tangential singularity $q \in \Sigma^t$ is a $\Sigma-$*fold point* of X if $X.\nabla f(q) = 0$ but $X^2.\nabla f(q) = X.\nabla(X.\nabla f)(q) \neq 0$. Moreover, $q \in \Sigma$ is a *visible* (resp. *invisible*) $\Sigma-$*fold point* of X if $X.\nabla f(q) = 0$ and $X^2.\nabla f(q) > 0$ (resp. $X^2.\nabla f(q) < 0$). We say that $q \in \Sigma^t$ is a $\Sigma-$*cusp point* of X if $X.\nabla f(q) = X^2.\nabla f(q) = 0$ but $X^3.\nabla f(q) \neq 0$. Moreover, $q \in \Sigma$ is a *natural* (resp. *inverse*) $\Sigma-$*cusp point* of X if

$X.\nabla f(q) = X^2.\nabla f(q) = 0$ and $X^3.\nabla f(q) > 0$ (resp. $X^3.\nabla f(q) < 0$).

A pseudo equilibrium $q \in \Sigma^p$ is a $\Sigma-$*saddle* provided one of the following condition is satisfied: (i) $q \in \Sigma_2$ and q is an attractor for Z^Σ or (ii) $q \in \Sigma_3$ and q is a repeller for Z^Σ. A pseudo equilibrium $q \in \Sigma^p$ is a $\Sigma-$*repeller* (resp. $\Sigma-$*attractor*) provided $q \in \Sigma_2$ (resp. $q \in \Sigma_3$) and q is a repeller (resp. attractor) equilibrium point for Z^Σ.

3 Regularization

In this section we present the concept of $\epsilon-$regularization of non$-$smooth vector fields. It was introduced by Sotomayor and Teixeira in [13]. The regularization gives the mathematical tool to study the stability of these systems, according with the program introduced by Peixoto in [12]. The method consists in the analysis of the regularized vector field which is a smooth approximation of the non$-$smooth vector field. Using this process we get a $1-$parameter family of vector fields $Z_\epsilon \in \mathfrak{X}^r$ such that for each $\epsilon_0 > 0$ fixed it satisfies that:

(i) Z_{ϵ_0} is equal to X in all points of Σ_+ whose distance to Σ is bigger than ϵ_0;

(ii) Z_{ϵ_0} is equal to Y in all points of Σ_- whose distance to Σ is bigger than ϵ_0.

Definition 2. *A $C^\infty-$function $\varphi : \mathbb{R} \longrightarrow \mathbb{R}$ is a transition function if $\varphi(x) = -1$ for $x \leqslant -1$, $\varphi(x) = 1$ for $x \geqslant 1$ and $\varphi'(x) > 0$ if $x \in (-1,1)$. The $\epsilon-$regularization of $Z = (X,Y)$ is the $1-$parameter family $Z_\epsilon \in \mathfrak{X}^r$ given by*

$$Z_\epsilon(q) = \left(\frac{1}{2} + \frac{\varphi_\epsilon(f(q))}{2}\right) X(q) + \left(\frac{1}{2} - \frac{\varphi_\epsilon(f(q))}{2}\right) Y(q). \tag{4}$$

with $\varphi_\epsilon(x) = \varphi(x/\epsilon)$, for $\epsilon > 0$.

4 Singular Perturbations

Definition 3. *Let $U \subseteq \mathbb{R}^2$ be an open subset and take $\epsilon \geqslant 0$. A singular perturbation problem in U (SP$-$Problem) is a differential system which can be written like*

$$x' = dx/d\tau = l(x,y,\epsilon), \quad y' = dy/d\tau = \epsilon m(x,y,\epsilon) \tag{5}$$

or equivalently, after the time re$-$scaling $t = \epsilon\tau$

$$\epsilon\dot{x} = \epsilon dx/dt = l(x,y,\epsilon), \quad \dot{y} = dy/dt = m(x,y,\epsilon), \tag{6}$$

with $(x,y) \in U$ and l,m smooth in all variables.

The understanding of the phase portrait of the vector field associated to a SP$-$problem is the main goal of the *geometric singular perturbation theory*. The techniques of this theory can be used to obtain information on the dynamics of (5) for small values of $\epsilon > 0$, mainly in searching limit cycles. System (5) is called the *fast system*, and (6) the *slow system* of SP−problem. Observe that for $\epsilon > 0$ the phase portraits of the fast and the slow systems coincide. For $\epsilon = 0$, let $\mathcal{S}$ be the set

$$\mathcal{S} = \{(x,y) : l(x,y,0) = 0\}$$

of all singular points of (5). We call $\mathcal{S}$ the slow critical manifold of the singular perturbation problem and it is important to notice that equation (6) defines a dynamical system, on $\mathcal{S}$, called the *reduced problem*:

$$l(x,y,0) = 0, \quad \dot{y} = m(x,y,0).$$

Combining results on the dynamics of these two limiting problems, with $\epsilon = 0$, one obtains information on the dynamics of Z_ϵ for small values of ϵ. We refer to [4] for an introduction to the general theory of singular perturbations. Related problems can be seen in [1], [3] and [14].

5 Boundary Bifurcations

Consider $Z = (X,Y) \in \Omega^r$. In this section we assume that the trajectories of the smooth vector field X is transversal to Σ and that Y has either a hyperbolic saddle or a hyperbolic focus or a $\Sigma-$cusp point in Σ. This configuration is clearly structurally unstable. We present here its topological normal forms and unfoldings.

We emphasize that the content of this section proves Theorem 1.

5.1 Codimension One Normal Forms

Take Σ as the $y-$axis, i.e., $f(x,y) = x$ and consider the parameter $\lambda \in (-1,1)$. The specific topological normal forms presented below can be found in [6] or [8].

