Solutions to abstract integral equations and infinite delay evolution equations *

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Dedicated to Jan Van Casteren on the occasion of his 65th birthday

Abstract

We establish new and general existence theorems of Lipschitz continuous solutions to integral equations with infinite delay in Banach spaces, as well as of strong and classical solutions to the Cauchy problem for some evolution equations with infinite delay. An example is given to illustrate the abstract result.

1 Introduction

In this paper, we consider the Cauchy problem for integral equations with infinite delay in a Banach space X

$$\begin{cases} u(t) = \mathcal{G}(t) + \int_\sigma^t \mathcal{F}(t,s,u(s),u_s)ds \ \ (\sigma \le t \le T), \\ u_\sigma = \phi, \end{cases} \tag{1.1}$$

and the Cauchy problem for semilinear evolution equations with infinite delay in X

$$\begin{cases} u'(t) = Au(t) + f(t,u(t),u_t), \quad \sigma \le t \le T, \\ u_\sigma = \phi, \end{cases} \tag{1.2}$$

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where $\phi \in \mathcal{P}$ (an admissible phase space, see the definition below), $0 \le \sigma < T$, $u_t(\theta) = u(t+\theta)$ $(\theta \in R^-)$, A is a given operator, $\mathcal{G}(t) \in C([\sigma,T],X)$, $\mathcal{F} \in C([\sigma,T]\times[\sigma,T]\times X\times\mathcal{P},X)$ and $f \in C([\sigma,T]\times X\times\mathcal{P},X)$ are given functions.

We first establish general criteria to judge the existence of Lipschitz continuous solutions to the Cauchy problem for integral equations with infinite delay in Banach spaces in Section 2. Then, in Section 3 we give new existence theorems of strong and classical solutions to the Cauchy problem for some evolution equations with infinite delay by means of the criteria obtained in Section 2.

This is a continuation of our previous work (see, e.g., [5, 6, 15, 16]). Related literature includes, for instance, [1, 4, 7, 8] and references therein.

Throughout this paper, we denote by R, R^-, and N the set of real numbers, the set of nonpositive numbers, and the set of natural numbers, respectively. For a linear operator A, $\mathscr{D}(A)$ is the domain, and $\mathcal{R}(A)$ the range. We denote by $\mathbf{L}(X)$ the space of all bounded and linear operators from X to itself.

What follows is a basic notion in the theory of delay equations.

Definition 1.1. A linear space $\mathcal{P}$ consisting of functions from R^- into X, with semi-norm $\|\cdot\|_{\mathcal{P}}$, is called an *admissible phase space* if $\mathcal{P}$ has the following properties.

(H1) For any $t_0 \in R$ and $a > 0$, if $x : (-\infty, t_0+a] \to X$ is continuous on $[t_0, t_0+a]$ and $x_{t_0} \in \mathcal{P}$, then $x_t \in \mathcal{P}$ and x_t is continuous in $t \in [t_0, t_0+a]$.

(H2) There exists a continuous function $K(t) > 0$ and a locally bounded function $M(t) \ge 0$ in $t \ge 0$ such that

$$\|x_t\|_{\mathcal{P}} \le K(t-t_0) \max_{s\in[t_0,t]} \|x(s)\| + M(t-t_0)\|x_{t_0}\|_{\mathcal{P}}$$

for $t \in [t_0, t_0+a]$ and x as in (H1).

(H3) The quotient space $\mathcal{P}/\|\cdot\|_{\mathcal{P}}$ is a Banach space.

For each $\tau > \sigma$, we write

$$\mathcal{P}^{[\sigma,\tau]} := \left\{ u : (-\infty,\tau] \to X;\ u\Big|_{[\sigma,\tau]} \in C([\sigma,\tau],X) \text{ and } u_\sigma \in \mathcal{P} \right\}.$$

2 Lipschitz continuity of solutions

Let $0 \le \sigma \le T$, we define

$$\begin{aligned} \mathcal{Q}^{[\sigma,T]} := \Big\{ \phi : R^- \to X;\ & \text{there are constants } a_\phi > T \text{ and } L_{\phi,\mathcal{P}} \text{ such that} \\ & \phi(\cdot) \text{ is Lipschitz continuous on } [-a_\phi, 0], \phi_{-a_\phi} \in \mathcal{P} \\ & \text{and } \left\| \phi_{-a_\phi+\tau} - \phi_{-a_\phi} \right\|_{\mathcal{P}} \le L_{(\phi,\mathcal{P})}\tau \text{ for } \tau \in [0, T-\sigma] \Big\}. \end{aligned} \tag{2.1}$$

Remark 2.1. Clearly, by (H1), we have $\mathcal{Q}^{[\sigma,T]} \subset \mathcal{P}$ and the set

$$\mathcal{Q}_0 = \{\phi(\theta);\ \phi : R^- \to X \text{ is Lipschitz continuous with compact support}\}$$

is a subset of $\mathcal{Q}^{[\sigma,T]}$.

