Quasilinearization Method for Functional Differential Equations with Delayed Arguments

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Abstract

We apply the quasilinearization method to boundary value problems for first order functional differential equations with delayed arguments. We formulate sufficient conditions for semi-quadratic or quadratic convergence of corresponding monotone sequences to a unique solution.

1 Introduction

Let us consider the problem

$$\begin{cases} x'(t) &= f(t, x(t), x(\alpha(t))), \quad t \in J = [0, T], \\ x(0) &= \lambda_1 x(T) + \lambda_2, \quad \lambda_1 \in [0, 1). \end{cases} \tag{1.1}$$

where $f \in C^2(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ and function $\alpha \in C(J, J)$ is such that $\alpha(t) \leq t$ for $t \in J$. By $C^2(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ we mean the space of functions $f = f(t, x, y)$ such that $f, f_x, f_y, f_{xx}, f_{xy}, f_{yx}, f_{yy} \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$.

A fruitful technique for proving existence results for differential equations is the monotone iterative method (see [2] –[4], [6], [8] –[10]). It gives a constructive procedure for approximation of solutions. However, from the practical point of view it is important to have higher order of convergence of sequences of the approximate solutions. Therefore in this paper we will focus on quasilinearization method (for details see for example [7], see also [1], [5]) for the above boundary value problem for functional differential equation with delayed argument. It is a

Received by the editors August 2009.

Communicated by J. Mawhin.

2000 *Mathematics Subject Classification* : 34A45, 34B15.

Key words and phrases : Functional differential equations, quasilinearization, quadratic convergence, semi-quadratic convergence.

well known method for problems without deviating arguments. According to my knowledge it haven't been applied for boundary value problems with deviated arguments so my results are new.

The plan of this paper is as follows. In Section 2 we formulate Lemma which is needed in succeeding sections. In Section 3 we consider the case when $\lambda_1 = 0$ i.e. the case when we have an initial value problem. Under natural assumptions, we prove quadratic convergence of monotone sequences to a unique solution. In Section 4 we consider the case $\lambda_1 \in (0,1)$. First we show the semi-quadratic convergence of corresponding monotone sequences to a unique solution. Next, under a little more restricted assumptions, we prove quadratic convergence. Note that the corresponding sequences are defined differently in each case. In the last section we give an example to verify the required assumptions.

2 Preliminaries

Lemma 1. *Assume that $\alpha \in C(J,J)$, $\alpha(t) \le t$ on J, $K \in C(J,\mathbb{R})$, $p \in C^1(J,\mathbb{R})$, and system*

$$\begin{array}{rcl} p'(t) & \le & K(t)p(t) + L(t)p(\alpha(t)), \quad t \in J, \\ p(0) & \le & \lambda p(T), \quad \lambda \in [0,1) \end{array}$$

is satisfied where nonnegative function L, integrable on J, is such that

$$\tilde{\lambda} + \int_0^T L(s) e^{-\int_{\alpha(s)}^s K(\tau)d\tau} ds < 1, \tag{2.1}$$

where $\tilde{\lambda} = \lambda e^{\int_0^T K(s)ds}$.
Then $p(t) \le 0$ on J.

Proof. Define

$$q(t) = e^{-\int_0^t K(s)ds} p(t).$$

We have $q(0) \le \tilde{\lambda} q(T)$ and

$$q'(t) = e^{-\int_0^t K(s)ds}(-K(t)p(t) + p'(t)) \le q(\alpha(t))L(t)e^{-\int_{\alpha(t)}^t K(s)ds}.$$

Hence

$$q(t) \le q(0) + \int_0^t q(\alpha(s))L(s)e^{-\int_{\alpha(s)}^s K(\tau)d\tau} ds,$$

so

$$q(0) \le \frac{\tilde{\lambda}}{1-\tilde{\lambda}} \int_0^T q(\alpha(s))L(s)e^{-\int_{\alpha(s)}^s K(\tau)d\tau} ds$$

and finally

$$q(t) \le \frac{\tilde{\lambda}}{1-\tilde{\lambda}} \int_0^T q(\alpha(s))L(s)e^{-\int_{\alpha(s)}^s K(\tau)d\tau} ds + \int_0^t q(\alpha(s))L(s)e^{-\int_{\alpha(s)}^s K(\tau)d\tau} ds.$$

Conversely, assume that there exists $t_0 \in J$ such that $p(t_0) > 0$, and consequently $q(t_0) > 0$. Put $q(t_1) = \max_{t \in J} q(t) > 0$. Then we get

$$q(t_1) \le q(t_1) \frac{1}{1-\tilde{\lambda}} \int_0^T L(s)e^{-\int_{\alpha(s)}^s K(\tau)d\tau} ds$$

Thus

$$q(t_1)\left[1 - \frac{1}{1-\tilde{\lambda}} \int_0^T L(s)e^{-\int_{\alpha(s)}^s K(\tau)d\tau}\, ds\right] \le 0,$$

which is contrary to the assumption. This proves that $p(t) \le 0$ on J. ■

Remark 1. If $K(t) \ge 0$, $t \in J$ and $\tilde{\lambda} + \int_0^T L(s)ds < 1$ then condition (2.1) holds. Note that this condition does not depend on α.