- *Regular−saddle*: Assume that the trajectories of X are transverse to Σ and that Y has a hyperbolic saddle in Σ with the eigenspaces transverse to Σ. The following topological normal form generically unfolds this configuration.

$$Z(x,y) = Z_\lambda(x,y) = \begin{cases} X(x,y) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & \text{for} \quad (x,y) \in \Sigma_+, \\ Y_\lambda(x,y) = \begin{pmatrix} -y \\ -x-\lambda \end{pmatrix}, & \text{for} \quad (x,y) \in \Sigma_-. \end{cases}$$

- *Regular−focus*: Assume that the trajectories of X are transverse to Σ and that Y has a hyperbolic focus in Σ. The following topological normal form generically unfolds this configuration.

$$Z(x,y) = Z_\lambda(x,y) = \begin{cases} X(x,y) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & \text{for} \quad (x,y) \in \Sigma_+, \\ Y_\lambda(x,y) = \begin{pmatrix} x+y+\lambda \\ -x+y-\lambda \end{pmatrix}, & \text{for} \quad (x,y) \in \Sigma_-. \end{cases}$$

- *Regular−cusp*: Assume that the trajectories of X are transverse to Σ and that Y has a $\Sigma-$cusp point. The following topological normal form generically unfolds this configuration.

$$Z(x,y) = Z_\lambda(x,y) = \begin{cases} X(x,y) = \begin{pmatrix} 1 \\ 1 \end{pmatrix}, & \text{for} \quad (x,y) \in \Sigma_+, \\ Y_\lambda(x,y) = \begin{pmatrix} -y^2+\lambda \\ 1 \end{pmatrix}, & \text{for} \quad (x,y) \in \Sigma_-. \end{cases}$$

5.2 Regular−saddle Bifurcation

Consider the regular−saddle topological normal form given in the previous subsection. The regularized vector field becomes

$$\begin{aligned} \dot{x} &= \frac{1-y}{2} + \varphi\left(\tfrac{x}{\epsilon}\right)\frac{1+y}{2}, \\ \dot{y} &= \frac{1-x-\lambda}{2} + \varphi\left(\tfrac{x}{\epsilon}\right)\frac{1+x+\lambda}{2}, \end{aligned}$$

where $\varphi(x/\epsilon)$ is the transition function. Making the polar blow up

$$x = r\cos\theta \qquad \text{and} \qquad \epsilon = r\sin\theta, \tag{7}$$

we obtain

$$\begin{aligned} r\dot{\theta} &= -\sin\theta\left(\frac{1-y}{2} + \varphi(\cot\theta)\frac{1+y}{2}\right), \\ \dot{y} &= \frac{1-r\cos\theta-\lambda}{2} + \varphi(\cot\theta)\frac{1+r\cos\theta+\lambda}{2}. \end{aligned}$$

In the blowing up locus $r=0$ the fast dynamics is determined by the system

$$\theta' = -\sin\theta\left(\frac{1-y}{2} + \varphi(\cot\theta)\frac{1+y}{2}\right), \quad y' = 0;$$

and the slow dynamics on the slow critical manifold is determined by the reduced system

$$\frac{-1+y}{2} + \varphi(\cot\theta)\frac{-1-y}{2} = 0, \quad \dot{y} = \frac{1-\lambda}{2} + \varphi(\cot\theta)\frac{1+\lambda}{2}.$$

We remark that the slow critical manifold is given by the explicit form

$$y(\theta) = \frac{1+\varphi(\cot(\theta))}{1-\varphi(\cot(\theta))} \tag{8}$$

which do not depends on the parameter λ (see Figure 2). Moreover, the slow critical manifold $y(\theta)$ is such that

$$\lim_{\theta\longrightarrow\frac{\pi}{4}} y(\theta) = +\infty \quad \text{and} \quad \lim_{\theta\longrightarrow\frac{3\pi}{4}} y(\theta) = 0.$$

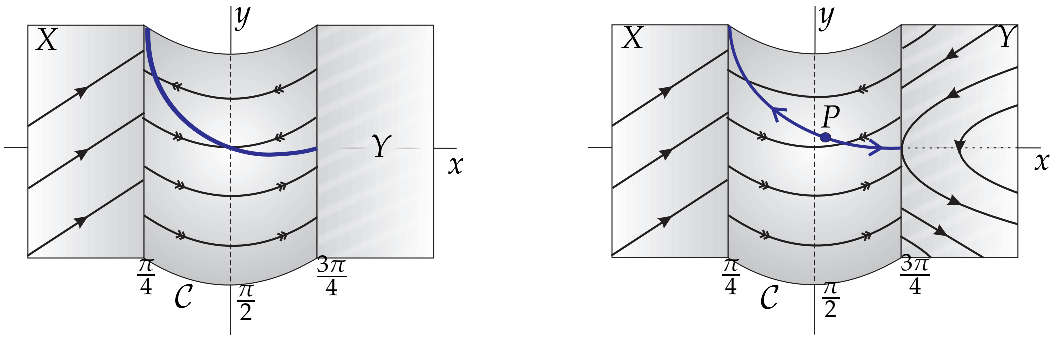

Figure 2: This figure is related to Subsection 5.2. Here it is pictured the slow critical manifold to the left and the case $\lambda < 0$ at the right. In both we consider $\theta \in (\pi/4, 3\pi/4)$.

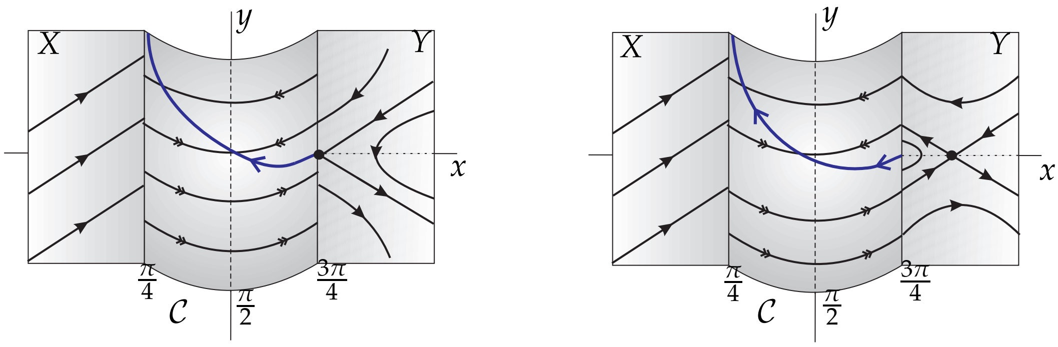

Figure 3: This figure is related to Subsection 5.2. Here it is pictured the case $\lambda = 0$ to the left and the case $\lambda > 0$ at the right. In both we consider $\theta \in (\pi/4, 3\pi/4)$.

The dynamics on the slow critical manifold depends on λ. In fact, if either $\lambda > 0$ or if $\lambda = 0$ then $\dot{y} \neq 0$ (see Figure 3) and if $\lambda > 0$ so $\dot{y}$ has a unique repeller critical point P (see Figure 2) given implicitly by the equation $\varphi(\cot\theta) = (-1+\lambda)/(1+\lambda)$.

In Figure 2 and in the next ones, double arrow over a curve means that it is a trajectory of the fast dynamical system, and simple arrow means that it is a trajectory of the one dimensional slow dynamical system. Moreover, we emphasize that after the polar blowing up it appears the half cylinder $\mathcal{C} = \{(\theta, y)\,;\, \theta \in (\pi/4, 3\pi/4), y \in \mathbb{R}\}$.