Theorem 2.2. *Let $0 \le \sigma \le T$, $\mathcal{P}$ be an admissible phase space, and $\mathcal{F} \in C([\sigma,T] \times [\sigma,T] \times X \times \mathcal{P}, X)$. Suppose that*

(i) *for every $r > 0$, there exists a constant $H(r)$ such that*

$$\begin{aligned} \|\mathcal{F}(t,s,x(s),x_s) - \mathcal{F}(t,s,y(s),y_s)\| \le H(r)\left(\|x(s)-y(s)\| + \|x_s - y_s\|_{\mathcal{P}}\right), \\ \text{for all } t,s \in [\sigma,T],\ x(\cdot),y(\cdot) \in \mathcal{P}^{[\sigma,T]} \text{ with} \\ \max_{s\in[\sigma,T]} \{\|x(s)\|, \|y(s)\|, \|x_s\|_{\mathcal{P}}, \|y_s\|_{\mathcal{P}}\} \le r. \end{aligned} \tag{2.2}$$

(ii) *$\phi \in \mathcal{Q}^{[\sigma,T]}$, $\mathcal{G}(t) : [\sigma,T] \to X$ is Lipschitz continuous with $\mathcal{G}(\sigma) = \phi(0)$, and there is a constant $L_{\mathcal{F}}$ such that*

$$\begin{aligned} \int_\sigma^t \|\mathcal{F}(t+\eta, s+\eta, x(s), x_s) - \mathcal{F}(t,s,x(s),x_s)\| ds \le \eta L_{\mathcal{F}}, \\ \text{for } t \in [\sigma,T],\ \eta \in [0, T-t],\ x(\cdot) \in \mathcal{P}^{[\sigma,T]}. \end{aligned} \tag{2.3}$$

Then (1.1) *has a unique solution $u(t)$ on $[\sigma, T_{\sup}(\sigma,\phi,\mathcal{G},\mathcal{F}))$, which is Lipschitz continuous on $[\sigma,\tau_0]$ for every $\tau_0 \in [\sigma, T_{\sup}(\sigma,\phi,\mathcal{G},\mathcal{F}))$. Here,*

$$T_{\sup}(\sigma,\phi,\mathcal{G},\mathcal{F}) := \sup\{\tau > \sigma;\ (1.1) \textit{ has a unique solution } u(\cdot) \textit{ on } [\sigma,\tau)\},$$

From the proof of [5, Theorem 3.2] it follows that $[\sigma, T_{\sup}(\sigma,\phi,\mathcal{G},\mathcal{F})) > 0$, so that (1.1) has a unique solution $u(t)$ on the half-open interval $[\sigma, T_{\sup}(\sigma,\phi,\mathcal{G},\mathcal{F}))$.

The proof of Theorem 2.2. Let $u(t)$ be the unique solution to (1.1) on the interval $[\sigma, T_{\sup}(\sigma,\phi,\mathcal{G},\mathcal{F}))$. Next, we prove $u(t)$ is Lipschitz continuous on $[\sigma,\tau_0]$ for every $\tau_0 \in [\sigma, T_{\sup}(\sigma,\phi,\mathcal{G},\mathcal{F}))$.

Let $L_{\mathcal{G}}$ be the Lipschitz constant for $\mathcal{G}$ and $\tau_0 \in [\sigma, T_{\sup}(\sigma,\phi,\mathcal{G},\mathcal{F}))$. Then by

(2.2) and (2.3) we deduce that for each $t \in [\sigma, \tau_0]$, $\eta \in [0, \tau_0 - t]$,

