Stability of difference problems generated by infinite systems of quasilinear parabolic functional differential equations

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Abstract

The paper deals with infinite weakly coupled systems of quasilinear parabolic differential functional equations. Initial boundary conditions of the Robin type are considered. We construct an explicit Euler type approximation method based on an infinite system of difference functional equations. Next we apply the truncation method to obtain a finite difference scheme corresponding to the original differential problem.

We present a complete convergence analysis for the methods. The results are based on a comparison technique with nonlinear estimates of the Perron type for given functions.

1 Introduction

For any metric spaces X and Y we denote by $C(X, Y)$ the class of all continuous functions from X into Y. Let $\mathbf{N}$ and $\mathbf{Z}$ be the sets of natural numbers and integers respectively. Denote by l^∞ the class of all real sequences $p = \{p_\mu\}_{\mu\in\mathbf{N}}$ such that $\|p\|_\infty = \sup\{|p_\mu| : \mu \in \mathbf{N}\} < \infty$. For simplicity we will write $p = \{p_\mu\}$ instead of $p = \{p_\mu\}_{\mu\in\mathbf{N}}$. If $p, q \in l^\infty$, $p = \{p_\mu\}$, $q = \{q_\mu\}$, then we set $p * q = \{p_\mu q_\mu\}$. Denote by $\mathcal{M}^\infty_{n\times n}$ the set of all $P = [p_{ij}]_{i,j=1,\dots,n}$ such that $p_{ij} \in l^\infty$, $1 \le i, j \le n$. Put $\mathcal{R}^\infty_n$ to denote the set of all $q = (q_1, \dots, q_n)$, such that $q_j \in l^\infty$, $1 \le j \le n$.

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We will use vectorial inequalities with the understanding that the same inequalities hold between their corresponding components. Analogously we understand inequalities between infinite sequences. We use the symbol $M_{n\times n}$ to denote the set of all real $n \times n$ matrices. Inequalities between symmetric $n \times n$ matrices are interpreted by means of quadratic forms.

Let $a > 0,\ b = (b_1, \ldots, b_n) \in R^n,\ b_j > 0$ for $j = 1, \ldots, n$, be given. Define the sets

$$E = [0,a] \times [-b,b],\ \ E_0 = \{0\} \times [-b,b],\ \ \partial_0 E = E \setminus \Big([0,a] \times (-b,b)\Big).$$

Set $\Xi = E \times C(E, l^\infty)$ and $\Sigma = E \times C(E, l^\infty) \times R^n$. Suppose that

$$\varrho : \Xi \to \mathcal{M}^\infty_{n\times n},\ \varrho = [\varrho_{ij}]_{i,j=1,\ldots,n},\ \varrho_{ij} = \{\varrho^{(\mu)}_{ij}\},\ 1 \le i,j \le n,$$

$$f : \Sigma \to l^\infty,\ f = \{f^{(\mu)}\},\ \ \varphi : E_0 \to l^\infty,\ \varphi = \{\varphi_\mu\},$$
$$\beta, \psi : \partial_0 E \to \mathcal{R}^\infty_n,$$
$$\beta = (\beta_1, \ldots, \beta_n),\ \psi = (\psi_1, \ldots, \psi_n),\ \beta_j = \{\beta_{j.\mu}\},\ \psi_j = \{\psi_{j.\mu}\},\ 1 \le j \le n,$$

are given functions. For the function $z : E \to l^\infty$, $z = \{z_\mu\}$, of the variables (t,x), $x = (x_1, \ldots, x_n)$, and for $1 \le j \le n$ we write

$$\partial_t z = \{\partial_t z_\mu\},\ \ \partial_{x_j} z = \{\partial_{x_j} z_\mu\},\ \ \ F[z] = \{F^{(\mu)}[z]\},$$

