ε-simultaneous approximation and invariant points

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Abstract

In this paper we generalize and extend Brosowski-Meinardus type results on invariant points from the set of best approximation to the set of ε-simultaneous approximation. As a consequence some results on ε-approximation and best approximation are also deduced. The results proved extend and generalize some of the results of R.N. Mukherjee and V. Verma [Bull. Cal. Math. Soc. 81(1989) 191-196; Publ. de l'Inst. Math. 49(1991) 111-116], T.D. Narang and S. Chandok [Mat. Vesnik 61(2009) 165-171; Selçuk J. Appl. Math. 10(2009) 75-80; Indian J. Math. 51(2009) 293-303], G.S. Rao and S.A. Mariadoss [Serdica-Bulgaricae Math. Publ. 9(1983) 244-248] and of few others.

1 Introduction and Preliminaries

The idea of applying fixed point theorems to approximation theory was initiated by G. Meinardus [9]. Meinardus introduced the notion of invariant approximation in normed linear spaces. Generalizing the result of Meinardus, Brosowski [2] proved the following theorem on invariant approximation using fixed point theory:

Theorem 1.1. *Let T be a linear and nonexpansive operator on a normed linear space E. Let C be a T-invariant subset of E and x a T-invariant point. If the set $P_C(x)$ of best*

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C-approximants to x is non-empty, compact and convex, then it contains a T-invariant point.

Subsequently, various generalizations of Brosowski's results appeared in the literature. Singh [18] observed that the linearity of the operator T and convexity of the set $P_C(x)$ in Theorem 1.1 can be relaxed and proved the following:

Theorem 1.2. *Let $T : E \to E$ be a nonexpansive self mapping on a normed linear space E. Let C be a T-invariant subset of E and x a T-invariant point. If the set $P_C(x)$ is non-empty, compact and starshaped, then it contains a T-invariant point.*

Singh [19] further showed that Theorem 1.2 remains valid if T is assumed to be nonexpansive only on $P_C(x) \cup \{x\}$. Since then, many results have been obtained in this direction (see Mukherjee and Som [10], Mukherjee and Verma [12], Narang and Chandok ([13] [14] [15]), Rao and Mariadoss [16] and references cited therein).

In this paper we prove some similar types of results on T-invariant points for the set of ε-simultaneous approximation to a pair of points x_1, x_2 in a metric space (X, d) from a set C, which is not necessarily starshaped but has a jointly continuous contractive family. Some results on T-invariant points for the set of ε-approximation and best approximation are also deduced. The results proved in the paper generalize and extend some of the results of [11], [12], [13], [14], [15], [16] and of few others.

Let G be a non-empty subset of a metric space (X, d),$x_1, x_2 \in X$ and $\varepsilon > 0$. An element $g_0 \in G$ is said to be **ε-simultaneous approximation**(respectively, **ε-simultaneous coapproximation**) if $d(x_1, g_0) + d(x_2, g_0) \leq r + \varepsilon$, where $r = \inf\{d(x_1, g) + d(x_2, g) : g \in G\}$ (respectively, $d(g_0, g) + \varepsilon \leq \max\{d(x_1, g) + d(x_2, g) : g \in G\}$ for all $g \in G$). We shall denote by $P_G(x_1, x_2, \varepsilon)$ (respectively, $R_G(x_1, x_2, \varepsilon)$) the set of all ε-simultaneous approximation (respectively, ε-simultaneous coapproximation) to x_1, x_2.

It can be easily seen that for $\varepsilon = 0$, the set $P_G(x_1, x_2, \varepsilon)$ (respectively, $R_G(x_1, x_2, \varepsilon)$) is the set of best simultaneous approximations (respectively, best simultaneous coapproximations) of x_1, x_2 in G and further if $x_1 = x_2 = x$, then it reduces to the set of best approximations (respectively, best coapproximations) of x in G.

It can be easily seen that for $\epsilon > 0$, the set $P_G(x_1, x_2, \varepsilon)$ is always a non-empty bounded set and is closed if G is closed.

A sequence $< y_n >$ in G is called a **ε-minimizing sequence** for x_1, x_2 if $\lim_{n\to\infty} [d(x_1, y_n) + d(x_2, y_n)] \leq \inf\{d(x_1, y) + d(x_2, y) : y \in G\} + \varepsilon$. The set G is said to be **ε-simultaneous approximatively compact** if for every pair $x_1, x_2 \in X$, each ε-minimizing sequence $< y_n >$ in G has a subsequence $< y_{n_i} >$ converging to an element of G.

Let (X, d) be a metric space. A continuous mapping $W : X \times X \times [0, 1] \to X$ is said to be a **convex structure** on X if for all $x, y \in X$ and $\lambda \in [0, 1]$,

$$d(u, W(x, y, \lambda)) \leq \lambda d(u, x) + (1 - \lambda) d(u, y)$$

holds for all $u \in X$. The metric space (X, d) together with a convex structure is called a **convex metric space** [22].

A convex metric space (X,d) is said to satisfy **Property (I)** [5] if for all $x,y,p \in X$ and $\lambda \in [0,1]$,

$$d(W(x,p,\lambda),W(y,p,\lambda)) \leq \lambda d(x,y).$$

A normed linear space and each of its convex subset are simple examples of convex metric spaces. There are many convex metric spaces which are not normed linear spaces (see [22]). Property (I) is always satisfied in a normed linear space.