$$
\begin{aligned}
& \|u(t+\eta) - u(t)\| \\
\le\ & \|\mathcal{G}(t+\eta) - \mathcal{G}(t)\| + \int_\sigma^{\sigma+\eta} \|\mathcal{F}(t+\eta, s, u(s), u_s)\| ds \\
& + \left\| \int_{\sigma+\eta}^{t+\eta} \mathcal{F}(t+\eta, s, u(s), u_s) ds - \int_\sigma^t \mathcal{F}(t, s, u(s), u_s) ds \right\| \\
\le\ & \|\mathcal{G}(t+\eta) - \mathcal{G}(t)\| + \int_\sigma^{\sigma+\eta} \|\mathcal{F}(t+\eta, s, u(s), u_s)\| ds \\
& + \int_\sigma^t \|\mathcal{F}(t+\eta, s+\eta, u(s+\eta), u_{s+\eta}) - \mathcal{F}(t+\eta, s+\eta, u(s), u_s)\| ds \\
& + \int_\sigma^t \|\mathcal{F}(t+\eta, s+\eta, u(s), u_s) - \mathcal{F}(t, s, u(s), u_s)\| ds \\
=\ & \|\mathcal{G}(t+\eta) - \mathcal{G}(t)\| + \int_\sigma^{\sigma+\eta} \|\mathcal{F}(t+\eta, s, u(s), u_s)\| ds \\
& + \int_\sigma^t \|\mathcal{F}(t+\eta, s+\eta, \widetilde{u}(s+\eta), \widetilde{u}_{s+\eta}) - \mathcal{F}(t+\eta, s+\eta, \overline{u}(s), \overline{u}_s)\| ds \\
& + \int_\sigma^t \|\mathcal{F}(t+\eta, s+\eta, \overline{u}(s), \overline{u}_s) - \mathcal{F}(t, s, \overline{u}(s), \overline{u}_s)\| ds \\
\le\ & \left[L_g + \max_{t,s \in [\sigma, \tau_0]} \|\mathcal{F}(t, s, u(s), u_s)\| + L_{\mathcal{F}} \right] \eta \\
& + H\left(\max_{[\sigma, \tau_0]} \{\|u(t)\|,\ \|u_t\|_{\mathcal{P}}\} \right) \int_\sigma^t \left[\|u(s+\eta) - u(s)\| + \|u_{s+\eta} - u_s\|_{\mathcal{P}} \right] ds,
\end{aligned}
$$

where

$$
\widetilde{u}(s+\eta) = \begin{cases} u(t+\eta), & s \in (t,\ T], \\ u(s+\eta), & s \in [\sigma,\ t], \end{cases}
$$

$$
\widetilde{u}_{s+\eta} = \begin{cases} u_{t+\eta}, & s \in (t,\ T], \\ u_{s+\eta}, & s \in [\sigma,\ t], \end{cases}
$$

$$
\overline{u}(s) = \begin{cases} u(t), & s \in (t,\ T], \\ u(s), & s \in [\sigma,\ t], \end{cases}
$$

$$
\overline{u}_s = \begin{cases} u_t, & s \in (t,\ T], \\ u_s, & s \in [\sigma,\ t]. \end{cases}
$$

Noting $\phi \in \mathcal{Q}^{[\sigma, T]}$ and letting L_ϕ be the Lipschitz constant for ϕ on $[-a_\phi, 0]$, we obtain, by (2.1) and (H2), for every $s \in [\sigma, t]$ ($t \in [\sigma, \tau_0]$), $\eta \in [0, \tau_0 - t]$,

$$
\begin{aligned}
&\|u_{s+\eta} - u_s\|_{\mathcal{P}} \\
\leq\ & K(s + a_\phi - \sigma) \max_{\zeta \in [-a_\phi + \sigma, s]} \|u(\eta + \zeta) - u(\zeta)\| \\
&\qquad + M(s + a_\phi - \sigma) \left\| \phi_{-a_\phi + \eta} - \phi_{-a_\phi} \right\|_{\mathcal{P}} \\
\leq\ & K(s + a_\phi - \sigma) \Bigg[\max_{\zeta \in [-a_\phi, -\eta]} \|\phi(\eta + \zeta) - \phi(\zeta)\| \\
&\qquad + \max_{\zeta \in [-\eta + \sigma, \sigma]} \|u(\zeta + \eta) - \phi(\zeta - \sigma)\| \\
&\qquad\quad + \max_{\zeta \in [\sigma, s]} \|u(\eta + \zeta) - u(\zeta)\| \Bigg] + M(s + a_\phi - \sigma) L_{(\phi, \mathcal{P})} \eta \\
\leq\ & K(s + a_\phi - \sigma) \Bigg\{ L_\phi \eta + \max_{\zeta \in [-\eta + \sigma, \sigma]} \|\mathcal{G}(\zeta + \eta) - \mathcal{G}(\sigma)\| \\
&\qquad + \max_{\zeta \in [-\eta + \sigma, \sigma]} \|\phi(0) - \phi(\zeta - \sigma)\| \\
&\qquad + \max_{\zeta \in [-\eta + \sigma, \sigma]} \left\| \int_\sigma^{\eta + \zeta} \mathcal{F}(\zeta + \eta, \xi, u(\xi), u_\xi) d\xi \right\| \\
&\qquad + \max_{\zeta \in [\sigma, s]} \|u(\eta + \zeta) - u(\zeta)\| \Bigg\} + \sup_{s \in [\sigma, T]} M(s + a_\phi - \sigma) L_{(\phi, \mathcal{P})} \eta \\
\leq\ & \Bigg\{ \max\nolimits_{s \in [\sigma, T]} K(s + a_\phi - \sigma) \left[2L_\phi + L_{\mathcal{G}} + \max_{t, s \in [\sigma, \tau_0]} \|\mathcal{F}(t, s, u(s), u_s)\| \right] \\
&\qquad + \sup_{s \in [\sigma, T]} M(s + a_\phi - \sigma) L_{(\phi, \mathcal{P})} \Bigg\} \eta \\
&\qquad + \max\nolimits_{s \in [\sigma, T]} K(s + a_\phi - \sigma) \max\nolimits_{\zeta \in [\sigma, s]} \|u(\eta + \zeta) - u(\zeta)\|.
\end{aligned}
$$