3 Case 1: $\lambda_1 = 0$

Let us define functions V_1, V_2 by

$$V_1(t,u,v) = F_y(t,u(t),u(\alpha(t))) + G_y(t,v(t),v(\alpha(t))),$$

$$V_2(t,u,v) = F_z(t,u(t),u(\alpha(t))) + G_z(t,v(t),v(\alpha(t))),$$

for corresponding functions F_y, G_y, F_z, G_z.

Theorem 1. *Let $\lambda_1 = 0$. Assume that*

1. $\alpha \in C(J,J)$, $\alpha(t) \le t$, $f = F + G$, *where* $F, G \in C^2(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$
2. $y_0, z_0 \in C^1(J,\mathbb{R})$, $y_0(t) \le z_0(t), t \in J$ *and satisfy the system*

$$\begin{aligned} y_0'(t) &\le f(t,y_0(t),y_0(\alpha(t))), \quad y_0(0) \le \lambda_2 \\ z_0'(t) &\ge f(t,z_0(t),z_0(\alpha(t))), \quad z_0(0) \ge \lambda_2, \end{aligned}$$

3. *for* $y_0(t) \le u \le z_0(t)$, $y_0(\alpha(t)) \le v \le z_0(\alpha(t))$ *we have*

$$\begin{aligned} &F_{yy}(t,u,v) \ge 0, \quad F_{yz}(t,u,v) \ge 0, \quad F_{zz}(t,u,v) \ge 0 \\ &G_{yy}(t,u,v) \le 0, \quad G_{yz}(t,u,v) \le 0, \quad G_{zz}(t,u,v) \le 0, \end{aligned}$$

4. $V_2(t,y_0,z_0) \ge 0$

5. *assumption* (2.1) *is satisfied for* $\lambda = \lambda_1$ *and functions* $K \in C(J, \mathbb{R})$, $L \in C(J, \mathbb{R}_+)$ *such that*
$$V_1(t, y_0, z_0) \geq K(t),$$
$$V_2(t, z_0, y_0) \leq L(t)$$
on J.

Then there exist sequences $\{y_n\}, \{z_n\} \subset C^1(J, \mathbb{R})$ *converging to a unique solution* $x \in C^1(J, \mathbb{R})$ *of problem* (1.1) *in the sector* $[y_0, z_0]_* = \{w \in C^1(J, \mathbb{R}) : y_0(t) \leq w(t) \leq z_0(t), t \in J\}$. *Moreover the convergence is quadratic i.e. there exist nonnegative constants* $c_1, c_2, \bar{c_1}, \bar{c_2}$ *such that for* $n = 0, 1, \ldots$

$$\max_{t\in J} |x(t) - y_{n+1}(t)| \leq c_1 \max_{t\in J} |x(t) - y_n(t)|^2 + c_2 \max_{t\in J} |z_n(t) - x(t)|^2$$

and

$$\max_{t\in J} |z_{n+1}(t) - x(t)| \leq \bar{c_1} \max_{t\in J} |x(t) - y_n(t)|^2 + \bar{c_2} \max_{t\in J} |z_n(t) - x(t)|^2.$$

Proof. For $n = 0, 1, \ldots$, define sequences $\{y_n\}, \{z_n\}$ as follows

$$\begin{cases} y'_{n+1}(t) &= f(t, y_n(t), y_n(\alpha(t))) + V_1(t, y_n, z_n)[y_{n+1}(t) - y_n(t)] \\ &+ V_2(t, y_n, z_n)[y_{n+1}(\alpha(t)) - y_n(\alpha(t))], \\ y_{n+1}(0) &= \lambda_2, \end{cases}$$

and

$$\begin{cases} z'_{n+1}(t) &= f(t, z_n(t), z_n(\alpha(t))) + V_1(t, y_n, z_n)[z_{n+1}(t) - z_n(t)] \\ &+ V_2(t, y_n, z_n)[z_{n+1}(\alpha(t)) - z_n(\alpha(t))], \\ z_{n+1}(0) &= \lambda_2. \end{cases}$$

Note that sequences $\{y_n\}, \{z_n\}$ are well defined as solutions of linear problems with initial conditions (use a Banach fixed point theorem with a corresponding norm).

First, we show that $y_0(t) \leq y_1(t) \leq z_1(t) \leq z_0(t)$. Put $p = y_0 - y_1$. Then $p(0) \leq 0$ and

$$\begin{aligned} p'(t) &\leq f(t, y_0(t), y_0(\alpha(t))) - f(t, y_0(t), y_0(\alpha(t))) \\ &- V_1(t, y_0, z_0)[y_1(t) - y_0(t)] \\ &- V_2(t, y_0, z_0)[y_1(\alpha(t)) - y_0(\alpha(t))] \\ &= V_1(t, y_0, z_0)p(t) + V_2(t, y_0, z_0)p(\alpha(t)), \end{aligned}$$

by assumption (2). In view of Lemma 1 (with $\lambda = 0$), we obtain that $p(t) \leq 0$ on J, which means that $y_0(t) \leq y_1(t)$. Analogically we can show that $z_1(t) \leq z_0(t)$, $t \in J$.

