On some properties of the class of semi-compact operators

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Abstract

We investigate Banach lattices for which the class of positive semi-compact operators coincides with that of L-weakly compact (resp. M-weakly compact) operators, and we give some consequences.

1 Introduction and notation

It is well known that each L-weakly compact (resp. regular M-weakly compact) operator is semi-compact (Theorem 3.6.10 and Corollary 3.6.14 of [4]), but a semi-compact operator is not necessary L-weakly compact (resp. M-weakly compact). In fact, the identity operator $Id_{l^\infty} : l^\infty \longrightarrow l^\infty$ is semi-compact, but it is not L-weakly compact (resp. M-weakly compact). However, in [2], it is proved that if E and F are nonzero Banach lattices, then each semi-compact operator $T : E \longrightarrow F$ is L-weakly compact if and only if the norm of F is order continuous [2, Theorem 1]. Also, if F is σ-Dedekind complete, then each positive semi-compact operator $T : E \longrightarrow F$ is M-weakly compact if and only if the norms of E' and F are order continuous or E is finite dimensional [2, Theorem 2].

Our aim in this paper is to characterize Banach lattices for which every positive semi-compact operator is L-weakly compact and M-weakly compact. After that, we establish our second Theorem, with another hypothesis different from that of [2], for the M-weak compactness of semi-compact operators. Finally, we

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give a necessary and sufficient condition for which the square of a semi-compact operator is L-weakly compact (resp. M-weakly compact).

Recall from [4] that an operator T from a Banach space E into a Banach lattice F is said to be semi-compact if for each $\varepsilon > 0$, there exists some $u \in F^+$ such that $T(B_E) \subset [-u, u] + \varepsilon B_F$ where B_H is the closed unit ball of $H = E$ or F and $F^+ = \{y \in F : 0 \leq y\}$. The operator T is called L-weakly compact if for each disjoint sequence (y_n), in the solid hull of $T(B_E)$, we have $\lim_n \|y_n\| = 0$. Finally, an operator T from a Banach lattice E into a Banach space F is said to be M-weakly compact if for each norm bounded disjoint sequence (x_n) of E, we have $\lim_n \|T(x_n)\| = 0$. Note that an operator T, between two Banach lattices, is L-weakly compact (resp. M-weakly compact) if and only if its adjoint T' is M-weakly compact (resp. L-weakly compact) [4, Proposition 3.6.11]. We refer to [1] and [4] for any unexplained terms from Banach lattice theory.

2 Major results

A compact operator is not necessary L-weakly compact (resp. M-weakly compact). In fact, if we consider the operator $T : l^1 \longrightarrow l^\infty$ defined by

$$T((\lambda_n)) = (\textstyle\sum_{n=1}^{\infty} \lambda_n)\, e \text{ for all } (\lambda_n) \in l^1$$

where $e = (1, 1, \cdots)$ is the constant sequence with value 1 [1, p. 322]. It is clear that T is compact (because its rank is one) but it is neither L-weakly compact nor M-weakly compact.

Also, this example proves that a semi-compact operator is not necessary M-weakly compact nor L-weakly compact.

On the other hand, a Dunford-Pettis operator is not necessarily either M-weakly compact or L-weakly compact. For an example, we have to just take the preceding example or the identity operator $Id_{l^1} : l^1 \longrightarrow l^1$.

The following characterization follows immediately from Theorem 1 of [2] and its proof:

Theorem 2.1. *Let E and F be nonzero Banach lattices. Then the following assertions are equivalent:*

1. *Every positive semi-compact operator $T : E \longrightarrow F$ is L-weakly compact.*
2. *Every positive compact operator $T : E \longrightarrow F$ is L-weakly compact.*
3. *The norm of F is order continuous.*

As a consequence, we obtain the following characterization:

Theorem 2.2. *Let E and F be nonzero Banach lattices. Then the following assertions are equivalent:*

1. *Every positive semi-compact operator $T : E \longrightarrow F$ is L-weakly compact and M-weakly compact.*

2. *Every positive Dunford-Pettis operator $T : E \longrightarrow F$ is L-weakly compact and M-weakly compact.*

3. *Every positive compact operator $T : E \longrightarrow F$ is L-weakly compact and M-weakly compact.*

4. *The norms of E' and F are order continuous.*

Proof. The implications (1) $\Rightarrow$ (3) and (2) $\Rightarrow$ (3) are clear.