As a consequence, there are constants $\overline{H}$ and $\underline{H}$ such that

$$
\max_{\zeta \in [\sigma, t]} \|u(\zeta + \eta) - u(\zeta)\| \leq \overline{H} \eta + \underline{H} \int_\sigma^t \max_{\zeta \in [\sigma, s]} \|u(\zeta + \eta) - u(\zeta)\| ds.
$$

Using Gronwall-Bellman's inequality we have

$$
\max_{\zeta \in [\sigma, t]} \|u(\zeta + \eta) - u(\zeta)\| \leq \widehat{H} \eta, \quad t \in [\sigma, \tau_0],\ \eta \in [0, \tau_0 - t],
$$

for a constant $\widehat{H}$. This implies that $u(t)$ is Lipschitz continuous on $[\sigma, \tau_0]$.

Remark 2.3. Clearly, if F satisfies the "local Lipschitz condition" with respect to the third and fourth variable, then F satisfies the assumptions (i) and (ii) in Theorem 2.2.

As a direct corollary of Theorem 2.2, we have the following result.

Corollary 2.4. *Let $0 \le \sigma < T$, $\mathcal{P}$ be an admissible phase space, and $f \in C([\sigma,T] \times X \times \mathcal{P}, X)$ such that for every $r > 0$, there exists a constant $\widetilde{H}(r)$ such that for each $s \in [\sigma, T]$,*

$$\|f(s,x(s),x_s) - f(s,y(s),y_s)\| \le \widetilde{H}(r)\left(\|x(s)-y(s)\| + \|x_s - y_s\|_{\mathcal{P}}\right),$$
$$\text{for all } x(\cdot), y(\cdot) \in \mathcal{P}^{[\sigma,T]} \text{ with } \max_{s\in[\sigma,T]} \{\|x(s)\|, \|y(s)\|, \|x_s\|_{\mathcal{P}}, \|y_s\|_{\mathcal{P}}\} \le r. \tag{2.4}$$

Then for every $\phi \in \mathcal{Q}^{[\sigma,T]}$, $\mathcal{G}(t) : [\sigma,T] \to X$ being Lipschitz continuous with $\mathcal{G}(\sigma) = \phi(0)$, and strongly continuous family $\{E(t)\}_{\sigma \le t \le T} \subset \mathbf{L}(X)$, the solution of

$$\begin{cases} u(t) = \mathcal{G}(t) + \int_\sigma^t E(t-s) f(s,u(s),u_s)ds, & t \in [\sigma, T], \\ u_\sigma = \phi, \end{cases} \tag{2.5}$$

on $[\sigma, T_{\sup}(\sigma,\phi,\mathcal{G},E(\cdot),f))$ is Lipschitz continuous on $[\sigma,\tau_0]$ for every $\tau_0 \in [\sigma, T_{\sup}(\sigma,\phi,\mathcal{G},E(\cdot),f))$, where

$$T_{\sup}(\sigma,\phi,E(\cdot),f)) := \sup\{\tau > \sigma;\ (2.5) \text{ has a unique solution } u(\cdot) \text{ on } [\sigma,\tau)\}.$$