$$F^{(\mu)}[z](t,x) = \sum_{i,j=1}^{n} \varrho^{(\mu)}_{ij}(\,t,x,z\,)\partial_{x_i x_j} z_\mu(t,x) + f^{(\mu)}(\,t,x,z,\partial_x z_\mu(t,x)\,)$$

where $\partial_x z_\mu = (\,\partial_{x_1} z_\mu, \ldots, \partial_{x_n} z_\mu\,)$, $\mu \in \mathbf{N}$. We consider the system

$$\partial_t z(t,x) = F[z](t,x) \tag{1}$$

with the initial condition

$$z(t,x) = \varphi(t,x) \ \text{ on } \ E_0. \tag{2}$$

Write

$$\partial_{j.+}E = \Big\{(t,x) \in \partial_0 E :\ x_j = b_j\Big\},\ \ \partial_{j.-}E = \Big\{(t,x) \in \partial_0 E :\ x_j = -b_j\Big\},\ \ 1 \le j \le n.$$

The following boundary conditions are associated with (1) and (2)

$$\beta_j(t,x) * z(t,x) + \partial_{x_j} z(t,x) = \psi_j(t,x) \ \text{ on } \ \partial_{j.+}E, \tag{3}$$

$$\beta_j(t,x) * z(t,x) - \partial_{x_j} z(t,x) = \psi_j(t,x) \ \text{ on } \ \partial_{j.-}E, \tag{4}$$

where $1 \le j \le n$. The system (1) is weakly coupled. Each μ-th equation in (1) depends on the first and second order partial derivatives of z_μ. The problem (1)-(4) is said to be the *third boundary problem* on E. The conditions (3), (4) are a special type of the Robin conditions. We will assume that the functional dependence in (1) is of the Volterra type.

Assumption H[V]. The functions $\varrho : \Xi \to \mathcal{M}^{\infty}_{n\times n}$ and $f : \Sigma \to l^{\infty}$ satisfy the Volterra condition, i.e. for each $(t,x) \in E$, $q \in R^n$ and $w, \bar{w} \in C(E, l^{\infty})$ such that $w(\tau,y) = \bar{w}(\tau,y)$, $(\tau,y) \in E$, $\tau \leq t$, we have $\varrho(t,x,w) = \varrho(t,x,\bar{w})$ and $f(t,x,w,q) = f(t,x,\bar{w},q)$.

Differential equations with deviated variables and differential integral equations can be derived from a general model of functional dependence in (1) by specializing the given functions.

We consider classical solutions of (1)-(4) in the following sense. We say that a function $v : E \to l^{\infty}$, $v = \{v_\mu\}$, is a *regular solution* of the system (1) if the derivatives $\partial_t v = \{\partial_t v_\mu\}$, $\partial_{x_i x_j} v = \{\partial_{x_i x_j} v_\mu\}$, $1 \leq i,j \leq n$, exist on E, $\partial_t v, \partial_{x_i x_j} v \in C(E, l^{\infty})$, $1 \leq i,j \leq n$, and v satisfies (1) on E.

A regular solution v of (1) is said to be *parabolic* if for any symmetric matrix $r \in M_{n\times n}$, $r = [r_{ij}]_{i,j=1,\ldots,n}$, such that $r \leq 0$ the inequality $\sum_{i,j=1}^{n} \varrho^{(\mu)}_{ij}(t,x,v) r_{ij} \leq 0$ is satisfied for $(t,x) \in E$, $\mu \in \mathbf{N}$. The parabolic solution v of (1) such that the conditions (2)-(4) hold, is called $\mathcal{P}$*-solution* of (1)-(4).

In recent years a number of papers concerned with problems for infinite systems of parabolic functional differential equations were published. The monograph [1] contains the exposition of existence results for such problems. Various applications of infinite systems of parabolic differential integral equations are also listed in [1]. Uniqueness criteria for infinite parabolic problems can be found in [12, 13] and [3].