5.3 Regular−focus Bifurcation

Consider the regular−focus topological normal form given in Subsection 5.1. The regularized vector field becomes

$$\begin{aligned} \dot{x} &= \frac{1+\lambda+x+y}{2} + \varphi\left(\frac{x}{\epsilon}\right)\frac{1-\lambda-x-y}{2}, \\ \dot{y} &= \frac{1-\lambda-x+y}{2} + \varphi\left(\frac{x}{\epsilon}\right)\frac{1+\lambda+x-y}{2}. \end{aligned}$$

Similarly to the previous case, considering the polar blow−up given in (7), we get

$$\begin{aligned} r\dot{\theta} &= -\sin\theta\left(\frac{1+\lambda+y+r\cos\theta}{2}+\varphi(\cot\theta)\frac{1-\lambda-y-r\cos\theta}{2}\right), \\ \dot{y} &= \frac{1+\lambda+y-r\cos\theta}{2}+\varphi(\cot\theta)\frac{1-\lambda-y+r\cos\theta}{2}. \end{aligned}$$

Putting $r=0$, the fast dynamics is determined by the system

$$\theta' = \sin\theta\left(\frac{-1-\lambda-y}{2}+\varphi(\cot\theta)\frac{-1+\lambda+y}{2}\right), \quad y'=0;$$

and the slow dynamics on the slow critical manifold is determined by the reduced system

$$\frac{-1-\lambda-y}{2}+\varphi(\cot\theta)\frac{-1+\lambda+y}{2}=0, \quad \dot{y}=\frac{1-\lambda+y}{2}+\varphi(\cot\theta)\frac{1+\lambda-y}{2}.$$

In this case, the slow critical manifold (see Figure 4) depends on the parameter λ. In fact, we get the explicit expression for the slow critical manifold

$$y(\theta) = 1-\lambda+\frac{2}{-1+\varphi(\cot(\theta))}. \tag{9}$$

The slow critical manifold $y(\theta)$, given in the previous equation, satisfies

$$\lim_{\theta\longrightarrow\frac{\pi}{4}} y(\theta) = -\infty \quad \text{and} \quad \lim_{\theta\longrightarrow\frac{3\pi}{4}} y(\theta) = -\lambda.$$

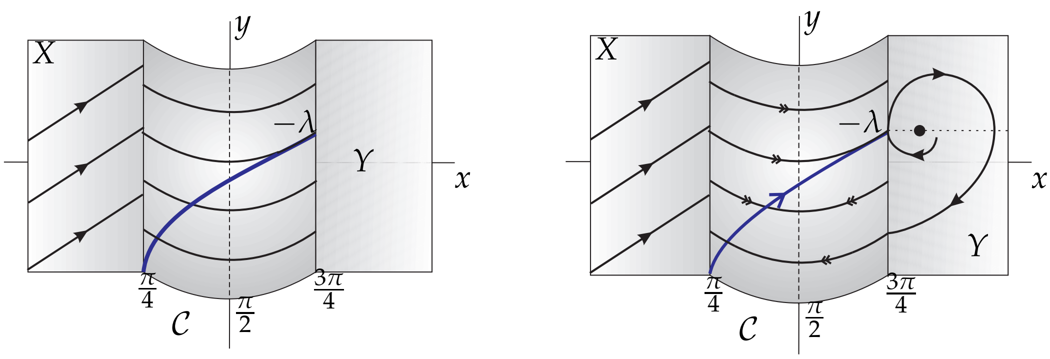

Figure 4: This figure is related to Subsection 5.3. Here it is pictured the slow critical manifold to the left and the case $\lambda < 0$ at the right. In both we consider $\theta \in (\pi/4, 3\pi/4)$.

We give now the dynamics on the slow critical manifold. If $\lambda < 0$ so $\dot{y} \neq 0$ (see Figure 4), if $\lambda > 0$ so $\dot{y}$ has an unique critical point P, given implicitly as the solution of $\varphi(\cot\theta) = (-1+\lambda+y)/(1+\lambda-y)$, which is an attractor (see Figure 5) and if $\lambda = 0$ so $\dot{y} \neq 0$ (see Figure 5).

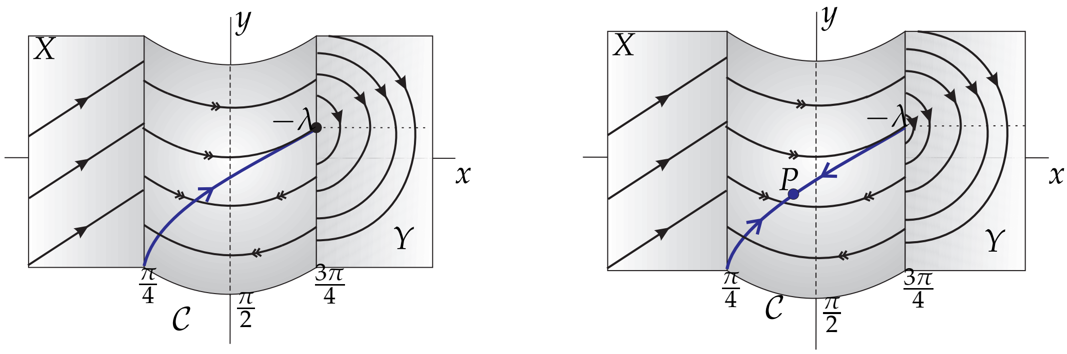

Figure 5: This figure is related to Subsection 5.3. Here it is pictured the case $\lambda = 0$ to the left and the case $\lambda > 0$ at the right. In both we consider $\theta \in (\pi/4, 3\pi/4)$.

5.4 Regular−Cusp Bifurcation

Consider the regular−cusp topological normal form given in Subsection 5.1. The regularized vector field becomes

$$\begin{aligned} \dot{x} &= \frac{1+\lambda-y^2}{2} + \varphi\left(\frac{x}{\epsilon}\right)\frac{1-\lambda+y^2}{2}, \\ \dot{y} &= \varphi\left(\frac{x}{\epsilon}\right). \end{aligned}$$

Making the polar blow−up given in (7), we get

$$\begin{aligned} r\dot{\theta} &= \frac{\sin\theta}{2}\left(\varphi(\cot\theta)(-1+\lambda-y^2) - 1 - \lambda + y^2\right), \\ \dot{y} &= \varphi(\cot\theta). \end{aligned}$$

Putting $r = 0$ the fast dynamics is determined by the system

$$\theta' = \frac{\sin\theta}{2}(\varphi(\cot\theta)(-1+\lambda-y^2) - 1 - \lambda + y^2), \quad y' = 0;$$

and the slow dynamics on the slow critical manifold is determined by the reduced system

$$\varphi(\cot\theta)(-1+\lambda-y^2) - 1 - \lambda + y^2 = 0, \quad \dot{y} = \varphi(\cot\theta).$$