(3) $\Rightarrow$ (4) Assume that (3) holds. It follows from Theorem 2.1 that the norm of F is order continuous.

Assume by way of contradiction that the norm of E' is not order continuous. It follows from Theorem 4.14 of [1] that there exists some $f \in (E')^+$ and there exists a disjoint sequence $(f_n) \subset [0, f]$ which does not converge to zero in norm. Pick some $y \in F^+$ with $\|y\| = 1$. By Theorem 39.3 of [9] there exists some $\psi \in (F')^+$ such that $\|\psi\| = 1$ and $\psi(y) = \|y\| = 1$.

Now, we consider the positive operator $T : E \to F$ defined by

$$T(x) = f(x)\, y \text{ for each } x \in E.$$

It is clear that T is compact (it has rank one).

On the other hand, we claim that T is not M-weakly compact. By Theorem 3.6.11 of [4], it suffices to show that its adjoint $T' : F' \to E'$ is not L-weakly compact. Note that $T'(\varphi) = \varphi(y) f$ for each $\varphi \in F'$. In particular, $T'(\psi) = \psi(y) f = f$. So, $f \in T'(B_{F'})$. From $(f_n) \subset [0, f]$, it follows that (f_n) is a disjoint sequence in the solid hull of $T'(B_{F'})$. Since (f_n) is not norm convergent to zero, then T' is not L-weakly compact. Hence T is not M-weakly compact. But this is in contradiction with our hypothesis (3). So, the norm of E' is order continuous.

(4) $\Rightarrow$ (1) We have just to apply Theorem 2.1 and Theorem 3.6.17 of [4].

(4) $\Rightarrow$ (2) We have just to apply Theorem 3.7.10 and Theorem 3.6.17 of [4]. ■

To establish another characterization of the M-weak compactness of semi-compact operators, we need to give some Lemmas. The first one is just a characterization of infinite-dimensional Banach lattices.

Lemma 2.3. *Let E be a Banach lattice. Then E is infinite-dimensional if and only if there exists a positive disjoint sequence (x_n) of E^+ such that $\|x_n\| = 1$ for all n.*

Proof. Assume that there exists a disjoint sequence (x_n) of E^+ such that $\|x_n\| = 1$ for each n. It follows from Corollary 2 [5, p. 53] that the subset $A = \{x_n : n \in \mathbb{N}\}$ is linearly independent. Then E is infinite-dimensional.

Conversely, assume that E is infinite-dimensional. By Proposition 0.2.11 of [7], there exists a disjoint positive sequence (y_n) of E such that $y_n \neq 0$ for all n. For every n, pick $x_n = \frac{1}{\|y_n\|} y_n$. Then the sequence (x_n) satisfies the desired properties. ■

Lemma 2.4. *Let E be a Banach lattice, and let (x_n) be a disjoint sequence of E. If (f_n) is a sequence of E', then there exists a disjoint sequence (g_n) of E' such that $|g_n| \leq |f_n|$, $g_n(x_n) = f_n(x_n)$ for all n and $g_n(x_m) = 0$ for $n \neq m$.*

Moreover, if (f_n) is a positive sequence of E' then we may take (g_n) in $(E')^+$.

Proof. Follows immediately from Proposition 0.3.11 of [7] and its proof. ■

Lemma 2.5. *Let E be a Banach lattice. If (x_n) is a positive disjoint sequence of E such that $\|x_n\| = 1$ for all n, then there exists a positive disjoint sequence (g_n) of E' with $\|g_n\| = 1$ such that $g_n(x_n) = 1$ for all n and $g_n(x_m) = 0$ for $n \neq m$.*

Proof. It follows from Theorem 39.3 of [9] that for each n there exists $f_n \in (E')^+$ such that $\|f_n\| = 1$ and $f_n(x_n) = \|x_n\| = 1$. Now, by applying Lemma 2.4 to the two sequences (x_n) and (f_n), there exists a positive disjoint sequence (g_n) of E' with $0 \leq g_n \leq f_n$ such that $g_n(x_n) = f_n(x_n) = 1$ for all n and $g_n(x_m) = 0$ for $n \neq m$.