Approximate methods for parabolic differential or functional differential equations were considered by many authors and under various assumptions. The main problem in these investigations is to find suitable difference or functional difference equations which are consistent with respect to the original problem and stable. It is not our aim to show a full review of papers concerning difference methods for parabolic functional differential problems. Bibliographical informations can be found in [5, 7, 9, 10, 11].

We are interested in establishing a numerical discretization method for solving the differential functional problem (1)-(4). We propose upwind difference explicit Euler type schemes which consist of replacing partial derivatives in (1) by suitable difference operators. The choice of the difference operators approximating mixed derivatives is locally determined by the sign of the coefficients in the differential equations. The approximation of the Robin boundary conditions (3), (4) requires an extension of the mesh outside the set E (see the definition of the sets $\partial_0^+ E_h$ and E_h^+). The same extended mesh was applied in [2] for the scalar quasilinear parabolic differential functional equation with the Neumann initial boundary conditions (see also [8]).

The first part of the present paper deals with an infinite system of difference functional equations generated by (1)-(4). In a general case this scheme is a theoretical approximation. If the original differential problem is reduced to the finite one then the difference method is also finite and it is practically solvable.

In the next part of the paper we consider truncated finite differential functional problems corresponding to (1)-(4) and difference functional methods related to them. A convergence analysis for these methods is given.

Results presented in the paper are new also in the case of infinite systems without a functional dependence.

To illustrate the theory we show results of numerical experiments.

2 Infinite systems of difference equations

We formulate a difference problem corresponding to (1)-(4). Denote by $\mathcal{F}(A,B)$ the class of all functions defined on A and taking values in B where A and B are arbitrary sets. If $x \in R^n$ then we put $\|x\| = |x_1| + \ldots + |x_n|$. We define a mesh on the set E in the following way. Suppose that (h_0, h') where $h' = (h_1, \ldots, h_n)$, $h_i > 0$, $0 \le i \le n$, stand for steps of the mesh. For $h = (h_0, h')$ and $(r,m) \in \mathbf{Z}^{1+n}$ where $m = (m_1, \ldots, m_n)$, we define nodal points as follows:

$$t^{(r)} = rh_0, \ \ x^{(m)} = (x_1^{(m_1)}, \ldots, x_n^{(m_n)}) = (m_1 h_1, \ldots, m_n h_n).$$

Denote by Δ the set of all $h = (h_0, h')$ such that there are $N_0 \in \mathbf{N}$ and $N = (N_1, \ldots, N_n) \in \mathbf{N}^n$ with the properties: $N_0 h_0 = a$ and $(N_1 h_1, \ldots, N_n h_n) = b$. Let

$$R_h^{1+n} = \{ \, (t^{(r)}, x^{(m)}) : \ (r,m) \in \mathbf{Z}^{1+n} \, \},$$

$$E_h = E \cap R_h^{1+n}, \ \ E_{0.h} = E_0 \cap R_h^{1+n}, \ \ \partial_0 E_h = \partial_0 E \cap R_h^{1+n}$$

and

$$E'_h = \Big\{ \, (t^{(r)}, x^{(m)}) \in E_h : \ \ 0 \le r \le N_0 - 1 \, \Big\}.$$

For every $(t^{(r)}, x^{(m)}) \in \partial_0 E_h$ we define the set $S^{(m)}$ of $s = (s_1, \ldots, s_n)$ such that $\|s\| = 1$ or $\|s\| = 2$ and

$$\text{if } \ m_j = N_j \ \text{ then } \ s_j \in \{0,1\}, \quad \text{if } \ m_j = -N_j \ \text{ then } \ s_j \in \{0,-1\},$$

$$\text{and if } \ -N_j < m_j < N_j \ \text{ then } \ s_j = 0,$$

where $1 \le j \le n$. Let

$$\partial_0^+ E_h = \{ (t^{(r)}, x^{(m+s)}) : \ (t^{(r)}, x^{(m)}) \in \partial_0 E_h, \ s \in S^{(m)} \} \quad \text{and} \quad E_h^+ = \partial_0^+ E_h \cup E_h.$$