Using a mean value theorem and monotonicity of F_y, F_z, G_y, G_z (condition (3)), we have

$$\begin{aligned} &f(t, y_0(t), y_0(\alpha(t))) - f(t, z_0(t), z_0(\alpha(t))) \\ &= -[F_y(t, \xi(t), z_0(\alpha(t))) + G_y(t, \xi_1(t), z_0(\alpha(t)))][z_0(t) - y_0(t)] \end{aligned}$$

$$
\begin{aligned}
&-[F_z(t,y_0(t),\zeta(t)) + G_z(t,y_0(t),\zeta_1(t))][z_0(\alpha(t)) - y_0(\alpha(t))] \\
&\leq -V_1(t,y_0,z_0)[z_0(t) - y_0(t)] - V_2(t,y_0,z_0)[z_0(\alpha(t)) - y_0(\alpha(t))], \quad (3.1)
\end{aligned}
$$

where $y_0(t) < \xi(t) < z_0(t)$, $y_0(t) < \xi_1(t) < z_0(t)$, $y_0(\alpha(t)) < \zeta(t) < z_0(\alpha(t))$, $y_0(\alpha(t)) < \zeta_1(t) < z_0(\alpha(t))$ for $t \in J$.

Put $p = y_1 - z_1$. Then $p(0) = 0$. Using the definition of y_1, z_1 and relation (3.1) we see that

$$p'(t) \quad \leq \quad V_1(t,y_0,z_0)p(t) + V_2(t,y_0,z_0)p(\alpha(t)).$$

In view of Lemma 1, we get $p(t) \leq 0$ on J. It proves that

$$y_0(t) \leq y_1(t) \leq z_1(t) \leq z_0(t), \quad t \in J.$$

Now we are going to show that y is a lower solution of problem (1.1). Using definition of y_1, assumption (3) and the relation (3.1) with y_1 instead of z_0, we obtain

$$
\begin{aligned}
y_1'(t) \quad &\leq \quad -V_1(t,y_0,y_1)[y_1(t) - y_0(t)] - V_2(t,y_0,y_1)[y_1(\alpha(t)) - y_0(\alpha(t))] \\
&+ \quad V_1(t,y_0,z_0)[y_1(t) - y_0(t)] + V_2(t,y_0,z_0)[y_1(\alpha(t)) - y_0(\alpha(t))] \\
&+ \quad f(t,y_1(t),y_1(\alpha(t))) \leq f(t,y_1(t),y_1(\alpha(t))).
\end{aligned}
$$

It proves that y_1 is a lower solution of (1.1). Analogically, we can prove that z_1 is an upper solution of (1.1).

Now, using mathematical induction we can show that

$$y_0(t) \leq y_1(t) \leq \ldots \leq y_n(t) \leq z_n(t) \leq \ldots \leq z_1(t) \leq z_0(t), \quad n = 0, 1, \ldots, \quad t \in J.$$

Sequences $\{y_n\}, \{z_n\}$ are uniformly bounded and equicontinuous on J. In view of Arzela-Ascoli theorem there exist subsequences $\{y_{n_k}\}$, $\{z_{n_k}\}$ of $\{y_n\}$, $\{z_n\}$ converging uniformly on J to some continuous functions y, z respectively. Functions y_{n_k}, z_{n_k} satisfy integral equations

$$
\begin{cases}
y_{n_k+1}(t) & = \quad y_{n_k+1}(0) + \int_0^t f(s, y_{n_k}(s), y_{n_k}(\alpha(s)))ds \\
& + \quad \int_0^t V_1(s, y_{n_k}, z_{n_k})[y_{n_k+1}(s) - y_{n_k}(s)]ds \\
& + \quad \int_0^t V_2(s, y_{n_k}, z_{n_k})[y_{n_k+1}(\alpha(s)) - y_{n_k}(\alpha(s))]ds, \\
y_{n_k+1}(0) & = \quad \lambda_2,
\end{cases}
$$

and

$$\begin{cases} z_{n_k+1}(t) &= z_{n_k+1}(0) + \int\limits_0^t f(s, z_{n_k}(s), z_{n_k}(\alpha(t)))ds \\ &+ \int\limits_0^t V_1(s, y_{n_k}, z_{n_k})[z_{n_k+1}(s) - z_{n_k}(s)]ds \\ &+ \int\limits_0^t V_2(s, y_{n_k}, z_{n_k})[z_{n_k+1}(\alpha(s)) - z_{n_k}(\alpha(s))]ds, \\ z_{n_k+1}(0) &= \lambda_2. \end{cases}$$

If $n_k \to \infty$, we get

$$\begin{cases} y(t) &= y(0) + \int\limits_0^t f(s, y(s), y(\alpha(s)))ds, \\ y(0) &= \lambda_2, \end{cases}$$

and

$$\begin{cases} z(t) &= z(0) + \int\limits_0^t f(s, z(s), z(\alpha(s)))ds, \\ z(0) &= \lambda_2 \end{cases}$$

because f is continuous. Hence

$$\begin{aligned} y'(t) &= f(t, y(t), y(\alpha(t))), \\ z'(t) &= f(t, z(t), z(\alpha(t))), \end{aligned}$$

so $y, z \in C^1(J, \mathbb{R})$ and are solutions of (1.1) in $[y_0, z_0]_*$.

Now, we show that $y = z$ is a unique solution of (1.1). We have

$$y_0(t) \le y(t) \le z(t) \le z_0(t).$$

Put $p = z - y$. Then $p(t) \ge 0$ on J, $p(0) = 0$ and

$$p'(t) \le V_1(t, z, y)p(t) + V_2(t, z, y)p(\alpha(t))$$

in view of a mean value theorem and condition (3). Lemma 1 yields that $p(t) \le 0$ on J. It proves that $y = z$ on J.