A subset K of a convex metric space (X,d) is said to be

i) a **convex set** [22] if $W(x,y,\lambda) \in K$ for all $x,y \in K$ and $\lambda \in [0,1]$;

ii) p**-starshaped** [7] where $p \in K$, provided $W(x,p,\lambda) \in K$ for all $x \in K$ and $\lambda \in [0,1]$;

iii) **starshaped** if it is p-starshaped for some $p \in K$.

Clearly, each convex set is starshaped but not conversely.

A self map T on a metric space (X,d) is said to be

i) **contraction** if $d(Tx,Ty) \leq kd(x,y)$ for all $x,y \in X$ and $0 \leq k < 1$;

ii) **nonexpansive** if $d(Tx,Ty) \leq d(x,y)$ for all $x,y \in X$;

iii) **quasi-nonexpansive** if the set $F(T)$ of fixed points of T is non-empty and $d(Tx,p) \leq d(x,p)$ for all $x \in X$ and $p \in F(T)$.

A nonexpansive mapping T on X with $F(T) \neq \emptyset$ is quasi-nonexpansive, but not conversely (see [20], p.27).

Let C be a subset of a metric space (X,d) and $\mathfrak{F} = \{f_\alpha : \alpha \in C\}$ a family of functions from $[0,1]$ into C, having the property $f_\alpha(1) = \alpha$, for each $\alpha \in C$. Such a family $\mathfrak{F}$ is said to be **contractive** if there exists a function $\phi : (0,1) \to (0,1)$ such that for all $\alpha, \beta \in C$ and for all $t \in (0,1)$,we have

$$d(f_\alpha(t), f_\beta(t)) \leq \phi(t)d(\alpha,\beta).$$

Such a family $\mathfrak{F}$ is said to be **jointly continuous** if $t \to t_\circ$ in $[0,1]$ and $\alpha \to \alpha_\circ$ in C imply $f_\alpha(t) \to f_{\alpha_\circ}(t_\circ)$ in C.

In normed linear spaces these notions were discussed by Dotson [4]. It was observed in [4] that if C is a starshaped subset (of a normed linear space) with star-center p then the family $\mathfrak{F} = \{f_\alpha : \alpha \in C\}$ defined by $f_\alpha(t) = (1-t)p + t\alpha$ is contractive if we take $\phi(t) = t$ for $0 < t < 1$, and is jointly continuous. The same is true for starshaped subsets of convex metric spaces with Property (I), by taking $f_\alpha(t) = W(\alpha,p,t)$ and so the class of subsets of X with the property of contractiveness and joint continuity contains the class of starshaped sets which in turn contains the class of convex sets.

2 Main Results

To start with, we prove the following proposition on ε-simultaneous approximation which will be used in the sequel.

Proposition 2.1. *If C a non-empty ε-simultaneous approximatively compact subset of a metric space (X, d), $x_1, x_2 \in X$, then the set $P_C(x_1, x_2, \varepsilon)$ is a non-empty compact subset of C.*

Proof. Since $\varepsilon > 0$, $P_C(x_1, x_2, \varepsilon)$ is non-empty.

We now show that $P_C(x_1, x_2, \varepsilon)$ is compact. Let $< y_n >$ be a sequence in $P_C(x_1, x_2, \varepsilon)$. Then $\lim[d(x_1, y_n) + d(x_2, y_n)] \le \inf\{d(x_1, y) + d(x_2, y) : y \in C\} + \varepsilon$, i.e. $< y_n >$ is an ε-minimizing sequence for the pair x_1, x_2 in C. Since C is ε-simultaneous approximatively compact, there is a subsequence $< y_{n_i} >$ such that $< y_{n_i} > \to y \in C$. Consider

$$\begin{aligned} d(x_1, y) + d(x_2, y) &= d(x_1, \lim y_{n_i}) + d(x_2, \lim y_{n_i}) \\ &= \lim\{d(x_1, y_{n_i}) + d(x_2, y_{n_i})\} \\ &\le \inf\{d(x_1, y) + d(x_2, y) : y \in C\} + \varepsilon. \end{aligned}$$

This implies that $y \in P_C(x_1, x_2, \varepsilon)$. Thus we get a subsequence $< y_{n_i} >$ of $< y_n >$ converging to an element $y \in P_C(x_1, x_2, \varepsilon)$. Hence $P_C(x_1, x_2, \varepsilon)$ is compact. ■

If $x_1 = x_2 = x$, we have the following result on ε-approximation.

Corollary 2.2. *(see [13]) If C is an ε-approximatively compact set in a metric space (X, d) then $P_C(x, \varepsilon)$ is a non-empty compact set.*

Further, if $\varepsilon = 0$, we have the following result.

Corollary 2.3. *(see, [14]) Let C be an approximatively compact subset of a metric space (X, d), $x \in X$ and $P_C(x) = \{y \in C : d(x, y) = d(x, C)\}$ is the set of best approximant to x in C then $P_C(x)$ is a non-empty compact subset of C.*

We shall be using the following result of Hardy and Rogers [6] in proving our first theorem.

