On entire solutions of $f^2(z) + cf'(z) = h(z)$

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Abstract

We investigate the existence of entire solutions of non-linear differential equations of type $f^2(z) + cf'(z) = h(z)$, where $h(z)$ is a given entire function, whose zeros form an $A-$set. As a by-product of the studies, we give a negative answer to an open question raised in [4].

1 Introduction

We assume that the reader is familiar with the usual notations and basic results of the Nevanlinna theory, see, e.g. [2]. As an application of the theory and a study on the growth of an entire function $f(z)$, when $f(z)$ and its lth $(l \geq 2)$ derivative $f^{(l)}(z)$ have only a finite number of zeros, the following special case was obtained.

Theorem A([1]). *If $f(z)$ is an entire function with the property that $f(z)$ and $f''(z)$ have only a finitely many number of zeros, then $f(z) = P(z)e^{Q(z)}$, where $P(z)$ and $Q(z)$ are polynomials.*

In the same paper, the following result was derived.

Theorem B. *Let $f(z)$ be an entire function and $ff'' \neq 0$. Then $f(z) = e^{az+b}$, where a and b are constants.*

The above result was extended as follows.

Theorem C([4]). *Let $f(z)$ be a non-constant entire function with $f(z) \neq 0$. If $f''(z)$ can be expressed as $f''(z) = [H(z)]^m$ for some entire function $H(z)$ and an integer $m \geq 3$,*

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then $f(z) = e^{az+b}$, when m is even, and where a and b are constants, while when m is odd, $f(z) = e^{p(z)}$, where $p(z)$ is a polynomial.

Remark. *By examining the proof Theorem C more carefully, one can easily find that even when m is odd ≥ 3, the polynomial $p(z)$ in the theorem, in fact, must be linear. Thus only the case $m = 2$ has been left to be resolved.*

That is, we have

Theorem D. *Let $f(z)$ be a transcendental entire function such that $f(z) \neq 0$ and $f''(z) = [H(z)]^m$ for some entire function $H(z)$ and an integer $m \geq 3$. Then $f(z) = e^{az+b}$, for some constants $a(\neq 0)$ and b.*

Moreover, the following question was raised in [4].

Conjecture. *Let $f(z)$ be a transcendental entire function with $f(z) \neq 0$. Suppose that $f''(z) = h^2(z)$ for some entire function $h(z)$, then $f(z)$ must be of order 1 and has the form $f = e^{az+b}$, for some constants $a(\neq 0)$ and b.*

2 Notations and the main result

Here, we give a negative answer to the conjecture, by constructing a counter-example as follows.

Example. *Let $f(z) = e^{g(z)}$, where $g(z)$ is an entire function. Then*

$$f''(z) = \{g'^2(z) + g''(z)\}e^{g(z)}. \tag{2.1}$$

By setting $G(z) = g'(z)$ in the above equation, we consider the following differential equation:

$$G^2(z) + G'(z) = (G(z) + c)^2, \tag{2.2}$$

where c denotes a constant.

It follows that $G' - 2cG - c^2 = 0$, and hence $G(z) = -\frac{c}{2} + \frac{1}{2c}e^{2cz}$. Thus

$$f''(z) = \{G^2(z) + G'(z)\}e^{\int Gdz} = \{[G(z) + c]e^{\frac{1}{2}\int Gdz}\}^2 = h^2(z),$$

where $h(z) = [G(z) + c]e^{\frac{1}{2}\int Gdz}$. Note if $c \neq 0$, then $f(z)$ is of infinite order.

Remarks 1. *When $c \neq 0$, $G(z) + c = \frac{c}{2} + \frac{1}{2c}e^{2cz}$, which is of order 1 and whose zeros lie on a straight line.* **2**. *Clearly, if the constant c in the equation (2.2) is replaced by an arbitrary given entire function $A(z)$, then the equation (2.2) always has some entire solution. Moreover, if $A(z)$ is not a constant, then the solution is of order no less than 1.*

Before stating our main result, we introduce the following notion.

Definition. *A sequence $\{a_n\}$ of complex numbers is called a generalized $A-$ set, if there exists a linear function $L(z) = az + b$ such that*

$$\sum_{L(a_n)\neq 0} \mid Im\frac{1}{L(a_n)} \mid < +\infty. \tag{2.3}$$

Remark. *When $L(z) \equiv z$, then a generalized $A-$set is called an $A-$set. Particularly, if all except a finitely many of $\{a_n\}$ lie on a straight line, then $\{a_n\}$ forms an $A-$ set ([3]).*

Theorem 2.1. *Let $h(z)$ be a given entire function of order greater than 1 or order 1 of maximal-type, with all its zeros $\{a_n\}$ forming a generalized $A-$set. Then for any non-zero constant c, there exists no entire function $f(z)$ that satisfies the following differential equation*

$$f^2(z) + cf'(z) = h(z). \tag{2.4}$$

Corollary 2.2. *Let c denote a non-zero constant, $p(z)$ a non-zero polynomial, and $\Gamma(z)$ the Gamma function. Then the following differential equation*

$$f^2(z) + cf'(z) = \frac{p(z)}{\Gamma(z)}$$

has no entire solution.

Here as an extension of Theorem 2.1, we would like to pose the following:
Conjecture: *For any non-constant polynomial $c(z)$ and a non-zero polynomial $p(z)$, the following differential equation*

$$f^2(z) + c(z)f'(z) = \frac{p(z)}{\Gamma(z)}$$

has no entire solution.

3 Proof of the Theorem

In order to prove our result, the following lemma will be used.

Lemma 3.1. *([3, Theorem 6]) Suppose that $f(z)$ is meromorphic and of the form*

$$f(z) = \frac{P_1(z)}{P_2(z)} e^{Q(z)}, \tag{3.1}$$

where $P_1(z), P_2(z)$ and $Q(z)$ are entire functions. Assume that

$$\int_1^{+\infty} \frac{\log T(t, P_1) + \log T(t, P_2)}{t^2} dt < +\infty. \tag{3.2}$$

If, in addition, the zeros of $ff^{(n)}$, for some integer $n \geq 2$, form an $A-$set, then $Q(z)$ is of exponential type and

$$\log T(r, f) = O(r).$$

Remark. *Clearly, from the proof of the lemma, the assertion of the lemma remains to be valid if the zeros of $ff^{(n)}$ form a generalized $A-$set.*

Now we proceed with the proof of the theorem.

Assume that $f(z)$ is an entire solution of the eq. (2.4) and set

$$F(z) = e^{k\int f(z)dz},$$

where k is a constant such that $1/k = c$. Then

$$F''(z) = k^2 f^2(z) + kf'(z) = k^2\{f^2(z) + cf'(z)\}.$$

Note the zeros of $F''(z)$ are the zeros of $f^2(z) + cf'(z)$, which, by assumption, form a generalized $A-$set. It follows that the zeros of FF'' form a generalized $A-$set. Hence, by the lemma , one concludes immediately that $k\int f(z)dz$ is of exponential type, and so is $f(z)$. On the other hand, from the eq. (2.4), f has an order greater than 1 or order 1 of maximal-type, a contradiction. This also proves the theorem.

Finally, we conclude the paper with the following:

Question: *Let $f(z)$ be a transcendental entire function. Then for any integer $n \geq 3$, can $f^{(n)}$ be expressed as h^n, for some entire function $h(z)$?*

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