It remains to show that it is a unique solution. Let $x \in [y_0, z_0]_*$ be any solution of (1.1). By the method of mathematical induction, it is easy to show that

$$y_n(t) \le x(t) \le z_n(t), \quad t \in J, \quad n = 0, 1, \ldots$$

If $n \to \infty$, then $y = z = x$. It means that $\{y_n\}$, $\{z_n\}$ converge to a unique solution x of (1.1).

Finally we show quadratic convergence of $\{y_n\}$ and $\{z_n\}$ to x. Define

$$p_{n+1}(t) = x(t) - y_{n+1}(t) \geq 0 \quad \text{and} \quad q_{n+1}(t) = z_{n+1}(t) - x(t) \geq 0, \quad t \in J.$$

We have $p_{n+1}(0) = q_{n+1}(0) = 0$ and

$$\begin{aligned} p'_{n+1}(t) &= f(t,x(t),x(\alpha(t))) - f(t,y_n(t),y_n(\alpha(t))) \\ &- V_1(t,y_n,z_n)[y_{n+1}(t) - y_n(t)] - V_2(t,y_n,z_n)[y_{n+1}(\alpha(t)) - y_n(\alpha(t))], \\ &= [F_y(t,\xi(t),y_n(\alpha(t))) + G_y(t,\xi_1(t),y_n(\alpha(t)))]p_n(t) \\ &+ [F_z(t,x(t),\zeta(t)) + G_z(t,x(t),\zeta_1(t))]p_n(\alpha(t)) \\ &+ V_1(t,y_n,z_n)[p_{n+1}(t) - p_n(t)] + V_2(t,y_n,z_n)[p_{n+1}(\alpha(t)) - p_n(\alpha(t))], \end{aligned}$$

where

$$y_n(t) < \xi(t) < x(t), \quad y_n(t) < \xi_1(t) < x(t)$$

$$y_n(\alpha(t)) < \zeta(t) < x(\alpha(t)), \quad y_n(\alpha(t)) < \zeta_1(t) < x(\alpha(t))$$

for $t \in J$. Using again the mean value theorem we obtain

$$\begin{aligned} p'_{n+1}(t) &\leq V_1(t,y_n,z_n)p_{n+1}(t) + V_2(t,y_n,z_n)p_{n+1}(\alpha(t)) \\ &+ p_n(t)F_{yy}(t,\mu_1(t),y_n(\alpha(t)))p_n(t) \\ &- p_n(t)G_{yz}(t,y_n(t),\mu_2(t))[q_n(\alpha(t)) + p_n(\alpha(t))] \\ &- p_n(t)G_{yy}(t,\mu_3(t),z_n(\alpha(t)))[q_n(t) + p_n(t)] \\ &+ p_n(\alpha(t))F_{zy}(t,\mu_4(t),x(\alpha(t)))p_n(t) \\ &+ p_n(\alpha(t))F_{zz}(t,y_n(t),\mu_5(t))p_n(\alpha(t)) \\ &- p_n(\alpha(t))G_{zy}(t,\mu_6(t),z_n(\alpha(t)))q_n(t) \\ &- p_n(\alpha(t))G_{zz}(t,x(t),\mu_7(t))[q_n(\alpha(t)) + p_n(\alpha(t))], \end{aligned}$$

where

$$y_n(t) < \mu_1(t) < x(t), \quad y_n(\alpha(t)) < \mu_2(t) < z_n(\alpha(t)), \quad y_n(t) < \mu_3(t) < z_n(t),$$

$$y_n(t) < \mu_4(t) < x(t), \quad y_n(\alpha(t)) < \mu_5(t) < x(\alpha(t)), \quad x(t) < \mu_6(t) < z_n(t),$$

$$y_n(\alpha(t)) < \mu_7(t) < z_n(\alpha(t)) \quad t \in J.$$

Hence

$$\begin{aligned} p'_{n+1}(t) &\leq V_1(t,y_n,z_n)p_{n+1}(t) + V_2(t,y_n,z_n)p_{n+1}(\alpha(t)) \\ &+ (|F_{yy}(t,\mu_1(t),y_n(\alpha(t)))| + |G_{yy}(t,\mu_3(t),z_n(\alpha(t)))|)p_n^2(t) \\ &+ \frac{1}{2}|G_{yy}(t,\mu_3(t),z_n(\alpha(t)))|[p_n^2(t) + q_n^2(t)] \\ &+ \frac{1}{2}(|G_{yz}(t,y_n(t),\mu_2(t)) + |F_{zy}(t,\mu_4(t),x(\alpha(t)))|)[p_n^2(t) + p_n^2(\alpha(t))] \\ &+ \frac{1}{2}|G_{yz}(t,y_n(t),\mu_2(t))|[p_n^2(t) + q_n^2(\alpha(t))] \\ &+ (|F_{zz}(t,y_n(t),\mu_5(t))| + |G_{zz}(t,x(t),\mu_7(t))|)p_n^2(\alpha(t)) \\ &+ \frac{1}{2}|G_{zy}(t,\mu_6(t),z_n(\alpha(t)))|[p_n^2(\alpha(t)) + q_n^2(t)] \end{aligned}$$

$$
\begin{aligned}
&+ \quad \frac{1}{2}|G_{zz}(t, x(t), \mu_7(t))|[p_n^2(\alpha(t)) + q_n^2(\alpha(t))] \\
&\leq \quad M_1 p_{n+1}(t) + M_2 p_{n+1}(\alpha(s)) \\
&+ \quad a_1 p_n^2(t) + a_2 p_n^2(\alpha(t)) + a_3 q_n^2(t) + a_4 q_n^2(\alpha(t)),
\end{aligned} \tag{3.2}
$$