Lemma 2.4. *Let F be a mapping from a complete metric space (X, d) into itself satisfying*

$$d(Fx, Fy) \le a[d(x, Fx) + d(y, Fy)] + b[d(y, Fx) + d(x, Fy)] + cd(x, y), \tag{2.1}$$

for any $x, y \in X$ where a, b and c are non-negative numbers such that $2a + 2b + c \le 1$. Then F has a unique fixed point u in X. In fact for any $x \in X$, the sequence $\{F^n x\}$ converges to u.

Theorem 2.5. *Let T be a continuous self map on a complete metric space (X, d) satisfying (2.1), C a T-invariant subset of X. Let $Tx_i = x_i$ $(i = 1, 2)$ for some x_1, x_2 not in $cl(C)$. If $P_C(x_1, x_2, \varepsilon)$ is compact and has a contractive jointly continuous family $\mathfrak{F}$, then it contains a T-invariant point.*

Proof. Let $D = P_C(x_1, x_2, \varepsilon)$ i.e.

$$D = \{z \in C : d(x_1, z) + d(x_2, z) \leq d(x_1, y) + d(x_2, y) + \varepsilon, \text{ for every } y \in C\}. \tag{2.2}$$

Let $z \in D$ be arbitrary. Then by (2.1), we have

$$\begin{aligned} d(x_1, Tz) + d(x_2, Tz) &= d(Tx_1, Tz) + d(Tx_2, Tz) \\ &\leq a[d(x_1, Tx_1) + d(z, Tz)] + b[d(z, Tx_1) + d(x_1, Tz)] + \\ &\quad cd(x_1, z) + a[d(x_2, Tx_2) + d(z, Tz)] + b[d(z, Tx_2) + \\ &\quad d(x_2, Tz)] + cd(x_2, z) \\ &= 2ad(z, Tz) + (b + c)[d(x_1, z) + d(x_2, z)] + b[d(x_1, Tz) + \\ &\quad d(x_2, Tz)] \\ &= a[d(z, Tz) - d(x_1, Tz)] + a[d(z, Tz) - d(x_2, Tz)] + \\ &\quad a[d(x_1, Tz) + d(x_2, Tz)] + (b + c)[d(x_1, z) + d(x_2, z)] + \\ &\quad b[d(x_1, Tz) + d(x_2, Tz)]. \end{aligned}$$

This gives,

$$(1 - a - b)[d(x_1, Tz) + d(x_2, Tz)] \leq (a + b + c)[d(x_1, z) + d(x_2, z)].$$

Hence

$$d(x_1, Tz) + d(x_2, Tz) \leq d(x_1, z) + d(x_2, z) \tag{2.3}$$

since $2a + 2b + c \leq 1$. Also, using (2.2), we get

$$d(x_1, Tz) + d(x_2, Tz) \leq d(x_1, y) + d(x_2, y) + \varepsilon \tag{2.4}$$

for all $y \in C$. Hence $Tz \in D$. Therefore T is a self map on D. Define $T_n : D \to D$ as $T_n x = f_{Tx}(\lambda_n)$, $x \in D$ where $< \lambda_n >$ is a sequence in $(0, 1)$ such that $\lambda_n \to 1$. Also

$$\begin{aligned} d(T_n x, T_n y) &= d(f_{Tx}(\lambda_n), f_{Ty}(\lambda_n)) \\ &\leq \phi(\lambda_n) d(Tx, Ty) \\ &\leq \phi(\lambda_n)[a[d(x, Tx) + d(y, Ty)] + b[d(y, Tx) + d(x, Ty)] + cd(x, y)] \end{aligned}$$

where $\phi(\lambda_n)[2a + 2b + c] \leq 1$. Therefore by Lemma 2.4, each T_n has a unique fixed point z_n in D. Since D is compact, there is a subsequence $< z_{n_i} >$ of $< z_n >$ such that $z_{n_i} \to z_\circ \in D$. We claim that $Tz_\circ = z_\circ$. Consider $z_{n_i} = T_{n_i} z_{n_i} = f_{Tz_{n_i}}(\lambda_{n_i}) \to f_{Tz_\circ}(1)$ as the family $\mathfrak{F}$ is jointly continuous and T is also continuous. Thus $z_{n_i} \to Tz_\circ$ and consequently, $Tz_\circ = z_\circ$ i.e. $z_\circ \in D$ is a T-invariant point. ∎

Since for an ε-simultaneous approximatively compact subset C of a metric space (X, d) the set of ε-simultaneous C-approximant is nonempty and compact (Proposition 2.1), we have the following result.