Theorem 2.5. *Let $0 \le \sigma < T$, $\mathcal{P}$ be an admissible phase space, $\phi \in \mathcal{P}$, $\mathcal{G}(t) : [\sigma,T] \to X$ be Lipschitz continuous with $\mathcal{G}(\sigma) = \phi(0)$, and $\mathcal{F} \in C([\sigma,T] \times [\sigma,T] \times X \times \mathcal{P}, X)$. Suppose that there is a constant $\widetilde{L}_{\mathcal{F}}$ such that*

$$\int_\sigma^t \|\mathcal{F}(t+\eta,s,x(s),x_s) - \mathcal{F}(t,s,x(s),x_s)\|ds \le \eta \widetilde{L}_{\mathcal{F}},$$
$$\text{for } t \in [\sigma,T],\ \eta \in [0, T-t],\ x(\cdot) \in \mathcal{P}^{[\sigma,T]}, \tag{2.6}$$

and (1.1) has a solution $u(t)$ on $[\sigma, T_{\sup}(\sigma,\phi,\mathcal{G},\mathcal{F}))$. Then $u(t)$ is Lipschitz continuous on $[\sigma,\tau_0]$ for every $\tau_0 \in [\sigma, T_{\sup}(\sigma,\phi,\mathcal{G},\mathcal{F}))$.

Proof. Fix $\tau_0 \in [\sigma, T_{\sup}(\sigma,\phi,\mathcal{G},\mathcal{F}))$ and let $L_{\mathcal{G}}$ be the Lipschitz constant for $\mathcal{G}$. Then by (2.6) we get for each $t \in [\sigma,\tau_0]$, $\eta \in [0, \tau_0 - t]$,

$$\begin{aligned} \|u(t+\eta) - u(t)\| &\le \|\mathcal{G}(t+\eta) - \mathcal{G}(t)\| + \int_t^{t+\eta} \|\mathcal{F}(t+\eta,s,u(s),u_s)\|ds \\ &\quad + \int_\sigma^t \|\mathcal{F}(t+\eta,s,u(s),u_s) - \mathcal{F}(t,s,u(s),u_s)\|ds \\ &= \|\mathcal{G}(t+\eta) - \mathcal{G}(t)\| + \int_t^{t+\eta} \|\mathcal{F}(t+\eta,s,u(s),u_s)\|ds \\ &\quad + \int_\sigma^t \|\mathcal{F}(t+\eta,s,\overline{u}(s),\overline{u}_s) - \mathcal{F}(t,s,\overline{u}(s),\overline{u}_s)\|ds \\ &\le \left(L_{\mathcal{G}} + \max_{t,s\in[\sigma,\tau_0]} \|\mathcal{F}(t,s,u(s),u_s)\| + \widetilde{L}_{\mathcal{F}}\right)\eta, \end{aligned}$$

where

$$\overline{u}(s) = \begin{cases} u(t), & s \in (t,\ T], \\ u(s), & s \in [\sigma,\ t], \end{cases}$$

$$\overline{u}_s = \begin{cases} u_t, & s \in (t,\ T], \\ u_s, & s \in [\sigma,\ t]. \end{cases}$$

Therefore, the solution $u(t)$ of (1.1) (with respect to every $\phi \in \mathcal{P}$) is Lipschitz continuous on $[\sigma, \tau_0]$. ∎

Theorem 2.6. *Let $T > 0$, $\{U(t,s)\}_{0\le s\le t\le T} \subset \mathbf{L}(X)$ be a Lipschitz evolution system ($cf.$ [9, 10]), i.e., satisfying*

$$\|U(t,s) - I\| \le (t-s)\overline{\overline{H}}e^{\omega(t-s)}, \quad 0 \le s \le t \le T, \tag{2.7}$$

for some constants $\overline{\overline{H}}$, $\omega \ge 0$. Let $\mathcal{P}$ be an admissible phase space and $f \in C([0,T] \times X \times \mathcal{P}, X)$ satisfying (2.4) (for $\sigma = 0$). Then for each $\phi \in \mathcal{P}$, the solution $u(t)$ of

$$u(t) = \begin{cases} U(t,0)\phi(0) + \int_0^t U(t,s)f(s,u(s),u_s)ds, & t \in [0,T], \\ \phi(t), \quad t \in (-\infty, 0] \end{cases} \tag{2.8}$$

(if exists on $[0, T_{\sup}(\phi))$) is Lipschitz continuous on $[0,\tau_0]$ for every $\tau_0 \in [0, T_{\sup}(\phi))$.