because $F_y, F_z, G_y, G_z, F_{yy}, F_{yz}, F_{zy}, F_{zz}, G_{yy}, G_{yz}, G_{zy}, G_{zz}$ are bounded. Put

$$
\begin{cases}
w'(t) &= M_1 p_{n+1}(t) + M_2 p_{n+1}(\alpha(t)) + a_1 p_n^2(t) + a_2 p_n^2(\alpha(t)) \\
&+ \quad a_3 q_n^2(t) + a_4 q_n^2(\alpha(t)), \\
w(0) &= p_{n+1}(0) = 0
\end{cases}
$$

Then we have

$$
w'(t) \leq (M_1 + M_2) w(t) + a_1 p_n^2(t) + a_2 p_n^2(\alpha(t)) + a_3 q_n^2(t) + a_4 q_n^2(\alpha(t))
$$

because $w'(t) \geq 0$ and $p_{n+1}(t) \leq w(t)$ for $t \in J$. Thus

$$
\begin{aligned}
w(t) &\leq e^{(M_1+M_2)t} w(0) \\
&+ \int_0^t e^{(M_1+M_2)(t-s)} [a_1 p_n^2(s) + a_2 p_n^2(\alpha(s)) + a_3 q_n^2(s) + a_4 q_n^2(\alpha(s))] ds \\
&\leq \frac{1}{M_1 + M_2} \left(e^{(M_1+M_2)T} - 1 \right) [(a_1 + a_2) \max_{t \in J} p_n^2(t) + (a_3 + a_4) \max_{t \in J} q_n^2(t)]
\end{aligned}
$$

and consequently

$$
\max_{t \in J} p_{n+1}(t) \leq c_1 \max_{t \in J} p_n^2(t) + c_2 \max_{t \in J} q_n^2(t),
$$

where

$$
c_1 = \frac{a_1 + a_2}{M_1 + M_2} \left(e^{(M_1+M_2)T} - 1 \right), c_2 = \frac{a_3 + a_4}{M_1 + M_2} \left(e^{(M_1+M_2)T} - 1 \right).
$$

Analogically we can show that

$$
\max_{t \in J} q_{n+1}(t) \leq c_3 \max_{t \in J} q_n^2(t) + c_4 \max_{t \in J} p_n^2(t). \qquad \blacksquare
$$

4 Case 2: $0 < \lambda_1 < 1$

Theorem 2. *Let $\lambda_1 \in (0, 1)$. Assume that assumptions (1) and (3)–(5) of Theorem 1 are satisfied. Moreover assume that*

1. *$y_0, z_0 \in C^1(J, \mathbb{R})$, $y_0(t) \leq z_0(t), t \in J$ and satisfy the system*

$$
\begin{aligned}
y_0'(t) &\leq f(t, y_0(t), y_0(\alpha(t))), \quad y_0(0) \leq \lambda_1 y_0(T) + \lambda_2 \\
z_0'(t) &\geq f(t, z_0(t), z_0(\alpha(t))), \quad z_0(0) \geq \lambda_1 z_0(T) + \lambda_2
\end{aligned}
$$

Then there exist sequences $\{y_n\}, \{z_n\} \subset C^1(J, \mathbb{R})$ *converging to a unique solution* $x \in C^1(J, \mathbb{R})$ *of problem* (1.1) *in the sector* $[y_0, z_0]_*$. *The convergence is semi-quadratic i. e. there exist nonnegative constants* $c_1, c_2, c_3, \bar{c}_1, \bar{c}_2, \bar{c}_3$ *such that*

$$\max_{t\in J} |x(t) - y_{n+1}(t)| \le c_1 \max_{t\in J} |x(t) - y_n(t)| + c_2 \max_{t\in J} |x(t) - y_n(t)|^2 + \\ c_3 \max_{t\in J} |z_n(t) - x(t)|^2$$

and

$$\max_{t\in J} |z_{n+1}(t) - x(t)| \le \bar{c}_1 \max_{t\in J} |z_n(t) - x(t)| + \bar{c}_2 \max_{t\in J} |x(t) - y_n(t)|^2 + \\ \bar{c}_3 \max_{t\in J} |z_n(t) - x(t)|^2.$$

Proof. Define sequences $\{y_n\}, \{z_n\}$ by

$$\begin{cases} y'_{n+1}(t) &= f(t, y_n(t), y_n(\alpha(t))) + V_1(t, y_n, z_n)[y_{n+1}(t) - y_n(t)] \\ &+ V_2(t, y_n, z_n)[y_{n+1}(\alpha(t)) - y_n(\alpha(t))], \\ y_{n+1}(0) &= \lambda_1 y_n(T) + \lambda_2, \end{cases}$$

and

$$\begin{cases} z'_{n+1}(t) &= f(t, z_n(t), z_n(\alpha(t))) + V_1(t, y_n, z_n)[z_{n+1}(t) - z_n(t)] \\ &+ V_2(t, y_n, z_n)[z_{n+1}(\alpha(t)) - z_n(\alpha(t))], \\ z_{n+1}(0) &= \lambda_1 z_n(T) + \lambda_2. \end{cases}$$

Note that in this case $y_{n+1}(0) = k_n \in \mathbb{R}$, $z_{n+1}(0) = l_n \in \mathbb{R}$, so sequences $\{y_n\}$, $\{z_n\}$ have similar structures as in Theorem 1.