If $A_h \subset R_h^{1+n}$ and $z : A_h \to l^\infty$, $\omega : A_h \to R$ then we write $z^{(r,m)} = z(t^{(r)}, x^{(m)})$ and $\omega^{(r,m)} = \omega(t^{(r)}, x^{(m)})$ on A_h. Set $e_i = (0, \ldots, 0, 1, 0, \ldots, 0) \in R^n$ with 1 standing on i-th place. We define the difference operators δ_0, $\delta = (\, \delta_1, \ldots, \delta_n \,)$ and δ_i^+, δ_i^-, $1 \le i \le n$, in the following way. For $\omega : E_h^+ \to R$ and $(t^{(r)}, x^{(m)}) \in E'_h$ set

$$\delta_0 \omega^{(r,m)} = \frac{1}{h_0} (\, \omega^{(r+1,m)} - \omega^{(r,m)} \,), \tag{5}$$

$$\delta_i \omega^{(r,m)} = \frac{1}{2h_i} (\, \omega^{(r,m+e_i)} - \omega^{(r,m-e_i)} \,), \tag{6}$$

$$\delta_i^+ \omega^{(r,m)} = \frac{1}{h_i} (\, \omega^{(r,m+e_i)} - \omega^{(r,m)} \,), \quad \delta_i^- \omega^{(r,m)} = \frac{1}{h_i} (\, \omega^{(r,m)} - \omega^{(r,m-e_i)} \,) \tag{7}$$

where $1 \leq i \leq n$. Solutions of difference equations will be defined on the set E_h^+. Since system (1) contains the functional variable z which is an element of the space $C(E, l^\infty)$, we need an interpolating operator $\mathcal{T}_h : \mathcal{F}(E_h^+, l^\infty) \to C(E, l^\infty)$. Additional assumptions on $\mathcal{T}_h$ will be required in the next part of this paper. For $z : E_h^+ \to l^\infty$, $z = \{z_\mu\}$, we put on E_h'

$$\delta_0 z = \{\delta_0 z_\mu\}, \quad F_h[z] = \{F_{h.\mu}[z]\},$$

$$F_{h.\mu}[z]^{(r,m)} = \sum_{i,j=1}^{n} \varrho_{ij}^{(\mu)}(\, t^{(r)}, x^{(m)}, \mathcal{T}_h z\,)\, \delta_{ij} z_\mu^{(r,m)} + f^{(\mu)}(\, t^{(r)}, x^{(m)}, \mathcal{T}_h z,\, \delta z_\mu^{(r,m)}\,), \quad \mu \in \mathbf{N}.$$

The difference operators δ_{ij}, $1 \leq i, j \leq n$, are defined as follows. Write

$$\delta_{ii} z_\mu^{(r,m)} = \delta_i^+ \delta_i^- z_\mu^{(r,m)}, \quad 1 \leq i \leq n. \tag{8}$$

Put

$$J = \Big\{ (i,j) : 1 \leq i, j \leq n,\ i \neq j \Big\}.$$

The difference expressions $\delta_{ij} z_\mu^{(r,m)}$ for $(i,j) \in J$ are defined in the following way:

$$\text{if } \varrho_{ij}^{(\mu)}(\, t^{(r)}, x^{(m)}, \mathcal{T}_h z\,) \geq 0 \quad \text{then} \quad \delta_{ij} z_\mu^{(r,m)} = \frac{1}{2}\Big(\delta_i^+ \delta_j^+ z_\mu^{(r,m)} + \delta_i^- \delta_j^- z_\mu^{(r,m)} \Big), \tag{9}$$

$$\text{if } \varrho_{ij}^{(\mu)}(\, t^{(r)}, x^{(m)}, \mathcal{T}_h z\,) < 0 \quad \text{then} \quad \delta_{ij} z_\mu^{(r,m)} = \frac{1}{2}\Big(\delta_i^+ \delta_j^- z_k^{(r,m)} + \delta_i^- \delta_j^+ z_\mu^{(r,m)} \Big). \tag{10}$$