Proof. Let $\phi \in \mathcal{P}$. By (2.7) we get for every $t \in [0,\tau_0]$, $\eta \in [0, \tau_0 - t]$,

$$\begin{aligned} \|U(t+\eta,0)\phi(0) - U(t,0)\phi(0)\| &\le \|U(t+\eta,t) - I\|\|U(t,0)\phi(0)\| \\ &\le \overline{\overline{H}}e^T \max_{t\in[0,T]} \|U(t,0)\|\|\phi(0)\|\eta, \end{aligned}$$

and for $t \in [0,\tau_0]$, $\eta \in [0,\tau_0 - t]$, and $x(\cdot) \in \mathcal{P}^{[0,\tau_0]}$,

$$\begin{aligned} &\int_0^t \|[U(t+\eta,s) - U(t,s)]f(s,x(s),x_s)\|ds \\ &\qquad \le \int_0^t \|U(t+\eta,t) - I\|\|U(t,s)f(s,x(s),x_s)\|ds \\ &\qquad\qquad \le T\overline{\overline{H}}e^T \max_{t,s\in[0,T]} \|U(t,s)\| \max_{t\in[0,T]} \|f(t,x(t),x_t)\|\eta, \end{aligned}$$

i.e, (2.6) holds. Thus, by Theorem 2.5, the solution $u(t)$ of (2.8) (if exists) on $[0, T_{\sup}(\phi))$ is Lipschitz continuous on $[0,\tau_0]$ for every $\tau_0 \in [0, T_{\sup}(\phi))$. ∎

3 Regularity

We first recall some basic concepts used in this section.

Definition 3.1. Let $C \in \mathbf{L}(X)$ be an injective operator, and $\tau > 0$. An operator family $\{E(t)\}_{t\in[0,\tau]} \subset \mathbf{L}(X)$ is called a *local C-regularized semigroup* on X if

(i) $E(0) = C$ and $E(t+s)C = E(t)E(s)$ for $s,\ t,\ s+t \in [0,\tau]$,

(ii) $\{E(t)\}_{t\in[0,\tau]}$ is strongly continuous.

The operator A defined by

$$\mathcal{D}(A) = \{x \in X : \lim_{t\to 0^+} \frac{1}{t}(E(t)x - Cx) \text{ exists and is in } \mathcal{R}(C)\}$$

and

$$Ax = C^{-1} \lim_{t\to 0^+} \frac{1}{t}(E(t)x - Cx), \quad \text{for each } x \in \mathcal{D}(A),$$

is called the *generator* of $\{E(t)\}_{t\in[0,\tau]}$. We also say that A *generates* $\{E(t)\}_{t\in[0,\tau]}$.

Definition 3.2. Let $E \in \mathbf{L}(X)$, A a closed operator in X and $\tau > 0$. An operator family $\{E(t)\}_{t\in[0,\tau]} \subset \mathbf{L}(X)$ is called a *local E-existence family* for A if

(i) $\{E(t)\}_{t\in[0,\tau]}$ is strongly continuous,

(ii)
$$\int_0^t E(s)xds \in \mathcal{D}(A) \quad \text{for every } x \in X,\ t \in [0,\tau],$$
and
$$A\left(\int_0^t E(s)xds\right) = E(t)x - Ex. \tag{3.1}$$

We also say that the operator A *has* a local E-existence family $\{E(t)\}_{t\in[0,\tau]}$.

Hereafter, we suppose that

the zero function is the unique continuous solution of the following equation
$$x(t) = A\int_0^t x(s)ds, \quad t \geq 0, \tag{3.2}$$
where the operator A is the coefficient operator in (1.2). This means that for any $x \in X$, $E(\cdot)x$ is the unique continuous solution of
$$x(t) = Ex + A\int_0^t x(s)ds, \quad t \geq 0.$$

Remark 3.3. It is easy to see that (3.2) holds automatically for the generator A of a local C-regularized semigroup. For more information on regularized semigroups and existence families, which are natural generalizations of C_0 semigroups (cf., e.g., [3, 11]), we refer the reader to, e.g., [12, 13, 14].

Definition 3.4. Let $\phi(0) \in \mathcal{R}(E)$. A function $u : (-\infty, a) \to X$ is called a *mild solution* of (1.2) on $[0,a)$ if $u \in C([0,a),X)$ satisfies
$$u(t) = \begin{cases} E(t)z + \int_0^t E(t-s)\widetilde{f}(s,u(s),u_s)ds, & t \in [0,a), \\ \phi(t), & t \in (-\infty, 0], \end{cases} \tag{3.3}$$
where $z \in X$ is such that $Ez = \phi(0)$, and $\widetilde{f} \in C([0,T]\times X \times \mathcal{P}, X)$ has the property that $E\widetilde{f} = f$.