Observe that the slow critical manifold depends on the parameter λ. We can obtain the explicit form. In fact, the slow critical manifold is composed by two branches (see Figure 6):

$$y_{\pm}^{\lambda}(\theta) = \pm\sqrt{\frac{\lambda(1-\varphi(\cot\theta)) + 1 + \varphi(\cot\theta)}{1-\varphi(\cot\theta)}}. \tag{10}$$

The slow critical manifold satisfies the properties:

(i) $\lim_{\theta\to\frac{\pi}{4}} y_{\pm}^{\lambda}(\theta) = \pm\infty;$

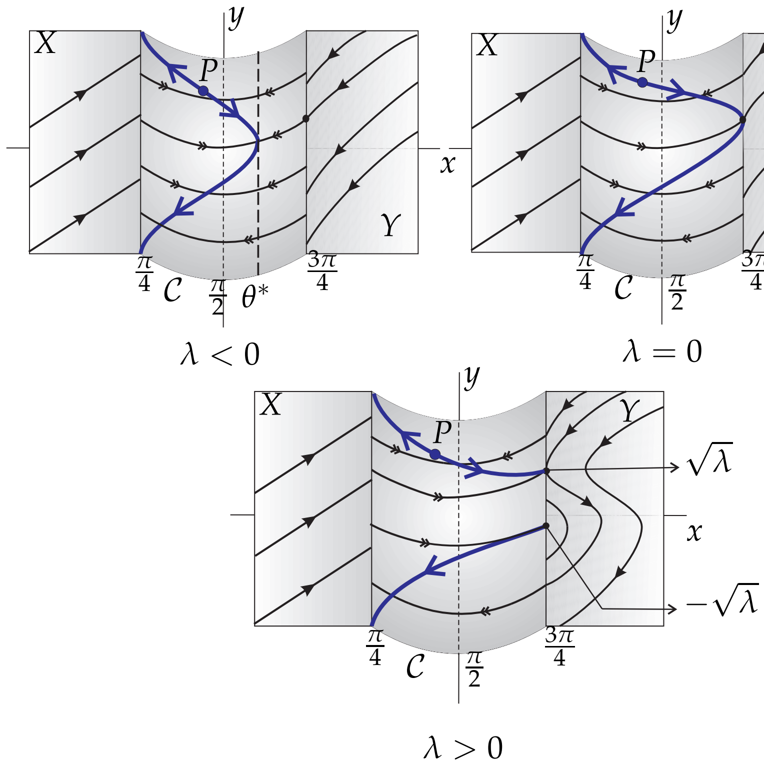

Figure 6: This figure is related to Subsection 5.4. Here it appears the Regular−cusp bifurcation Diagram.

(ii) If $\lambda < 0$ there exists $\theta^* \in (\pi/4, 3\pi/4)$ such that $y^{\lambda}_{\pm}(\theta^*) = 0$ and the slow critical manifold is not defined for $\theta \in (\theta^*, 3\pi/4)$. For $\theta \in (\pi/4, \theta^*)$ there exist homeomorphisms $\xi_{\pm}$ between each branch of the slow critical manifold and $\mathbb{R}\backslash\{0\}$. That is, for each $z \in \mathbb{R}\backslash\{0\}$ there exists $\theta(z) \in (\pi/4, \theta^*)$ such that $y^{\lambda}_{\pm}(\theta(z)) = z$;

(iii) If $\lambda \geq 0$ there exist homeomorphisms $\xi_{\pm}$ between each branch of the slow critical manifold and $\mathbb{R}\backslash\{0\}$. That is, for each $z \in \mathbb{R}\backslash\{0\}$ there exists $\theta(z) \in (\pi/4, 3\pi/4)$ such that $y^{\lambda}_{\pm}(\theta(z)) = z$. Moreover, $\lim\limits_{\theta \to \frac{3\pi}{4}} y^{\lambda}_{\pm}(\theta) = \pm\sqrt{\lambda}$ and the two branches of the slow critical manifold are not connected when $\lambda \neq 0$. This peculiar fact is result of the arising of the two $\Sigma-$fold points of Y.

In fact, the item (i) is a straightforward calculus. In order to prove the item (ii) observe Expression (10). Let θ^* be such that $\varphi(\cot\theta^*) = (1+\lambda)(\lambda-1)$. We have, $y^{\lambda}_{\pm}(\theta^*) = 0$ and the radical in (10) is negative for $\theta \in (\theta^*, \pi)$.

We define the maps:

$$\begin{aligned} \xi_{\pm} : \quad \mathbb{R}\backslash\{0\} &\to (\pi/4, \theta^*) \\ z \quad &\mapsto \theta(z) = \cot^{-1}\left(\varphi^{-1}\left(\frac{z^2-\lambda-1}{1-\lambda+z^2}\right)\right). \end{aligned} \tag{11}$$

Given $z \in \mathbb{R}\backslash\{0\}$ if we put $\xi(z) = \theta(z)$ in (11) we get $y^{\lambda}_{\pm}(\theta(z)) = z$. Note that $\xi_{\pm}$ are homeomorphisms.

The proof of the item (iii) is analogous.

The dynamics in the slow critical manifold is given by $\dot{y} = \varphi(\cot\theta)$. So there exists a unique critical point P given implicitly as the solution of $\varphi(\cot\theta_p) = 0$. Note that this critical point is a repeller because $\varphi(\cot\theta) < 0$ for $\theta < \theta_p$ and $\varphi(\cot\theta) > 0$ for $\theta > \theta_p$. See Figure 6.

5.5 Proof of Theorem 1

The proof follows directly in face of the previous discussion and Theorem 1.1 of [10]. Moreover, as we give the topological behavior of the cases $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ it is easy to construct the bifurcation diagram of (2).

6 Fold−fold Bifurcations

In this section we analyze the dynamics of a non−smooth dynamical system around a point q which is a $\Sigma-$fold point of both X and Y. We say that q is a *Fold−Fold singularity* of $Z \in \Omega^r$.

We emphasize that the content of this section proves Theorem 2.

We divide the fold−fold singularities in types according with the sign of $X^2.\nabla f(q)$ and $Y^2.\nabla f(q)$:

(a) **Elliptic case**: $X^2.\nabla f(q) > 0$ and $Y^2.\nabla f(q) < 0$. See Figure 7 (a).

(b) **Hyperbolic case**: $X^2.\nabla f(q) < 0, Y^2.\nabla f(q) > 0$. See Figure 7 (b).

$(c.1)$ **Parabolic case − Kind 1**: $X^2.\nabla f(q) > 0, Y^2.\nabla f(q) > 0$. See Figure 7 (c.1).