If $z : E_h^+ \to l^\infty$ and $(t^{(r)}, x^{(m)}) \in \partial_0 E_h$, $s \in S^{(m)}$, then we write

$$g_h[z]^{(r,m,s)} = 2 \sum_{j=1}^{n} s_j^2 h_j \psi_j(t^{(r)}, x^{(m)}) - (z^{(r,m+s)} + z^{(r,m-s)}) * \sum_{j=1}^{n} s_j^2 h_j \beta_j(t^{(r)}, x^{(m)}).$$

We will approximate solutions of (1)-(4) by means of solutions of the difference functional problem

$$\delta_0 z^{(r,m)} = F_h[z]^{(r,m)} \quad \text{on } E_h', \tag{11}$$

$$z^{(r,m)} = \varphi_h^{(r,m)} \quad \text{on } E_{0.h}, \tag{12}$$

$$z^{(r,m+s)} - z^{(r,m-s)} = g_h[z]^{(r,m,s)} \quad \text{on } \partial_0 E_h,\ s \in S^{(m)}, \tag{13}$$

where $\varphi_h : E_{0.h} \to l^\infty$, $\varphi_h = \{\varphi_{h.\mu}\}$, is a given function.

For $w \in C(E, R)$ and for $z \in \mathcal{F}(E_h^+, R)$ we put

$$|w|_t = \max \Big\{ |w(\tau, x)| : (\tau, x) \in E,\ \tau \leq t \Big\}, \quad t \in [0, a],$$

$$|z|_{(r)} = \max \Big\{ |z^{(\nu,m)}| : (t^{(\nu)}, x^{(m)}) \in E_h,\ \nu \leq r \Big\}, \quad 0 \leq r \leq N_0.$$

If $w \in C(E, l^\infty)$, $w = \{w_\mu\}$, and $z \in \mathcal{F}(E_h^+, l^\infty)$, $z = \{z_\mu\}$, then we set $|w|_t = \{|w_\mu|_t\}$, $t \in [0, a]$, and $|z|_{(r)} = \{|z_\mu|_{(r)}\}$, $0 \leq r \leq N_0$.

Assumption H[$\mathcal{T}_h$]. The operator $\mathcal{T}_h : \mathcal{F}(E_h^+, l^\infty) \to C(E, l^\infty)$ is linear, $\mathcal{T}_h z = \{T_h z_\mu\}$ for $z \in \mathcal{F}(E_h^+, l^\infty)$, $z = \{z_\mu\}$, and the mapping $T_h : \mathcal{F}(E_h^+, R) \to C(E, R)$ satisfies the conditions:

1) if $\omega, \bar{\omega} \in \mathcal{F}(E_h^+, R)$ and $\omega = \bar{\omega}$ on E_h then $T_h\omega = T_h\bar{\omega}$,

2) for $\omega : E_h^+ \to R$ and $0 \le r \le N_0$ we have $|T_h\omega|_{t^{(r)}} = |\omega|_{(r)}$,

3) if $w : E \to R$ is of class C^1 and w_h is the restriction of w to the set E_h then there exists $\tilde{\gamma} : \Delta \to R_+$ such that $| T_h w_h - w |_a \le \tilde{\gamma}(h)$ and $\lim_{h\to 0} \tilde{\gamma}(h) = 0$.