Analogically to the proof of Theorem 1, we can show that there exists a unique solution $x \in C^1(J, \mathbb{R})$ of problem (1.1) and that $\{y_n\}$, $\{z_n\}$ converge to x.

Now we show the semi-quadratic convergence of $\{y_n\}$, $\{z_n\}$ to x. Put

$$p_{n+1}(t) = x(t) - y_{n+1}(t) \ge 0 \quad \text{and} \quad q_{n+1}(t) = z_{n+1}(t) - x(t) \ge 0, \quad t \in J.$$

Note that

$$\begin{aligned} p_{n+1}(t) &= x(t) - y_{n+1}(t) + y_n(t) - y_n(t) = p_n(t) + y_n(t) - y_{n+1}(t) \\ &\le p_n(t) \end{aligned}$$

and

$$\begin{aligned} q_{n+1}(t) &= z_{n+1}(t) - x(t) + z_n(t) - z_n(t) = q_n(t) + z_{n+1}(t) - z_n(t) \\ &\le q_n(t). \end{aligned}$$

Using the above and the relation (3.2), we see that

$$p'_{n+1}(t) \le M_1 p_n(t) + M_2 p_n(\alpha(t)) + a_1 p_n^2(t) + a_2 p_n^2(\alpha(t)) + a_3 q_n^2(t) + a_4 q_n^2(\alpha(t)).$$

Integrating the above relation from 0 to t, we get

$$p_{n+1}(t) \quad \leq \quad c_1 \max_{t\in J} p_n(t) + c_2 \max_{t\in J} p_n^2(t) + c_3 \max_{t\in J} q_n^2(t),$$

where

$$c_1 = \lambda_1 + T(M_1 + M_2), \;\; c_2 = T(a_1 + a_2), \;\; c_3 = T(a_3 + a_4).$$

It yields

$$\max_{t\in J} p_{n+1}(t) \leq c_1 \max_{t\in J} p_n(t) + c_2 \max_{t\in J} p_n^2(t) + c_3 \max_{t\in J} q_n^2(t).$$

Analogically we can show that

$$\max_{t\in J} q_{n+1}(t) \leq \bar{c}_1 \max_{t\in J} q_n(t) + \bar{c}_2 \max_{t\in J} q_n^2(t) + \bar{c}_3 \max_{t\in J} p_n^2(t).$$ ■

To obtain the next result the following lemma is needed:

Lemma 2. *Assume that* $g \in C(J \times \mathbb{R} \times \mathbb{R}, \mathbb{R})$ *and satisfies a Lipschitz condition*

$$|g(t,u,v) - g(t,\bar{u},\bar{v})| \leq M_1|u - \bar{u}| + M_2|v - \bar{v}|,$$

with constants $M_1, M_2 > 0$ *such that*

$$\lambda_1 + T(M_1 + M_2) < 1.$$

Then problem

$$\begin{cases} y'(t) &= g(t,y(t),y(\alpha(t))), \;\; t \in J, \\ y(0) &= \lambda_1 y(T) + \lambda_2, \;\; \lambda_1 \in [0,1) \end{cases} \tag{4.1}$$

has exactly one solution.

Proof. First we show that solving (4.1) is equivalent to solving a fixed point problem. Let y be a solution of problem (4.1). Integrating (4.1) from 0 to t and using the boundary condition yields

$$y(t) = \frac{\lambda_2}{1-\lambda_1} + \int_0^T G(t,s)g(s,y(s),y(\alpha(s)))ds \equiv (Ay)(t),$$

where

$$G(t,s) = \begin{cases} \dfrac{1}{1-\lambda_1} & \text{for } 0 \leq s < t, \\ \dfrac{\lambda_1}{1-\lambda_1} & \text{for } t \leq s \leq T. \end{cases}$$

Similarly, it is easy to see that if y is any solution of $y = Ay$, then y is a solution of problem (4.1).

Put $\|u\| = \max_{t\in J} |u(t)|$. Let $u, v \in C(J, \mathbb{R})$. We have

$$\|Au - Av\| \;\; \leq \;\; \max_{t\in J} \int_0^T |G(t,s)||g(s,u(s),u(\alpha(s))) - g(s,v(s),v(\alpha(s))|ds$$

$$
\begin{aligned}
&\leq \|u-v\| \max_{t\in J} \int_0^T (M_1+M_2)|G(t,s)|ds \\
&\leq \|u-v\| \max_{t\in J} \int_0^T \frac{M_1+M_2}{1-\lambda_1} ds \\
&= \frac{T(M_1+M_2)}{1-\lambda_1}\|u-v\|,
\end{aligned}
$$

which proves that A is a contraction. In view of the Banach fixed point theorem, we have the conclusion. ∎

Now we can prove the following main result of this paper.