$(c.2)$ **Parabolic case − Kind 2**: $X^2.\nabla f(q) < 0, Y^2.\nabla f(q) < 0$. See Figure 7 (c.2).

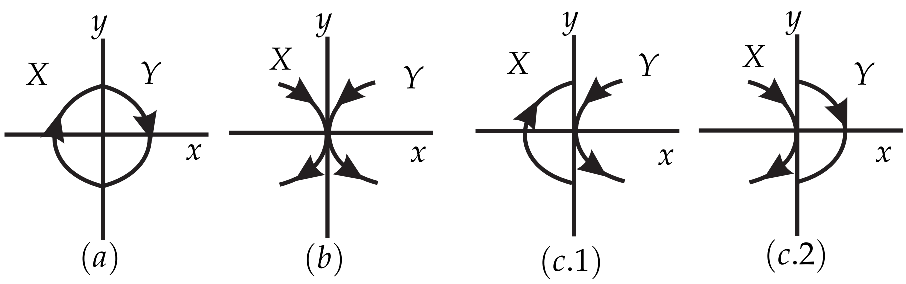

Figure 7: Fold−fold singularities.

Note that, we can define a first return map ψ_Z only in the elliptic case. Take Σ as the $y-$axis, i.e., $f(x, y) = x$ and consider the parameter $\lambda \in (-1, 1)$. Specific topological normal forms of the hyperbolic and parabolic fold−fold singularities are given in Table 1 . A specific topological normal form of the elliptic fold−fold singularity is given in Subsection 6.4. In both cases, related topological normal forms can be found in [6] or [8]. In this section we consider just the cases described above, with the particular choice of orientations of the trajectories of X and Y. In the other cases a similar approach can be done.

Hyperbolic	Parabolic − Kind 1	Parabolic − Kind 2
$X_\lambda(x,y) = (y-\lambda, -1)$ $Y(x,y) = (-y, -1)$	$X_\lambda(x,y) = (y-\lambda, 2)$ $Y(x,y) = (-y, -1)$	$X_\lambda(x,y) = (y-\lambda, -2)$ $Y(x,y) = (y, -1)$

Table 1:

In the next three subsections we study the dynamics of the hyperbolic and parabolic fold−fold singularities via geometric singular perturbations.

6.1 Hyperbolic Case

Consider the topological normal form of the hyperbolic fold−fold singularity given in Table 1. The regularized vector field is given by

$$\begin{aligned} \dot{x} &= -\frac{\lambda}{2} + \varphi\left(\frac{x}{\epsilon}\right)\left(\frac{-\lambda+2y}{2}\right), \\ \dot{y} &= -1. \end{aligned}$$

By the polar blow up we get

$$\begin{aligned} r\dot{\theta} &= \sin\theta\left(\frac{\lambda}{2} + \varphi(\cot\theta)\frac{\lambda-2y}{2}\right), \\ \dot{y} &= -1. \end{aligned}$$

Putting $r = 0$ the fast dynamics is determined by the system

$$\theta' = \sin\theta\left(\frac{\lambda}{2} + \varphi(\cot\theta)\left(\frac{\lambda-2y}{2}\right)\right), \quad y' = 0;$$

and the slow dynamics on the slow critical manifold is determined by the reduced system

$$\frac{\lambda}{2} + \varphi(\cot\theta)\left(\frac{\lambda-2y}{2}\right) = 0, \quad \dot{y} = -1.$$

In this case we obtain the explicit expression for the slow manifold:

$$y(\theta) = \frac{\lambda(1+\varphi(\cot\theta))}{2\varphi(\cot\theta)}. \tag{12}$$

Observe that, the slow critical manifold $y(\theta)$ is not defined for θ_0 such that $\varphi(\cot\theta_0) = 0$. So, for $\lambda \neq 0$, $y(\theta)$ has two branches and satisfies:

(a) $\lim\limits_{\theta\longrightarrow\theta_0^-} y(\theta) = -\infty$ for $\lambda < 0$ and $\lim\limits_{\theta\longrightarrow\theta_0^-} y(\theta) = +\infty$ for $\lambda > 0$;

(b) $\lim_{\theta \longrightarrow \theta_0^+} y(\theta) = +\infty$ for $\lambda < 0$ and $\lim_{\theta \longrightarrow \theta_0^+} y(\theta) = -\infty$ for $\lambda > 0$.

(c) For $\lambda = 0$ the slow critical manifold is given implicitly by $y\varphi(\cot\theta) = 0$, that is, $\{(\theta, y) \mid \theta = \theta_0\} \cup \{(\theta, y) \mid y = 0\}$ is the slow manifold.

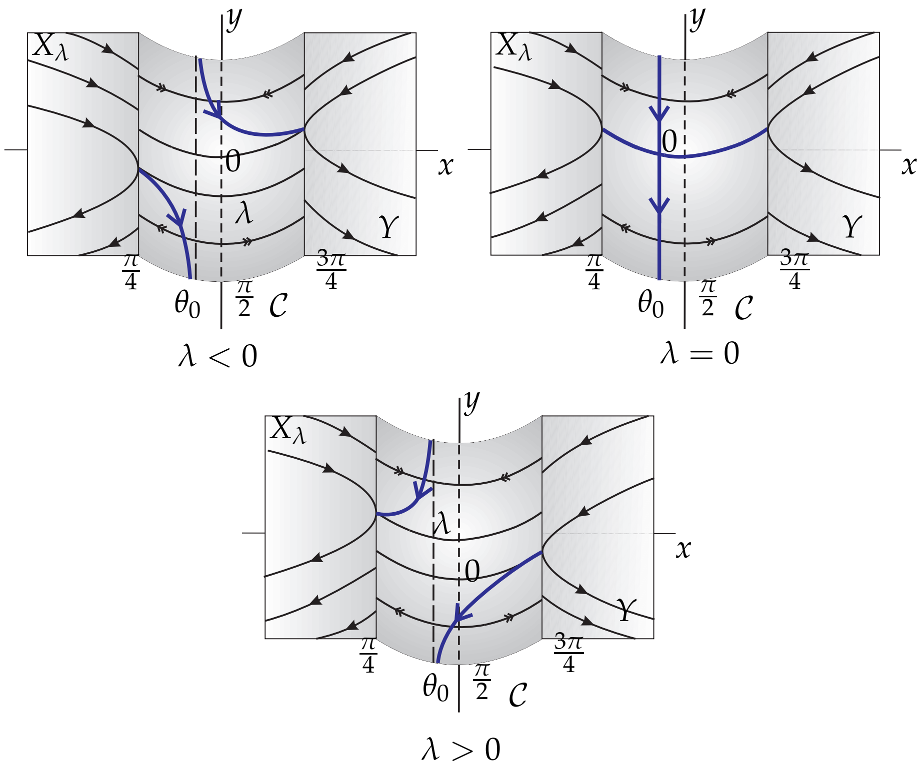

Figure 8: Slow manifold depending on the parameter λ.