Corollary 2.6. *Let T be a continuous self map on a complete metric space (X, d) satisfying (2.1), C an ε-simultaneous approximatively compact and T-invariant subset of X. Let $Tx_i = x_i$ $(i = 1, 2)$ for some x_1, x_2 not in $cl(C)$. If the set $D = P_C(x_1, x_2, \varepsilon)$ has a contractive jointly continuous family $\mathfrak{F}$, then it contains a T-invariant point.*

Corollary 2.7. *Let T be a continuous self map on a complete convex metric space (X,d) with Property (I) satisfying (2.1), C an ε-simultaneous approximatively compact and T-invariant subset of X. Let $Tx_i = x_i$ $(i = 1,2)$ for some x_1, x_2 not in $cl(C)$. If the set $D = P_C(x_1, x_2, \varepsilon)$ is p-starshaped, then it contains a T-invariant point.*

Proof. Define $f_\alpha : [0,1] \to D$ as $f_\alpha(t) = W(\alpha, p, t)$. Then

$$d(f_\alpha(t), f_\beta(t)) = d(W(\alpha, p, t), W(\beta, p, t)) \leq td(\alpha, \beta),$$

$\phi(t) = t$, $0 < t < 1$, i.e. D is a contractive jointly continuous family. Taking $\lambda_n = \frac{n}{n+1}$ and defining $T_n(x) = f_{Tx}(\lambda_n) = W(Tx, p, \lambda_n)$, we get the result using Theorem 2.5. ∎

For $\varepsilon = 0$ in Theorem 2.5, we have the following results.

Corollary 2.8. *Let T be a continuous self map on a complete metric space (X,d) satisfying (2.1), C a T-invariant subset of X. Let $Tx_i = x_i$ $(i = 1,2)$ for some x_1, x_2 not in $cl(C)$. If $P_C(x_1, x_2)$ is nonempty, compact and has a contractive jointly continuous family $\mathfrak{F}$, then it contains a T-invariant point.*

Corollary 2.9. *Let T be a continuous self map on a complete metric space (X,d) satisfying (2.1), C an approximatively compact and T-invariant subset of X. Let $Tx_i = x_i$ $(i = 1,2)$ for some x_1, x_2 not in $cl(C)$. If the set $D = P_C(x_1, x_2)$ has a contractive jointly continuous family $\mathfrak{F}$, then it contains a T-invariant point.*

Corollary 2.10. *Let T be a continuous self map on a complete convex metric space (X,d) with Property (I) satisfying (2.1), C an approximatively compact and T-invariant subset of X. Let $Tx_i = x_i$ $(i = 1,2)$ for some x_1, x_2 not in $cl(C)$. If the set D of best simultaneous C-approximants to x_1, x_2 is p-starshaped, then it contains a T-invariant point.*

Corollary 2.11. *(see [12]) Let T be a continuous self map on a Banach space X satisfying (2.1), C an approximatively compact and T-invariant subset of X. Let $Tx_i = x_i$ $(i = 1,2)$ for some x_1, x_2 not in $cl(C)$. If the set of best simultaneous C-approximants to x_1, x_2 is starshaped, then it contains a T-invariant point.*

If $a = b = 0$ in Corollary 2.8, the map T becomes nonexpansive, so we have the following result.

Corollary 2.12. *(see [15]) Let T be a mapping on a metric space (X,d), C a T-invariant subset of X and x a T-invariant point. If $P_C(x)$ is a non-empty, compact set for which there exists a contractive jointly continuous family $\mathfrak{F}$ of functions and T is non-expansive on $P_C(x) \cup \{x\}$ then $P_C(x)$ contains a T-invariant point.*

Corollary 2.13. *(see [10]-Theorem 2, [17]-Theorem 3.4) Let T be nonexpansive operator on a normed linear space X. Let C be a T-invariant subset of X and x a T-invariant point. If $P_C(x)$ is non-empty, compact and for which there exists a contractive jointly continuous family $\mathfrak{F}$ of functions, then it contains a T-invariant point.*

Since for an approximatively compact subset C of a metric space (X,d) the set $P_C(x)$ is non-empty and compact (Corollary 2.3), we have:

Corollary 2.14. *Let T be a mapping on a metric space (X, d), C an approximatively compact, T-invariant subset of X and x a T-invariant point. If there exists a contractive jointly continuous family $\mathfrak{F}$ of functions and T is nonexpansive on $P_C(x) \cup \{x\}$, then $P_C(x)$ contains a T-invariant point.*

Corollary 2.15. *Let T be a mapping on a convex metric space (X, d) with Property (I), C an approximatively compact, p-starshaped, T-invariant subset of X and x a T-invariant point. If T is nonexpansive on $P_C(x) \cup \{x\}$, then $P_C(x)$ contains a T-invariant point.*

Corollary 2.16. *(see [14]-Theorem 4) Let T be a quasi-nonexpansive mapping on a convex metric space (X, d) with Property (I), C a T-invariant subset of X and x a T-invariant point. If $P_C(x)$ is nonempty, compact and starshaped, and T is nonexpansive on $P_C(x)$, then $P_C(x)$ contains a T-invariant point.*

Corollary 2.17. *(see [14]-Theorem 5) Let T be a quasi-nonexpansive mapping on a convex metric space (X, d) with Property (I), C an approximatively compact, T-invariant subset of X and x a T-invariant point. If $P_C(x)$ is starshaped and T is nonexpansive on $P_C(x)$, then $P_C(x)$ contains a T-invariant point.*

Remark 2.1. 1. If $a = b = 0$ and $x_1 = x_2 = x$ in Theorem 2.5, then it improves and generalizes Theorem 1 of Narang and Chandok [13].