Remark 3.5. The integral equation (3.3) is independent of the choices of z and $\widetilde{f}$.

Definition 3.6. A function $u : (-\infty, a) \to X$ is called a *strong solution* of (1.2) if u is absolutely continuous on $[0,a)$ and differentiable a.e. on $[0,a)$ such that $u'(\cdot) \in L^1([0,a),X)$ satisfying (1.2) a.e. on $[0,a)$.

Definition 3.7. A function $u : (-\infty, a) \to X$ is called a *classical solution* of (1.2) if
$$u \in C^1([0,a),X) \cap C([0,a),[\mathcal{D}(A)])$$
satisfying (1.2) on $[0,a)$.

Definition 3.8. Let (Ω, Σ, μ) be a measure space. If μ is a measure on (Ω, Σ), then X has the Radon-Nikodym property with respect to μ if, for every countably-additive vector measure γ on (Ω, Σ) with values in Ω which has bounded variation and is absolutely continuous with respect to μ, there is a μ-integrable function $g : \Omega \to X$ such that

$$\gamma(E) = \int_E g d\mu$$

for every measurable set $E \in \Sigma$. The Banach space X has the Radon-Nikodym property if X has the Radon-Nikodym property with respect to every finite measure.

For instance, L^p-spaces, $1 < p < \infty$, are friendly enough to enjoy the Radon-Nikodym property. Moreover, spaces with Radon-Nikodym property include separable dual spaces (this is the Dunford-Pettis theorem) and reflexive spaces, which include, in particular, Hilbert spaces.

Theorem 3.9. *Let $T > 0$ and $\mathcal{P}$ be an admissible phase space. Let A be closed and have a local E-existence family $\{E(t)\}_{t\in[0,T]}$ satisfying that for each $z \in \mathcal{D}(A)$, $E(\cdot)z$ be an absolutely continuous X-valued function on $[0, T]$. Let (3.2) hold and $\widetilde{f} \in C([0, T] \times X \times \mathcal{P}, X)$ such that for every $r > 0$, there exists a constant $\widetilde{\widetilde{H}}(r)$ such that for each $s \in [0, T]$,*

$$\|\widetilde{f}(s, x(s), x_s) - \widetilde{f}(s, y(s), y_s)\| \le \widetilde{\widetilde{H}}(r) \left(\|x(s) - y(s)\| + \|x_s - y_s\|_{\mathcal{P}}\right),$$
$$\text{for all } x(\cdot), y(\cdot) \in \mathcal{P}^{[0,T]} \text{ with } \max_{s\in[0,T]} \{\|x(s)\|, \|y(s)\|, \|x_s\|_{\mathcal{P}}, \|y_s\|_{\mathcal{P}}\} \le r. \tag{3.4}$$

If X satisfies the Radon-Nikodym property, then for each $\phi \in \mathcal{Q}^{[0,T]}$ with $Ez = \phi(0)$ $(z \in \mathcal{D}(A))$, the corresponding mild solution of (1.2) is a strong solution of (1.2).

Proof. Let $\phi \in \mathcal{Q}^{[0,T]}$ with $Ez = \phi(0)$ for a $z \in \mathcal{D}(A)$, and let $u(t)$ be the corresponding mild solution of (1.2) on $[0, T_{\sup}(\phi))$.

By the Radon-Nikodym property of X, we get for all $z \in \mathcal{D}(A)$, $E(t)z$ is differentiable in t a.e. on $[0, T_{\sup}(\phi))$. Arguing as in the proof [2, Proposition 2.7] we deduce that $E(t)z \in \mathcal{D}(A)$ for a.e. $t \in [0, T_{\sup}(\phi))$ and

$$\int_0^t AE(s)z = E(t)z - Ez, \quad \text{a.e. } t \in [0, T_{\sup}(\phi)). \tag{3.5}$$

Moreover, by virtue of Theorem 2.2 and the Radon-Nikodym property of X, we have

$$u(t) \text{ is differentiable a.e. on } [0, T_{\sup}(\phi)). \tag{3.6}$$

By (3.1),

$$A \int_0^{t-s} E(\tau)\widetilde{f}(s, u(s), u_s)d\tau = E(t - s)\widetilde{f}(s, u(s), u_s) - f(s, u(s), u_s),$$
$$0 \le s \le t \le T_{\sup}(\phi).$$