Remark 1. *To define an operator $T_h : \mathcal{F}(E_h^+, R) \to C(E, R)$ satisfying the above conditions we can use the interpolating operator proposed in [6] for the construction of difference scheme corresponding to first order partial differential functional equations.*

If $p \in l^\infty$, $p = \{p_\mu\}$, then we write $|p| = \{|p_\mu|\}$. Let $\mathbf{0} \in l^\infty$ and $\mathbf{1} \in l^\infty$ be the sequences with all the elements equal to 0 and 1 respectively. Put $R_+ = [0, +\infty)$ and

$$l_+^\infty = \Big\{ p \in l^\infty : \; p = \{p_\mu\},\; p_\mu \in R_+,\; \mu \in \mathbf{N} \Big\},$$

$$l_0^\infty = \Big\{ p \in l_+^\infty : \; p = \{p_\mu\},\; \lim_{\mu\to\infty} p_\mu = 0 \Big\}.$$

Assumption H[σ_0]. The functions $f : \Sigma \to l^\infty$, $\beta, \psi : \partial_0 E \to \mathcal{R}_n^\infty$ and $\varphi : E_0 \to l^\infty$ satisfy the conditions:

1) there is $A_0 \in l_+^\infty$ such that $|\varphi(t, x)| \le A_0$ on E_0,

2) there is $\tilde{b} \in l^\infty$, $\tilde{b} = \{\tilde{b}_\mu\}$, such that $\beta_j(t, x) \ge \tilde{b} > \mathbf{0}$ on $\partial_0 E$, $1 \le j \le n$,

3) there exist $\sigma_0 \in C([0, a] \times l_+^\infty, l_+^\infty)$ and $L_0 \in l_+^\infty$ such that

   (i) σ_0 is nondecreasing with respect to both variables and $\sigma_0(t, p) \le L_0$ for $(t, p) \in [0, a] \times l_+^\infty$,

   (ii) there exists on $[0, a]$ a maximal solution $\omega_0 = \{\omega_{0.\mu}\}$ of the Cauchy problem
$$\omega'(t) = \sigma_0(\, t, \omega(t)\,), \;\; \omega(0) = A_0, \tag{14}$$

4) the estimates

$$|f(t, x, w, 0)| \le \sigma_0(t, |w|_t), \;\; (t, x, w) \in \Xi,$$

$$|\psi_j(t, x)| \le \tilde{b} * \omega_0(t), \;\; (t, x) \in \partial_0 E, \;\; 1 \le j \le n, \tag{15}$$

are satisfied.

Remark 2. *Suppose that Assumption H[σ_0] is satisfied. Then $\mathcal{P}$-solution $v : E \to l^\infty$ of problem (1)-(4) satisfies the estimate*

$$|v(t, x)| \le \omega_0(t) \;\; \text{on} \;\; E$$

where ω_0 is the maximal solution of (14). This assertion follows from the comparison theorem for infinite systems of parabolic functional differential equations (see [3]).

Let $E^+ = [0, a] \times (-b^+, b^+)$ where $b^+ \in R_+^n$ and $b^+ > b$.

Assumption H$_0$[Δ]. The functions $\varrho : \Xi \to \mathcal{M}_{n\times n}^\infty$, $f : \Sigma \to l^\infty$, $\beta : \partial_0 E \to \mathcal{R}_n^\infty$, $\varphi_h : E_{0.h} \to l^\infty$ and $h \in \Delta$ are such that

1) Assumption H[V] is satisfied,

2) there exist the derivatives $\partial_q f^{(\mu)} = (\partial_{q_1} f^{(\mu)}, \ldots, \partial_{q_n} f^{(\mu)})$ on Σ and $\partial_q f^{(\mu)}(t, x, w, \cdot) \in C(R^n, R^n)$ where $(t, x, w) \in \Xi$, $\mu \in \mathbf{N}$,

3) there is $A_0 \in l^\infty_+$, $A_0 = \{A_{0.\mu}\}$, such that $|\varphi_h^{(r,m)}| \le A_0$ on $E_{0.h}$,