2. Corollary 2.8 is a generalization and extension of Theorem 1 of Rao and Mariadoss [16] for a mapping T which maps the set D of best simultaneous C-approximants to $x_1, x_2 \in X$ into itself and the spaces undertaken are metric spaces.

We shall be using the following result of Bose and Mukherjee [1] in proving our next theorem.

Lemma 2.18. *Let $\{F_n\}$ be a sequence of self mappings of complete metric space (X, d) such that*

$$d(F_i x, F_j y) \le a_1 d(x, F_i x) + a_2 d(y, F_j y) + a_3 d(y, F_i x) + a_4 d(x, F_i y) + a_5 d(x, y), \quad (j > i) \tag{2.5}$$

for all $x, y \in X$ where $a_1, a_2, \ldots, a_5$ are non-negative numbers such that $\sum_{k=1}^{5} a_k < 1$ and $a_3 = a_4$. Then the sequence $\{F_n x\}$ has a unique common fixed point.

Theorem 2.19. *Let T_1 and T_2 be a pair of continuous self maps on a complete metric space (X, d) satisfying $d(T_1 x, T_2 y) \le d(x, y)$, for $x, y \in X$ $(x \neq y)$. Let the set C be T_i-invariant $(i = 1, 2)$ subset of X. Suppose that x_1 and x_2 are two common fixed points for the pair T_1 and T_2 not in $cl(C)$. If the set $D = P_C(x_1, x_2, \varepsilon)$ is compact and has a contractive jointly continuous family $\mathfrak{F}$, then it has a point which is both T_1- and T_2-invariant.*

Proof. Since x_1 and x_2 are common fixed points of T_1 and T_2, proceeding as in Theorem 2.5, we get that $T_1(D) \subseteq D$ and $T_2(D) \subseteq D$. Now we show that there is a point $z_\circ \in D$ such that $T_i z_\circ = z_\circ$ $(i = 1, 2)$. Define T_{1n} and T_{2n} as $T_{1n} x = f_{T_1 x}(\lambda_{1n})$, and $T_{2n} x = f_{T_2 x}(\lambda_{2n})$, $x \in D$ where $< \lambda_{1n} >$ and $< \lambda_{2n} >$ are sequences in $(0, 1)$ such that $< \lambda_{1n} >, < \lambda_{2n} > \to 1$. Then using Lemma 2.18, we have $T_{1n} z_n = T_{2n} z_n = z_n \in D$. Since D is compact, there is a subsequence

$< z_{n_i} >$ of $< z_n >$ such that $z_{n_i} \to z_\circ \in D$. We claim that $T_1 z_\circ = z_\circ = T_2 z_\circ$. Consider $z_{n_i} = T_{1n_i} z_{n_i} = f_{T_1 z_{n_i}}(\lambda_{1n_i}) \to f_{T_1 z_\circ}(1) = T_1 z_\circ$ as the family $\mathfrak{F}$ is jointly continuous and T_{1n} is continuous. Thus $z_{n_i} \to T_1 z_\circ$ and similarly, $z_{n_i} \to T_2 z_\circ$. Hence the result. ∎

Corollary 2.20. *Let T_1 and T_2 be a pair of continuous self maps on a complete metric space (X,d) satisfying $d(T_1x, T_2y) \leq d(x,y)$, for $x, y \in X$ $(x \neq y)$. Let C be an ε-simultaneous approximatively compact, T_i-invariant $(i = 1,2)$ subset of X. Suppose that x_1 and x_2 are two common fixed points for the pair T_1 and T_2 not in $cl(C)$. If the set $D = P_C(x_1, x_2, \varepsilon)$ has a contractive jointly continuous family $\mathfrak{F}$, then it has a point which is both T_1- and T_2-invariant.*

Corollary 2.21. *Let T_1 and T_2 be a pair of continuous self maps on a complete convex metric space (X,d) with Property (I) satisfying $d(T_1x, T_2y) \leq d(x,y)$, for $x, y \in X$ $(x \neq y)$. Let C be an ε-simultaneous approximatively compact, T_i-invariant $(i = 1,2)$ subset of X. Suppose that x_1 and x_2 are two common fixed points for the pair T_1 and T_2 not in $cl(C)$. If the set $D = P_C(x_1, x_2, \varepsilon)$ is starshaped, then it has a point which is both T_1- and T_2-invariant.*

For $\varepsilon = 0$, we have the following result.

Corollary 2.22. *Let T_1 and T_2 be a pair of continuous self maps on a complete metric space (X,d) satisfying $d(T_1x, T_2y) \leq d(x,y)$, for $x, y \in X$ $(x \neq y)$. Let the set C be T_i-invariant $(i = 1,2)$ subset of X. Suppose that x_1 and x_2 are two common fixed points for the pair T_1 and T_2 not in $cl(C)$. If the set $D = P_C(x_1, x_2)$ is nonempty, compact and has a contractive jointly continuous family $\mathfrak{F}$, then it has a point which is both T_1- and T_2-invariant.*