Finally, it is clear that $\|g_n\| = \|f_n\| = 1$ for all n. ■

If we replace "F is σ-Dedekind complete" by "E has an order continuous norm" in Theorem 2 of [2], we obtain the following characterization:

Theorem 2.6. *Let E and F be two Banach lattices such that E has an order continuous norm. Then the following assertions are equivalent:*

1. *Each positive semi-compact operator $T : E \longrightarrow F$ is M-weakly compact.*
2. *One of the following conditions holds:*
   - *(a) both E' and F have order continuous norms.*
   - *(b) E is finite-dimensional.*

Proof. (1) $\Rightarrow$ (2) Assume that (1) holds. If the norm of E' is not order continuous, then it follows from the proof of Theorem 2.2 that there exists a positive compact operator $T : E \longrightarrow F$ which is not M-weakly compact. Hence T is semi-compact but it is not M-weakly compact, and this gives a contradiction with our hypothesis (1). So, the norm of E' is order continuous.

Assume by way of contradiction that the norm of F is not order continuous. By Theorem 4.14 of [1], there exists some $u \in F^+$ and there exists a disjoint sequence $(u_n) \subset [0, u]$ which does not converge to zero in norm. We may assume that $\|u_n\| = 1$ for all n.

On the other hand, since E is an infinite-dimensional Banach lattice, it follows from Lemma 2.3 and Lemma 2.5 the existence of a positive disjoint sequence (x_n) in E^+ with $\|x_n\| = 1$ for all n and there exists a positive disjoint sequence (g_n) of E' with $\|g_n\| = 1$ for each n, such that

$$g_n(x_n) = 1 \text{ for all } n \text{ and } g_n(x_m) = 0 \text{ for } n \neq m. \qquad (*)$$

To finish the proof, we have to construct a positive semi-compact operator $T : E \longrightarrow F$ which is not M-weakly compact.

Since E has an order continuous norm, it follows from Corollary 2.4.3 of [4] that $g_n \to 0$ for $\sigma(E', E)$. Hence the positive operator $R : E \to c_0$ defined by

$$R(x) = (g_n(x))_{n=1}^{\infty} \text{ for each } x \in E,$$

is well defined and $R(B_E) \subset B_{c_0}$. Also, it follows from the proof of Theorem 117.1 of [8] that the positive operator

$$S : c_0 \longrightarrow F,\ (\alpha_1, \alpha_2, \cdots) \longmapsto \sum_{i=1}^{\infty} \alpha_i u_i$$

defines a lattice isomorphism from c_0 into F and $S(B_{c_0}) \subset [-u, u]$.

Next, we consider the composed operator

$$T = S \circ R : E \longrightarrow F, x \longmapsto \sum_{i=1}^{\infty} g_i(x) u_i.$$

It follows from $T(B_E) = S(R(B_E)) \subseteq S(B_{c_0}) \subseteq [-u, u]$ that T is semi-compact but the operator T is not M-weakly compact. In fact, by $(*)$ we have

$$T(x_n) = u_n \text{ for all } n.$$

Since (x_n) is a disjoint sequence of E^+ with $\|x_n\| = 1$ for all n and $\|T(x_n)\| = \|u_n\| = 1$ for all n, it follows that T is not M-weakly compact, and this gives a contradiction with our hypothesis (1). So, the norm of F is order continuous and this completes the proof of $(1) \Rightarrow (2)$.

$(a) \Rightarrow (1)$ Follows from Theorem 2.2.

$(b) \Rightarrow (1)$ In this case, every operator $T : E \longrightarrow F$ is M-weakly compact. In fact, if E is finite-dimensional then for every norm bounded disjoint sequence (x_n) of E there exists some n_0 such that $x_n = 0$ for all $n \geq n_0$. So, $T(x_n) = 0$ for all $n \geq n_0$. Then $\|T(x_n)\| \to 0$ and hence T is M-weakly compact. ■

Remark 2.7. *The assumption "the norm of E is order continuous" in Theorem 2.6 or "F is σ-Dedekind complete" in Theorem 2 of [2] is essential. For instance, take* $E = l^\infty$ *and* $F = c$*. It is clear that every operator* $T : l^\infty \longrightarrow c$ *is semi-compact (because c is an AM-space with unit). On the other hand, every operator* $T : l^\infty \longrightarrow c$ *is weakly compact (see the proof of Proposition 1 of [6]). Since* l^∞ *is an AM-space, T is M-weakly compact [1, Theorem 5.62], and then the class of semi-compact operators coincides with that of M-weakly compact operators from* l^∞ *into c. But the condition (2) of Theorem 2.6 (resp. Theorem 2 of [2]) is not satisfied.*

Finally, we observe that the square of a semi-compact operator $T : E \longrightarrow E$ is not necessary L-weakly compact (resp. M-weakly compact). In fact, the identity operator $Id_{l^\infty} : l^\infty \longrightarrow l^\infty$ is semi-compact but its square $(Id_{l^\infty})^2 = Id_{l^\infty}$ is not L-weakly compact (resp. M-weakly compact).