The dynamics on the slow critical manifold is given by $\dot{y} = -1$. Therefore, there are not critical points. See Figure 8.

6.2 Parabolic case – Kind 1

Consider the topological normal form of the parabolic–Kind 1 fold–fold singularity given in Table 1. The regularized vector field is

$$\dot{x} = -\frac{\lambda}{2} + \varphi\left(\frac{x}{\epsilon}\right)\left(\frac{-\lambda + 2y}{2}\right),$$

$$\dot{y} = \frac{1}{2} + \frac{3}{2}\varphi\left(\frac{x}{\epsilon}\right).$$

By the polar blow up we get

$$r\dot{\theta} = \sin\theta\left(\frac{\lambda}{2} + \varphi(\cot\theta)\left(\frac{\lambda - 2y}{2}\right)\right),$$

$$\dot{y} = \frac{1}{2} + \frac{3}{2}\varphi(\cot\theta).$$

Putting $r = 0$ the fast dynamics is determined by the system

$$\theta' = \sin\theta \left(\frac{\lambda}{2} + \varphi(\cot\theta) \left(\frac{\lambda - 2y}{2} \right) \right), \quad y' = 0;$$

and the slow dynamics on the slow critical manifold is determined by the reduced system

$$\lambda + \varphi(\cot\theta)(\lambda - 2y) = 0, \quad \dot{y} = \frac{1}{2} + \frac{3}{2}\varphi(\cot\theta).$$

The analysis is similar to the hyperbolic case. In the present case the dynamics on the slow critical manifold is given by $\dot{y} = 1/2 + 3\varphi(\cot\theta)/2$. For $\lambda < 0$ (respectively $\lambda > 0$) it presents a repeller critical point (respectively an attractor critical point). For $\lambda = 0$ the slow critical manifold is composed by two branches. See Figure 9.

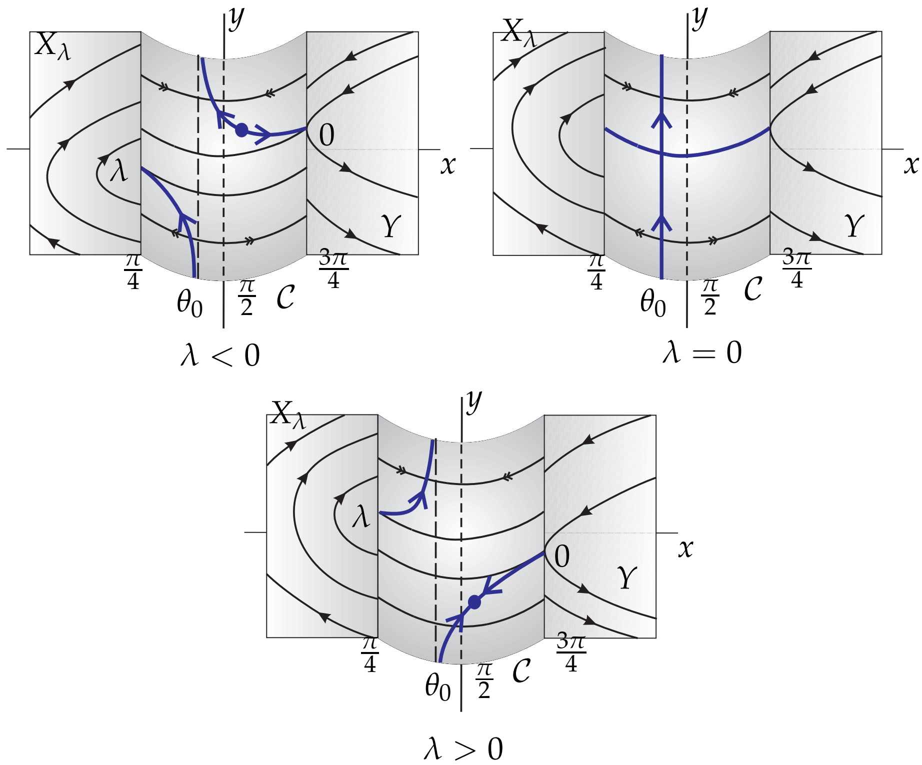

Figure 9: Bifurcation Diagram of the Parabolic−Kind 1 Fold−Fold Singularity.

6.3 Parabolic case − Kind 2

For this case, we get a distinct topological kind of bifurcation. Consider the topological normal form of the parabolic−Kind 2 fold−fold singularity given in Table 1. The regularized vector field is

$$\begin{aligned}\dot{x} &= -\frac{\lambda}{2}\varphi\left(\frac{x}{\epsilon}\right) + \frac{-\lambda + 2y}{2}, \\ \dot{y} &= -\frac{1}{2}\left(3 + \varphi\left(\frac{x}{\epsilon}\right)\right).\end{aligned}$$

By the polar blow up we get

$$\begin{aligned}r\dot{\theta} &= \frac{\sin\theta}{2}\left(\lambda\varphi(\cot\theta) + \lambda - 2y\right), \\ \dot{y} &= -\frac{1}{2}\left(3 + \varphi\left(\frac{x}{\epsilon}\right)\right).\end{aligned}$$

Putting $r = 0$ the fast dynamics is determined by the system

$$\theta' = \frac{\sin\theta}{2}\left(\lambda\varphi(\cot\theta) + \lambda - 2y\right), \quad y' = 0;$$

and the slow dynamics on the slow critical manifold is determined by the reduced system

$$\sin\theta\left(\frac{\lambda}{2}\varphi(\cot\theta) + \left(\frac{\lambda - 2y}{2}\right)\right) = 0, \quad \dot{y} = -\frac{1}{2}\left(3 + \varphi\left(\frac{x}{\epsilon}\right)\right).$$

We have the explicit expression for the slow critical manifold in this case:

$$y(\theta) = \frac{\lambda}{2}(1 + \varphi(\cot\theta)). \tag{13}$$

The analysis is similar to the previous cases and the bifurcation diagram is expressed in Figure 10.

6.4 Elliptic case

In this case, associated with the non−smooth vector fields, there exist the first return map $\psi_Z(p)$. Therefore, we need to analyze the structural stability of this one dimensional diffeomorphism (details about the function ψ_Z can be found in [16]).