Corollary 2.23. *Let T_1 and T_2 be a pair of continuous self maps on a complete metric space (X,d) satisfying $d(T_1x, T_2y) \leq d(x,y)$, for $x, y \in X$ $(x \neq y)$. Let C be an approximatively compact, T_i-invariant $(i = 1,2)$ subset of X. Suppose that x_1 and x_2 are two common fixed points for the pair T_1 and T_2 not in $cl(C)$. If the set D of best simultaneous C-approximants to x_1, x_2 has a contractive jointly continuous family $\mathfrak{F}$, then it has a point which is both T_1- and T_2-invariant.*

Corollary 2.24. *Let T_1 and T_2 be a pair of continuous self maps on a complete convex metric space (X,d) with Property (I) satisfying $d(T_1x, T_2y) \leq d(x,y)$, for $x, y \in X$ $(x \neq y)$. Let C be an approximatively compact, T_i-invariant $(i = 1,2)$ subset of X. Suppose that x_1 and x_2 are two common fixed points for the pair T_1 and T_2 not in $cl(C)$. If the set D of best simultaneous C-approximants to x_1, x_2 is starshaped, then it has a point which is both T_1- and T_2-invariant.*

Corollary 2.25. *(see [12]) Let T_1 and T_2 be a pair of continuous self maps on a Banach space X satisfying $d(T_1x, T_2y) \leq d(x,y)$, for $x, y \in X$ $(x \neq y)$. Let C be an approximatively compact, T_i-invariant $(i = 1,2)$ subset of X. Suppose that x_1 and x_2 are two common fixed points for the pair T_1 and T_2 not in $cl(C)$. If the set D of best simultaneous C-approximants to x_1, x_2 is starshaped, then it has a point which is both T_1- and T_2-invariant.*

Definition 2.1. A subset K of a metric space (X, d) is said to be **contractive** if there exists a sequence $\langle f_n \rangle$ of contraction mappings of K into itself such that $f_n y \to y$ for each $y \in K$.

Theorem 2.26. *Let T be a self mapping on a metric space (X, d), G a T-invariant subset of X and $Tx_i = x_i$ $(i = 1, 2)$ for some x_1, x_2 not in $cl(G)$. If T is nonexpansive and the set $D = P_G(x_1, x_2, \varepsilon)$ is compact and contractive, then D contains a T-invariant point.*

Proof. Proceeding as in Theorem 2.5, we can prove that T is a self map of D. Since D is contractive, there exists a sequence $\langle f_n \rangle$ of contraction mapping of D into itself such that $f_n z \to z$ for every $z \in D$.

Clearly, $f_n T$ is a contraction on the compact set D for each n and so by Banach contraction principle, each $f_n T$ has a unique fixed point, say z_n in D. Now the compactness of D implies that the sequence $\langle z_n \rangle$ has a subsequence $\langle z_{n_i} \rangle \to z_\circ \in D$. We claim that $z_\circ$ is a fixed point of T. Let $\varepsilon > 0$ be given. Since $z_{n_i} \to z_\circ$ and $f_n T z_\circ \to T z_\circ$, there exist a positive integer m such that for all $n_i \geq m$

$$d(z_{n_i}, z_\circ) < \frac{\varepsilon}{2} \text{ and } d(f_{n_i} T z_\circ, T z_\circ) < \frac{\varepsilon}{2}.$$

Again,

$$d(f_{n_i} T z_{n_i}, f_{n_i} T z_\circ) \leq d(z_{n_i}, z_\circ) < \frac{\varepsilon}{2}.$$

Hence

$$\begin{aligned} d(f_{n_i} T z_{n_i}, T z_\circ) &\leq d(f_{n_i} T z_{n_i}, f_{n_i} T z_\circ) + d(f_{n_i} T z_\circ, T z_\circ) \\ &< \frac{\varepsilon}{2} + \frac{\varepsilon}{2} \end{aligned}$$

i.e. $d(f_{n_i} T z_{n_i}, T z_\circ) < \varepsilon$ for all $n_i \geq m$ and so $f_{n_i} T z_{n_i} \to T z_\circ$. But $f_{n_i} T z_{n_i} = z_{n_i} \to z_\circ$ and therefore $T z_\circ = z_\circ$. ∎

Using Proposition 2.1 we have the following result.

Corollary 2.27. *Let T be a self mapping on a metric space (X, d), G an ε- simultaneous approximatively compact, T-invariant subset of X and $Tx_i = x_i$ $(i = 1, 2)$ for some x_1, x_2 not in $cl(G)$. If T is nonexpansive and the set $D = P_G(x_1, x_2, \varepsilon)$ is contractive, then D contains a T-invariant point.*

If $\varepsilon = 0$, we have the following results.

Corollary 2.28. *Let T be a self mapping on a metric space (X, d), G a T-invariant subset of X and $Tx_i = x_i$ $(i = 1, 2)$ for some x_1, x_2 not in $cl(G)$. If T is nonexpansive and the set $D = P_G(x_1, x_2)$ is nonempty compact, contractive, then D contains a T-invariant point.*

Corollary 2.29. *Let T be a self mapping on a metric space (X, d), G an ε- approximatively compact, T-invariant subset of X and $Tx_i = x_i$ $(i = 1, 2)$ for some x_1, x_2 not in $cl(G)$. If T is nonexpansive and the set $D = P_G(x_1, x_2)$ is contractive, then D contains a T-invariant point.*

Remark 2.2. Theorem 2.26 improves and generalizes the corresponding results of Brosowski [2], Mukherjee and Verma [11] [12], Narang and Chandok [13], Rao and Mariadoss [16], Singh [18] and of Subrahmanyam [21].