Theorem 3. *Let $\lambda_1 \in (0,1)$. Assume that*

1. *assumptions (1), (3)-(5) of Theorem 1 hold*
2. *$y_0, z_0 \in C^1(J,\mathbb{R})$, $y_0(t) \leq z_0(t), t \in J$ and satisfy the system*

$$
\begin{aligned}
y_0'(t) &\leq f(t, y_0(t), y_0(\alpha(t))), \quad y_0(0) \leq \lambda_1 y_0(T) + \lambda_2 \\
z_0'(t) &\geq f(t, z_0(t), z_0(\alpha(t))), \quad z_0(0) \geq \lambda_1 z_0(T) + \lambda_2
\end{aligned}
$$

3. *$\lambda_1 + (M_1+M_2)T < 1$, where $M_1, M_2 > 0$ are such that*

$$|V_1(t,u,v)| \leq M_1, \quad |V_2(t,u,v)| \leq M_2$$

for $y_0(t) \leq u \leq z_0(t)$, $y_0(\alpha(t)) \leq v \leq z_0(\alpha(t))$, $t \in J$.

Then there exist sequences $\{y_n\}, \{z_n\} \subset C^1(J,\mathbb{R})$ converging to a unique solution $x \in C^1(J,\mathbb{R})$ of problem (1.1) in the sector $[y_0, z_0]_*$. Moreover the convergence is quadratic.

Proof. Define

$$
\begin{cases}
y_{n+1}'(t) = f(t, y_n(t), y_n(\alpha(t))) + V_1(t, y_n, z_n)[y_{n+1}(t) - y_n(t)] \\
\qquad + V_2(t, y_n, z_n)[y_{n+1}(\alpha(t)) - y_n(\alpha(t))], \\
y_{n+1}(0) = \lambda_1 y_{n+1}(T) + \lambda_2,
\end{cases}
$$

and

$$
\begin{cases}
z_{n+1}'(t) = f(t, z_n(t), z_n(\alpha(t))) + V_1(t, y_n, z_n)[z_{n+1}(t) - z_n(t)] \\
\qquad + V_2(t, y_n, z_n)[z_{n+1}(\alpha(t)) - z_n(\alpha(t))], \\
z_{n+1}(0) = \lambda_1 z_{n+1}(T) + \lambda_2.
\end{cases}
$$

Lemma 2 yields that sequences $\{y_n\}, \{z_n\}$ are well defined.

First we show that

$$y_0(t) \leq y_1(t) \leq z_1(t) \leq z_0(t), \quad t \in J.$$

Put $p = y_0 - y_1$. Then $p(0) \leq \lambda_1 p(T)$ and

$$p'(t) \leq f(t, y_0(t), y_0(\alpha(t))) - f(t, y_0(t), y_0(\alpha(t)))$$

$$\begin{aligned} &- \quad V_1(t,y_0,z_0)[y_1(t)-y_0(t)] - V_2(t,y_0,z_0)[y_1(\alpha(t))-y_0(\alpha(t))] \\ &= \quad V_1(t,y_0,z_0)p(t) + V_2(t,y_0,z_0)p(\alpha(t)). \end{aligned}$$

In view of Lemma 1 we obtain that $p(t) \leq 0$ for $t \in J$, so $y_0(t) \leq y_1(t)$, $t \in J$. Analogically we show that $z_1(t) \leq z_0(t), t \in J$.

Now put $p = y_1 - z_1$. We have $p(0) = \lambda_1 p(T)$ and

$$\begin{aligned} p'(t) \quad &\leq \quad -V_1(t,y_0,z_0)[z_0(t)-y_0(t)] - V_2(t,y_0,z_0)[z_0(\alpha(t))-y_0(\alpha(t))] \\ &+ \quad V_1(t,y_0,z_0)[y_1(t)-y_0(t)] + V_2(t,y_0,z_0)[y_1(\alpha(t))-y_0(\alpha(t))] \\ &- \quad V_1(t,y_0,z_0)[z_1(t)-z_0(t)] - V_2(t,y_0,z_0)[z_1(\alpha(t))-z_0(\alpha(t))] \\ &= \quad V_1(t,y_0,z_0)p(t) + V_2(t,y_0,z_0)p(\alpha(t)). \end{aligned}$$

In view of Lemma 1 we have $y_1(t) \leq z_1(t)$, $t \in J$.

It is easy to show that

$$y_1'(t) \leq f(t, y_1(t), y_1(\alpha(t))) \text{ and } z_1'(t) \geq f(t, z_1(t), z_1(\alpha(t))), \quad t \in J.$$

By induction we can show that

$$y_0(t) \leq y_1(t) \leq \ldots \leq y_n(t) \leq z_n(t) \leq \ldots \leq z_1(t) \leq z_0(t), n = 0, 1, \ldots, t \in J.$$

Analogically as in Theorem 1 we can show that $\{y_n\}$, $\{z_n\}$ converge to a unique solution $x \in C^1(J, \mathbb{R})$ of problem (1.1).

It remains to show the quadratic convergence. Put

$$p_{n+1}(t) = x(t) - y_{n+1}(t) \geq 0 \quad \text{and} \quad q_{n+1}(t) = z_{n+1}(t) - x(t) \geq 0, \quad t \in J.$$

By (3.2) we have

$$\begin{aligned} p_{n+1}'(t) \quad &\leq \quad M_1 p_{n+1}(t) + M_2 p_{n+1}(\alpha(t)) \\ &+ \quad a_1 p_n^2(t) + a_2 p_n^2(\alpha(t)) + a_3 q_n^2(t) + a_4 q_n^2(\alpha(t)), \end{aligned}$$

(see the proof of Theorem 1).