Consider Z presenting an elliptic fold−fold singularity, $f(x,y) = x$ and

$$Z_\lambda(x,y) = \begin{cases} X_\lambda(x,y) = (y - \lambda, 1), & \text{for } (x,y) \in \Sigma_+, \\ Y(x,y) = (y, -1), & \text{for } (x,y) \in \Sigma_-. \end{cases} \tag{14}$$

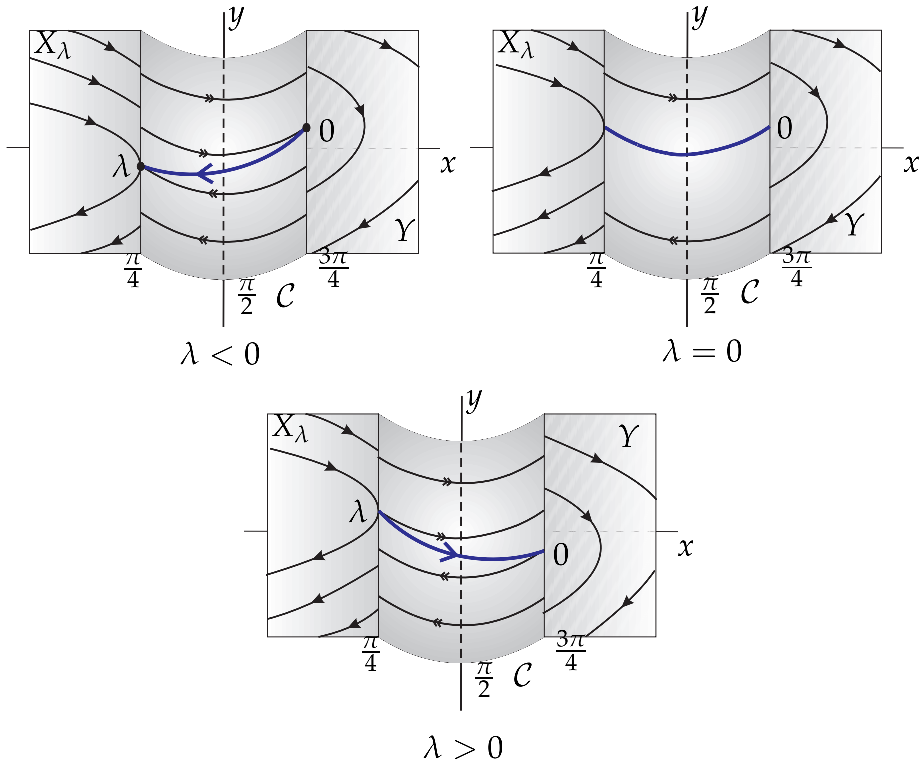

Figure 10: Bifurcation Diagram of the Parabolic−Kind 2 Fold−Fold Singularity.

The regularized vector field is

$$\dot{x} = -\varphi\left(\frac{x}{\epsilon}\right)\frac{\lambda}{2} + \frac{2y-\lambda}{2},$$

$$\dot{y} = \varphi\left(\frac{x}{\epsilon}\right).$$

By the polar blow up we get

$$r\dot{\theta} = \sin\theta\left(\frac{\lambda\varphi(\cot\theta)}{2} + \left(\frac{\lambda-2y}{2}\right)\right),$$

$$\dot{y} = \varphi(\cot\theta)$$

Putting $r = 0$ the fast dynamics is determined by the system

$$\theta' = \sin\theta\left(\frac{\lambda\varphi(\cot\theta)}{2} + \left(\frac{\lambda-2y}{2}\right)\right), \quad y' = 0;$$

and the slow dynamics on the slow critical manifold is determined by the reduced system

$$\frac{\lambda\varphi(\cot\theta)}{2} + \left(\frac{\lambda-2y}{2}\right) = 0, \quad \dot{y} = \varphi(\cot\theta).$$

In this case, for $\lambda = 0$, holds that $\Sigma_2 \cup \Sigma_3 = \emptyset$. The explicit expression for the slow critical manifold is

$$y(\theta) = \frac{\lambda(1+\varphi(\cot\theta))}{2} \quad (15)$$

and there exist an attractor critical point if $\lambda < 0$ and a repeller if $\lambda > 0$. See Figure 11.

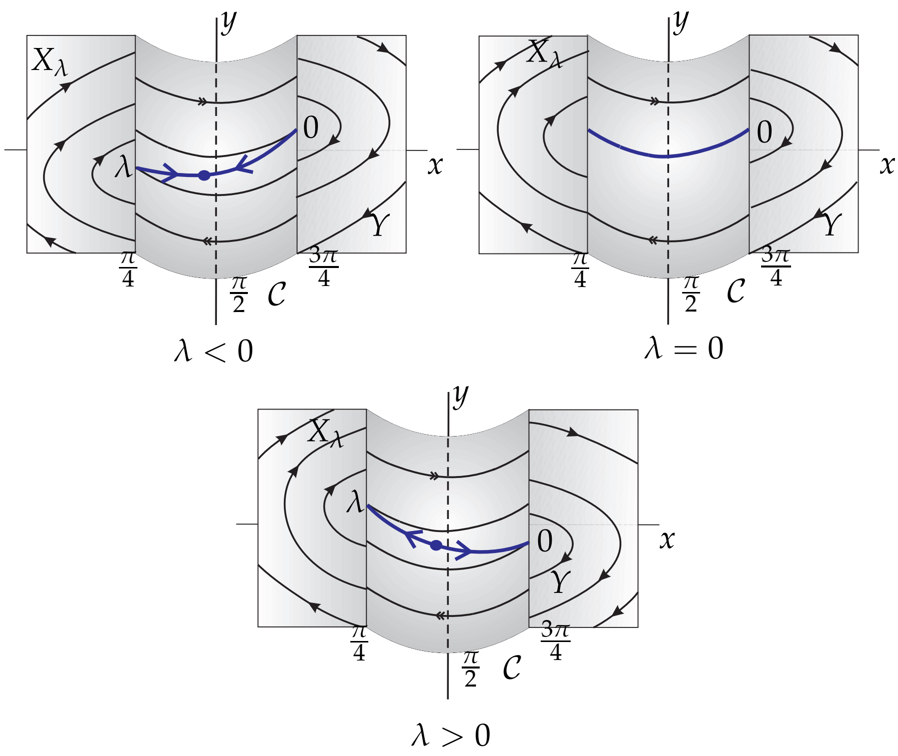

Figure 11: Bifurcation Diagram of the 1−parameter Elliptic Fold−Fold Singularity.

Distinctly of the hyperbolic and parabolic cases, the unfolding (14) does not give the generic unfolding of a non−smooth vector field presenting a elliptic fold−fold singularity.