Definition 2.2. For each bounded subset G of a metric space (X,d), the **Kuratowski's measure of noncompactness** of G, $\alpha[G]$ is defined as,

$$\alpha[G] = \inf\{\varepsilon > 0 : G \text{ is covered by a finite number of closed balls centered at points of } X \text{ of radius } \leq \varepsilon\}.$$

A mapping $T : X \to X$ is called **condensing** if for all bounded sets $G \subset X$, $\alpha[T(G)] \leq \alpha[G]$.

Lemma 2.30. *[3] Let X be a complete contractive metric space with contractions $\{f_n\}$. Let C be a closed bounded subsets of X and $f_n : C \to C$ is nonexpansive and condensing, then T has a fixed point in C.*

Theorem 2.31. *Let (X,d) be a complete, contractive metric space with contractions f_n. Let G be a closed and bounded subset of X. If T is a nonexpansive and condensing self map on X such that $Tx_1 = x_1$ and $Tx_2 = x_2$ for some $x_1, x_2 \in X$, then $D = P_G(x_1, x_2, \varepsilon)$ has a T-invariant point.*

Proof. As G is closed and bounded, D is nonempty, closed and bounded. Using Theorem 2.5, we can prove that T is a self map of D. Now a direct application of Lemma 2.30, gives a T-invariant point in D. ∎

Corollary 2.32. *([12]-Theorem 3.1) Let X be a complete, contractive metric space with contractions f_n. Let G be a closed and bounded subset of X. If T is a nonexpansive and condensing self map on X such that $Tx_1 = x_1$ and $Tx_2 = x_2$ for some $x_1, x_2 \in X$, and $D = P_G(x_1, x_2)$ is nonempty, then it has a T-invariant point.*

Corollary 2.33. *([16]-Theorem 4) Let X be a complete, contractive metric space with contractions f_n. Let G be a closed and bounded subset of X. If T is a nonexpansive and condensing self map on X such that $Tx = x$ for some $x \in X$, and $P_G(x)$ is nonempty, then it has a T-invariant point.*

Definition 2.3. A mapping T on a metric space (X,d) is called **Kannan**[8] if there exists $\alpha \in (0, \frac{1}{2})$ such that

$$d(Tx, Ty) \leq \alpha[d(x, Tx) + d(y, Ty)] \tag{2.6}$$

for all $x, y \in X$.

Kannan [8] proved that if X is complete, then every Kannan mapping has a unique fixed point.

Theorem 2.34. *Let G be an ε-simultaneous approximatively compact subset of a complete metric space (X,d). Let T be a self map on X with $Tx_1 = x_1$ and $Tx_2 = x_2$ for some $x_1, x_2 \in X \backslash G$ and let T^m satisfies*

$$d(T^m y, T^m z) \leq \alpha[d(y, T^m y) + d(z, T^m z)], \tag{2.7}$$

for some positive integer m, $y, z \in G$ and $0 < \alpha < \frac{1}{2}$. Then $D = P_G(x_1, x_2, \varepsilon)$ has a unique fixed point of T.

Proof. As $Tx_1 = x_1$, and $Tx_2 = x_2$, $T^n x_1 = x_1$ and $T^n x_2 = x_2$ for all positive integer n. Let $y_0 \in D$. Then, for $0 < \alpha < \frac{1}{2}$,

$$\begin{aligned} & d(x_1, T^m y_0) + d(x_2, T^m y_0) \\ & \quad = d(Tx_1, T^m y_0) + d(Tx_2, T^m y_0) \\ & \quad \leq \alpha[d(x_1, T^m x_1) + d(y_0, T^m y_0)] + \alpha[d(x_2, T^m x_2) + d(y_0, T^m y_0)] \\ & \quad = 2\alpha d(y_0, T^m y_0) \\ & \quad \leq \alpha[d(y_0, x_1) + d(x_1, T^m y_0)] + \alpha[d(y_0, x_2) + d(x_2, T^m y_0)], \end{aligned}$$

which implies that

$$d(x_1, T^m y_0) + d(x_2, T^m y_0) \leq \frac{\alpha}{1-\alpha} d(y_0, x_1) + \frac{\alpha}{1-\alpha} d(y_0, x_2).$$

Further, for all $y \in G$, we have

$$d(x_1, T^m y_0) + d(x_2, T^m y_0) \leq \frac{\alpha}{1-\alpha}[d(y, x_1) + d(y, x_2) + \varepsilon].$$

Therefore, $T^m y_0 \in D$, $T^m(D) \subset D$. Since T^m satisfies the conditions of Kannan map, T^m has a unique fixed point x_0 in D. Now, $T^m(Tx_0) = T(T^m x_0) = Tx_0$, implies that Tx_0 is a fixed point of T^m. But the fixed point of T^m is unique and equals x_0. Therefore $Tx_0 = x_0$ and hence x_0 is a unique fixed point of T in D. ∎

Remarks 2.1. i) If $\varepsilon = 0$, Theorem 2.34 extends Theorem 3.2 of Mukherjee and Verma [12] and further if $x_1 = x_2 = x$, then it extends Theorem 5 of Rao and Mariadoss [16].

ii)It is interesting to note that Theorem 2.34 gives a unique fixed point in the set $P_G(x_1, x_2, \varepsilon)$ and it also extends Brosowski's result to a generalized form (2.7) of Kannan map (2.6).