Integrating the above from 0 to t we get

$$\begin{aligned} p_{n+1}(t) \quad &\leq \quad \lambda_1 p_{n+1}(T) + \int_0^t [M_1 p_{n+1}(s) + M_2 p_{n+1}(\alpha(s))] ds \\ &+ \quad \int_0^t [a_1 p_n^2(s) + a_2 p_n^2(\alpha(s)) + a_3 q_n^2(s) + a_4 q_n^2(\alpha(s))] ds \\ &\leq \quad [\lambda_1 + T(M_1 + M_2)] \max_{t \in J} p_{n+1}(t) \\ &+ \quad T(a_1 + a_2) \max_{t \in J} p_n^2(t) + T(a_3 + a_4) \max_{t \in J} q_n^2(t). \end{aligned}$$

Hence

$$\begin{aligned} (1 - [\lambda_1 + T(M_1 + M_2)]) \max_{t \in J} p_{n+1}(t) \quad &\leq \quad T(a_1 + a_2) \max_{t \in J} p_n^2(t) \\ &+ \quad T(a_3 + a_4) \max_{t \in J} q_n^2(t). \end{aligned}$$

Thus
$$\max_{t\in J} p_{n+1}(t) \le c_1 \max_{t\in J} p_n^2(t) + c_2 \max_{t\in J} q_n^2(t),$$
where
$$c_1 = \frac{T(a_1+a_2)}{1-[\lambda_1 + T(M_1+M_2)]}, \quad c_2 = \frac{T(a_3+a_4)}{1-[\lambda_1 + T(M_1+M_2)]}.$$
Similarly we can show that
$$\max_{t\in J} q_{n+1}(t) \le c_3 \max_{t\in J} q_n^2(t) + c_4 \max_{t\in J} p_n^2(t).$$
This completes the proof. ∎

5 Example

Example 1. Let us consider the problem
$$\begin{cases} x'(t) &= \dfrac{1}{10}x(t) - \dfrac{1}{5}e^{-x(t)} - \dfrac{1}{6}e^{-2x(\frac{1}{2}t)} + 1, \quad t \in [0,1], \\ x(0) &= \dfrac{1}{10}x(1) + \dfrac{7}{10}. \end{cases} \tag{5.1}$$
Put
$$F(t,y,z) = \frac{1}{10}y + 1, \quad G(t,y,z) = -\frac{1}{5}e^{-y} - \frac{1}{6}e^{-2z}.$$
Note that here we have $\alpha(t) = \frac{1}{2}t$. Take $y_0(t) = \frac{1}{2}t + \frac{1}{2}$, $z_0(t) = 2t+1$. Then we have
$$\begin{aligned} f\left(t, y_0(t), y_0\left(\frac{1}{2}t\right)\right) &= \frac{1}{20}t + \frac{1}{20} - \frac{1}{5}e^{-\frac{1}{2}t-\frac{1}{2}} - \frac{1}{6}e^{-\frac{1}{2}t-1} + 1 \\ &\ge \frac{1}{20} - \frac{1}{5}e^{-\frac{1}{2}} - \frac{1}{6}e^{-1} + 1 \\ &> \frac{1}{20} - \frac{1}{5} - \frac{1}{6} + 1 > \frac{1}{2} = y_0'(t) \\ f\left(t, z_0(t), z_0\left(\frac{1}{2}t\right)\right) &= \frac{1}{5}t + \frac{1}{10} + 1 - \frac{1}{5}e^{-2t-1} - \frac{1}{6}e^{-2t-2} \\ &< \frac{1}{5}t + \frac{1}{10} + 1 < 2 = z_0'(t) \end{aligned}$$
and
$$\begin{aligned} \frac{1}{\lambda_1}y_0(1) + \lambda_2 = 10y_0(1) + \frac{7}{10} &= \frac{8}{10} > \frac{1}{2} = y_0(0) \\ \lambda_1 z_0(1) + \lambda_2 = \frac{1}{10}z_0(1) + \frac{7}{10} &= 1 = z_0(0). \end{aligned}$$

Moreover
$$V_1(t, y_0, z_0) = \frac{1}{10} + \frac{1}{5}e^{-2t-1} > \frac{1}{10}$$

$$\begin{aligned} V_2(t,z_0,y_0) &= \frac{1}{3}e^{-\frac{1}{2}t-1} \leq \frac{1}{3}e^{-1} \\ V_2(t,y_0,z_0) &= \frac{1}{3}e^{-2t-2} > 0. \end{aligned}$$

Defining $K(t) = \dfrac{1}{10}$, $L(t) = \dfrac{1}{3}e^{-1}$, $M_1 = \frac{3}{10}$, $M_2 = \frac{1}{3}$ we get

$$\begin{aligned} \tilde{\lambda}_1 + \int\limits_0^1 L(t)e^{-\int_{\alpha(t)}^t K(s)ds}dt &= \frac{1}{10}e^{\frac{1}{10}} + \int\limits_0^1 \frac{1}{3}e^{-1}e^{-\int_{\frac{1}{2}t}^t \frac{1}{10}ds}dt \\ &< \frac{1}{10}e^{\frac{1}{10}} + \frac{1}{3}e^{-1} < 1 \end{aligned} \tag{5.2}$$

and

$$\lambda + (M_1 + M_2)T = \frac{22}{30} < 1.$$

All assumptions of Theorem 3 are satisfied. Thus there exist monotone sequences $\{y_n\}$, $\{z_n\}$ converging quadratically to a unique solution of problem (5.1) in the sector $[y_0, z_0]_*$.

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