Assume $Z_\lambda = (X_\lambda, Y)$ like in (14). The expression of its first return map is

$$\psi_{Z_\lambda}(y) = \gamma_Y \circ \gamma_{X_\lambda}(y) = y - 2\lambda, \tag{16}$$

where $\gamma_{X_\lambda}(p)$ (respectively $\gamma_Y(p)$) is the first return to Σ of the trajectory of X_λ (respectively Y) that passes through p. Observe that we can change the order of γ_Y and γ_{X_λ} and consequently we get another expression of ψ_{Z_λ}. However both expressions produce the same results and we fix the first one.

Therefore, we conclude that all points on Σ are fixed point of ψ_{Z_λ}. This configuration is clearly structurally unstable. In order to obtain the generic unfolding of this case we need to unfold the first return map. So, the unfolding of the elliptic fold−fold singularity depends on two parameters. The first, λ, is responsible by the displacement of one fold along the $y-$axis and another one, ε, for the unfolding of ψ_{Z_λ}.

So, the flows of X_λ and Y_ε are:

$$\begin{aligned}\phi^t_{X_\lambda}(x_0, y_0) &= (x_0 + (y_0 - \lambda)t + t^2/2, y_0 + t),\\ \phi^t_{Y_\varepsilon}(x_0, y_0) &= \left(x_0 + \int_0^t h_2^\varepsilon(s)\, ds, y_0 + \int_0^t g_2^\varepsilon(s)\, ds\right)\end{aligned}$$

where

$$Y_\varepsilon(x(t), y(t)) = (h_2^\varepsilon(t), g_2^\varepsilon(t)). \tag{17}$$

Let $t^* \in \mathbb{R}\backslash\{0\}$ and $t_1 = 2(\lambda - y_0)$ such that

$$\int_0^{t^*} h_2^\varepsilon(s)\, ds = 0 \tag{18}$$

and $\phi_{X_\lambda}^{t_1}(0, y_0) = (0, -y_0 + 2\lambda) \in \Sigma$.

Observe that there exist t^* as in Equation (18) because 0 is an invisible fold singularity of Y. We suppose that $h_2^\varepsilon(.)$ and $g_2^\varepsilon(.)$ satisfies:

(a) h_2, g_2 are C^r-functions;

(b) $Y_\varepsilon.\nabla f(0,0) = h_2^\varepsilon(0) = 0$;

(c) $Y_\varepsilon^2.\nabla f(0,0) = h_2^\varepsilon(0)\dfrac{d}{dx}h_2^\varepsilon(0) + g_2^\varepsilon(0)\dfrac{d}{dy}h_2^\varepsilon(0) \neq 0$;

(d) $\int_0^{t^*} g_2^\varepsilon(s)\, ds = (2+\varepsilon)y + O(y^2)$.

The hypotheses expressed in (18) and in the item (d) above give us sufficient conditions to get the unfolding of the first return map ψ_{Z_λ}. In fact, as we said before, all points on Σ are fixed points of the first return map expressed in (16). This holds because the derivative of $\psi_{Z_\lambda}(y)$ in (16) is equal to 1. However, with the previous hypothesis the smooth vector fields X_λ and Y_ε exhibited in (14) and (17), respectively, supply the unfolding of ψ_Z given by

$$\psi_{Z_{\lambda,\varepsilon}}(y) = \phi_{Y_\varepsilon}^{t^*} \circ \phi_{X_\lambda}^{t_1}(0, y) = (1+\varepsilon)y + 2\lambda + O(y^2), \tag{19}$$

whose derivative is equal to $(1+\varepsilon)$.

Therefore, the generic unfolding for the non$-$smooth vector field Z presenting an elliptic fold$-$fold singularity at the origin is $Z_{\lambda,\varepsilon} = (X_\lambda, Y_\varepsilon)$ where

$$Z_{\lambda,\varepsilon}(x,y) = \begin{cases} X_\lambda(x,y) = (y-\lambda, 1), & \text{if } (x,y) \in \Sigma_+, \\ Y_\varepsilon(x,y) = (h_2^\varepsilon(x,y), g_2^\varepsilon(x,y)), & \text{if } (x,y) \in \Sigma_- \end{cases} \tag{20}$$

and the smooth function h_2^ε and g_2^ε satisfies the conditions $(a), (b), (c)$ and (d) given previously.

6.5 Proof of Theorem 2

In this theorem we extend Theorem 1.1 of [10] considering that can exists a point q such that $X.\nabla f(q) = Y.\nabla f(q) = 0$, $X^2.\nabla f(q) \neq 0$ and $Y^2.\nabla f(q) \neq 0$. In this way, q is a $\Sigma-$fold point of both X and Y.

Consider a non$-$smooth dynamical system $Z_\lambda = (X_\lambda, Y)$ where $\lambda \in \mathbb{R}$ is a parameter. If with the variation of $\lambda \in (-\varepsilon, \varepsilon)$, the following behaviors are observable then we consider that Z_λ presents a bifurcation, where $\varepsilon > 0$ and small. Consider $\lambda^+ \in (0, \varepsilon)$ and $\lambda^- \in (-\varepsilon, 0)$. The behaviors are:

(i) A change of stability on Σ, i.e., where Z_{λ^+} has a sliding region Σ_3 the non$-$smooth vector field Z_{λ^-} has an escaping region Σ_2.

(ii) A change of stability on $\dot{y}_\lambda$, i.e., there are components of Σ such that the induced flow on the slow manifold is such that $\dot{y}_{\lambda^+} > 0$ and $\dot{y}_{\lambda^-} < 0$.

(iii) A change of stability of the $\Sigma-$singularity, i.e., Z_{λ^+} presents a $\Sigma-$attractor and Z_{λ^-} presents a $\Sigma-$repeller.

(iv) A change of orientation on Σ_1 (the sewer region), i.e., Z_{λ^+} and Z_{λ^-} presents distinct orientations on Σ_1.

In face of these previous observations, Theorem 2 follows straightforward from Section 6.

Note that, as we give the topological behavior of the cases $\lambda < 0$, $\lambda = 0$ and $\lambda > 0$ it is easy to construct the bifurcation diagram of (3) when $\lambda \in \mathbb{R}$.

The case $\mu = (\lambda, \varepsilon) \in \mathbb{R}^2$ is used in Subsection 6.4 with analogous results.

Acknowledgments. We would like to thank the referee for his comments and suggestions that make us improve this paper. The first author is partially supported by FAPESP$-$BRAZIL, grants 2007/08707-5 and 2010/18190-2. This work is partially realized at UFG/Brazil as part of projects number 35796 and 35797.

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IBILCE–UNESP, CEP 15054–000 S. J. Rio Preto,
São Paulo, Brazil
email:ti-car@hotmail.com

Universidade Federal de Goiás, IME,
CEP 74001-970 Caixa Postal 131, Goiânia, GO, Brazil.
email:djtonon@mat.ufg.br