This, together with the closedness of A, implies that for $0 \le s \le t \le T_{\sup}(\phi)$,

$$\begin{aligned} & A\int_0^t\int_0^\tau E(\tau-s)\widetilde{f}(s,u(s),u_s)dsd\tau \\ = & A\int_0^t\int_0^{t-s} E(\tau)\widetilde{f}(s,u(s),u_s)d\tau ds \\ = & \int_0^t [E(t-s)\widetilde{f}(s,u(s),u_s) - f(s,u(s),u_s)]ds. \end{aligned}$$

Hence, by (3.5), we infer that

$$u(t) = A\int_0^t u(s)ds + \int_0^t f(s,u(s),u_s)ds + z, \quad 0 \le t \le T_{\sup}(\phi). \tag{3.7}$$

Using (3.6), (3.7) and the closedness of A, we obtain $u(t) \in \mathcal{D}(A)$ for a.e. $t \in [0, T_{\sup}(\phi))$ and

$$Au(t) = u'(t) - f(t,u(t),u_t), \quad \text{a.e. } t \in [0, T_{\sup}(\phi)).$$

This means that $u(t)$ is a strong solution of (1.2). ∎

Remark 3.10. Theorem 3.9 is new even for the corresponding case without delay (cf. [2]).

When $\{E(t)\}_{t\in[0,T]}$ is a local C-regularized semigroup, we have the result as follows.

Theorem 3.11. *Let $T > 0$, A be the generator of a local C-regularized semigroup $\{E(t)\}_{t\in[0,T]}$, and $\mathcal{P}$ an admissible phase space. Suppose that $\widetilde{f}$ is locally Lipschitz continuous in all the three variables. If X satisfies the Radon-Nikodym property, then for each $\phi \in \mathcal{Q}^{[0,T]}$ with $Cz = \phi(0)$ $(z \in \mathcal{D}(A))$, the corresponding mild solution of* (1.2) *is a classical solution of* (1.2).

Proof. Let $\phi \in \mathcal{Q}^{[0,T]}$ with $Cz = \phi(0)$ for a $z \in \mathcal{D}(A)$, and let $u(t)$ be the corresponding mild solution of (1.2) on $[0, T_{\sup}(\phi))$.

Since $\{E(t)\}_{t\in[0,T]}$ is a local C-regularized semigroup, we have $E(t)z$ is differentiable on $[0, T_{\sup}(\phi))$ for every $z \in \mathcal{D}(A)$. On the other hand, by virtue of the Radon-Nikodym property of X, the local Lipschitz continuity of $\widetilde{f}$, and Theorem 2.2, we obtain $\widetilde{f}(s,u(s),u_s)ds$ is differentiable a.e. on $[0, T_{\sup}(\phi))$. Therefore, it can be proved that

$$t \to \int_0^t E(t-s)\widetilde{f}(s,u(s),u_s)ds$$

is differentiable on $[0, T_{\sup}(\phi))$. This implies that Theorem 3.11 is true. ∎

Example 3.12. Consider the following Cauchy problem

$$\begin{cases} \dfrac{\partial u(t,\xi)}{\partial t} = i\Delta u(t,\xi) + \beta(\xi)\displaystyle\int_{-\infty}^t \int_{R^3} u^3(s,\sigma)d\sigma ds, & 0 \le t \le T,\ \xi \in R^3, \\ u_0 = \phi, \end{cases} \tag{3.8}$$

where $\beta \in H^1(R^3)$.

Take

$$X = L^3(R^3), \quad \mathcal{P} = L^3(R^-, L^3(R^3)),$$

$$A = i\Delta, \quad f(u_t) = \beta \int_{-\infty}^{t} \int_{R^3} u^3(s,\sigma) d\sigma ds,$$

$$C = (1-\Delta)^{-\frac{1}{2}}, \quad \widetilde{f}(u_t) = \left[(1-\Delta)^{\frac{1}{2}}\beta\right] \int_{-\infty}^{t} \int_{R^3} u^3(s,\sigma) d\sigma ds.$$

Then A generates a $\frac{1}{2}$-times integrated semigroup on X (cf. [12]), and therefore an C-regularized semigroup $\{E(t)\}_{t\in[0,T]}$ on X. Thus, we see that all the assumptions of Theorem 3.11 are satisfied. Therefore, there exists $\overline{T} \in (0,T]$ such that for any $\phi : R^- \to X$ being Lipschitz continuous with compact support and $\phi(0) \in H^3(R^3)$, (3.8) has a unique classical solution on $[0,\overline{T}]$.

Remark 3.13. In the example above, the condition that the "initial function" ϕ is of compact support can be relaxed. Here we employ this condition just for convenience.

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