We now prove a result for T-invariant points from the set of ε-simultaneous coapproximations.

A mapping $T : X \to X$ satisfies **condition (A)** (see [11]) if $d(Tx, y) \leq d(x, y)$ for all $x, y \in X$.

Theorem 2.35. *Let T be a self map satisfying condition (A) and inequality (2.1) on a convex metric space (X, d) satisfying Property (I), G a subset of X such that $R_G(x_1, x_2, \varepsilon)$ is compact and starshaped, then $R_G(x_1, x_2, \varepsilon)$ contains a T-invariant point.*

Proof. Let $g_\circ \in R_G(x_1, x_2, \varepsilon)$. Consider

$$d(Tg_\circ, g) + \varepsilon \leq d(g_\circ, g) + \varepsilon \leq \max\{d(x_1, g), d(x_2, g)\},$$

for all $g \in G$ and so $Tg_\circ \in R_G(x_1, x_2, \varepsilon)$ i.e. $T : R_G(x_1, x_2, \varepsilon) \to R_G(x_1, x_2, \varepsilon)$. Since $R_G(x_1, x_2, \varepsilon)$ is starshaped, there exists $p \in R_G(x_1, x_2, \varepsilon)$ such that $W(z, p, \lambda) \in R_G(x_1, x_2, \varepsilon)$ for all $z \in R_G(x_1, x_2, \varepsilon)$, $\lambda \in [0, 1]$. Let $< k_n >$, $0 \leq k_n < 1$, be a sequence of real numbers such that $k_n \to 1$ as $n \to \infty$. Define T_n as $T_n(z) = W(Tz, p, k_n)$, $z \in R_G(x_1, x_2, \varepsilon)$. Since T is a self map on $R_G(x_1, x_2, \varepsilon)$

and $R_G(x_1, x_2, \varepsilon)$ is starshaped, each T_n is a well defined and maps $R_G(x_1, x_2, \varepsilon)$ into $R_G(x_1, x_2, \varepsilon)$. Moreover,

$$\begin{aligned} d(T_n y, T_n z) &= d(W(Ty, p, k_n), W(Tz, p, k_n)) \\ &\le k_n d(Ty, Tz) \\ &\le k_n[a[d(y, Ty) + d(z, Tz)] + b[d(z, Ty) + d(y, Tz)] + cd(y, z)], \end{aligned}$$

where $k_n[2a + 2b + c] \le 1$. So by Lemma 2.4 each T_n has a unique fixed point $x_n \in R_G(x_1, x_2, \varepsilon)$ i.e. $T_n x_n = x_n$ for each n. Since $R_G(x_1, x_2, \varepsilon)$ is compact, $\langle x_n \rangle$ has a subsequence $x_{n_i} \to x \in R_G(x_1, x_2, \varepsilon)$. We claim that $Tx = x$. Consider,

$$\begin{aligned} d(x_{n_i}, Tx) &= d(T_{n_i} x_{n_i}, Tx) \\ &= d(W(Tx_{n_i}, p, k_{n_i}), Tx) \\ &\le k_{n_i} d(Tx_{n_i}, Tx) + (1 - k_{n_i}) d(p, Tx) \\ &\le k_{n_i}[a[d(x_{n_i}, Tx_{n_i}) + d(x, Tx)] + b[d(x, Tx_{n_i}) + d(x_{n_i}, Tx)] + \\ &\quad cd(x_{n_i}, x)] + (1 - k_{n_i}) d(p, Tx) \\ &\le k_{n_i}[a[d(x_{n_i}, x_{n_i}) + d(x, x)] + b[d(x, x_{n_i}) + d(x_{n_i}, x)] + cd(x_{n_i}, x)] \\ &\quad +(1 - k_{n_i}) d(p, x) \\ &\to 0, \end{aligned}$$

and so $x_{n_i} \to Tx$. Therefore $Tx = x$ i.e. x is T-invariant. Hence the result. ∎

If $\varepsilon = 0$ in the above theorem we have the following result.

Corollary 2.36. *Let T be a self map satisfying condition (A) and inequality (2.1) on a convex metric space (X, d) satisfying Property (I), G a subset of X such that $R_G(x_1, x_2)$ is nonempty compact and starshaped, then $R_G(x_1, x_2)$ contains a T-invariant point.*

Remarks 2.2. i) Taking $x_1 = x_2 = x$ and $a = b = 0$, we see that Theorem 2.35 improves and generalizes Theorem 4 of Narang and Chandok [13].

ii) Taking $x_1 = x_2 = x$, $a = b = 0$ and $\varepsilon = 0$, we see that Theorem 2.35 improves and generalizes Theorem 4.1 of Mukherjee and Verma [11].

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