In the following, we give a necessary and sufficient condition for which the square of a semi-compact operator is L-weakly compact (resp. M-weakly compact).

Theorem 2.8. *Let E be a Banach lattice. Then the following assertions are equivalent:*

1. *For every positive operators S and T from E into E such that* $0 \leq S \leq T$ *and T is semi-compact, the operator S is L-weakly compact.*
2. *Every positive semi-compact operator* $T : E \longrightarrow E$ *is L-weakly compact.*
3. *For every positive semi-compact operator T from E into E,* T^2 *is L-weakly compact.*

4. *The norm of E is order continuous.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (4) Assume by way of contradiction that the norm of E is not order continuous. By Theorem 4.14 of [1], there exists some $u \in E^+$ and there exists a disjoint sequence $(u_n) \subset [0, u]$ which does not converge to zero in norm. We may assume that $\|u\| = 1$.

On the other hand, it follows from Theorem 39.3 of [9] that there exists $f \in (E')^+$ such that $\|f\| = 1$ and $f(u) = \|u\| = 1$.

Now, we consider the positive operator $T : E \to E$ defined by

$$T(x) = f(x)\, u \text{ for each } x \in E.$$

It is clear that T is semi-compact (it has rank one) but the operator T^2 is not L-weakly compact. In fact, note that $T^2(u) = u$ and $\|u\| = 1$. So it follows from $(u_n) \subset [0, u]$ that (u_n) is a disjoint sequence in the solid hull of $T^2(B_E)$. Since (u_n) is not norm convergent to zero, then T is not L-weakly compact. But this is in contradiction with our hypothesis (3).

(4) $\Rightarrow$ (1) It follows from Theorem 5.72 of [1] that S is semi-compact and hence S is L-weakly compact by Theorem 1 of [2]. ■

Theorem 2.9. *Let E be a Banach lattice. Then the following assertions are equivalent:*

1. *For every positive operators S and T from E into E such that $0 \leq S \leq T$ and T is semi-compact, the operator S is M-weakly compact.*
2. *Every positive semi-compact operator $T : E \longrightarrow E$ is M-weakly compact.*
3. *For every positive semi-compact operator T from E into E, T^2 is M-weakly compact.*
4. *The norm of E' is order continuous.*

Proof. (1) $\Rightarrow$ (2) $\Rightarrow$ (3) Obvious.

(3) $\Rightarrow$ (4) Assume by way of contradiction that the norm of E' is not order continuous. By Theorem 4.14 of [1], there exists some $f \in (E')^+$ and there exists a disjoint sequence $(f_n) \subset [0, f]$ which does not converge to zero in norm. We may assume that $\|f\| = 1$. Pick some $u \in E^+$ such that $f(u) > 0$.

Now, we consider the positive operator $T : E \to E$ defined by

$$T(x) = \frac{f(x)}{f(u)} u \text{ for each } x \in E.$$

It is clear that T is semi-compact (it has rank one) but the operator T^2 is not M-weakly compact. In fact, by Theorem 3.6.11 of [4], it suffices to show that its adjoint $(T^2)' : E' \to E'$ is not L-weakly compact. Note that $T'(\varphi) = \frac{\varphi(u)}{f(u)} f$ for each $\varphi \in E'$. In particular, $T'(f) = f$ and hence $(T^2)'(f) = f$. Then it follows from $(f_n) \subset [0, f]$ that (f_n) is a disjoint sequence in the solid hull of $(T^2)'(B_{E'})$. Since (f_n) is not norm convergent to zero, then $(T^2)'$ is not L-weakly compact and hence T^2 is not M-weakly compact. But this is in contradiction with our hypothesis (3).

(4) $\Rightarrow$ (1) It follows from Theorem 5.72 of [1] that S is semi-compact and hence S is M-weakly compact by Theorem 2 of [2]. ■

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