4) $h \in \Delta$ is such that $E_h^+ \subset E^+$ and the inequalities

$$\mathbf{1} - 2\sum_{i=1}^{n} \frac{h_0}{h_i^2} \varrho_{ii}(t, x, w) + \sum_{(i,j)\in J} \frac{h_0}{h_i h_j} |\varrho_{ij}(t, x, w)| \ge \mathbf{0}, \tag{16}$$

$$\frac{1}{h_i} \varrho_{ii}(t, x, w) - \sum_{j=1, j\ne i}^{n} \frac{1}{h_j} |\varrho_{ij}(t, x, w)| - \frac{1}{2} |\partial_{q_i} f(t, x, w, q)| \ge \mathbf{0}, \;\; 1 \le i \le n, \tag{17}$$

hold with $(t, x, w) \in \Xi$, $q \in R^n$, where $\partial_{q_i} f = \{\partial_{q_i} f^{(\mu)}\}$, $1 \le i \le n$, and the inequality

$$\mathbf{1} - \sum_{j=1}^{n} h_j \beta_j(t, x) \ge \mathbf{0} \tag{18}$$

holds on $\partial_0 E$.

Lemma 1. *If Assumptions H[$\mathcal{T}_h$], H[σ_0] and H$_0$[Δ] are satisfied then there exists exactly one solution $u_h : E_h^+ \to l^\infty$ of problem (11)-(13) and*

$$|u_h^{(r,m)}| \le \omega_0(t^{(r)}) \;\; \text{on} \;\; E_h \tag{19}$$

where ω_0 is the maximal solution of (14).

Proof. It follows from Assumptions H[$\mathcal{T}_h$] and H[V] that there exists exactly one solution u_h of problem (11)-(13) and u_h is defined on E_h^+. We prove the estimate (19). Let $\zeta_h^{(r)} = \{\zeta_{h.\mu}^{(r)}\}$, $0 \le r \le N_0$, be given by

$$\zeta_{h.\mu}^{(r)} = \max\left\{ |u_{h.\mu}^{(\nu,m)}| : (t^{(\nu)}, x^{(m)}) \in E_h^+, \; \nu \le r \right\}, \;\; \mu \in \mathbf{N}.$$

We prove by induction that for any $0 \le r \le N_0$ the inequality

$$\zeta_h^{(r)} \le \omega_0(t^{(r)}) \tag{20}$$

is true. First we prove that $\zeta_h^{(0)} \le A_0$. If on the contrary, $\zeta_{h.\mu}^{(0)} > A_{0.\mu}$ for some $\mu \in \mathbf{N}$ then in view of the assumption 3) of H$_0$[Δ] there is $(0, x^{(\tilde m + \tilde s)}) \in \partial_0^+ E_h$ with $\tilde s \in S^{(\tilde m)}$ such that $\zeta_{h.\mu}^{(0)} = |u_{h.\mu}^{(0,\tilde m+\tilde s)}|$. Assume that $\zeta_{h.\mu}^{(0)} = u_{h.\mu}^{(0,\tilde m+\tilde s)}$. It follows that

$$\zeta_{h.\mu}^{(0)} \Big(1 + \sum_{j=1}^{n} \tilde s_j^2 h_j \beta_{j.\mu}(0, x^{(\tilde m)}) \Big) =$$

$$= \varphi_{h.\mu}^{(0,\tilde{m}-\tilde{s})}\Big(1-\sum_{j=1}^{n}\tilde{s}_j^2 h_j \beta_{j.\mu}(0,x^{(\tilde{m})})\Big)+2\sum_{j=1}^{n}\tilde{s}_j^2 h_j \psi_{j.\mu}(0,x^{(\tilde{m})}) \leq$$

$$\leq A_{0.\mu}\Big(1-\sum_{j=1}^{n}\tilde{s}_j^2 h_j \beta_{j.\mu}(0,x^{(\tilde{m})})\Big)+2\sum_{j=1}^{n}\tilde{s}_j^2 h_j |\psi_{j.\mu}(0,x^{(\tilde{m})})|.$$

Thus

$$\sum_{j=1}^{n}\tilde{s}_j^2 h_j|\psi_{j.\mu}(0,x^{(\tilde{m})})| > A_{0.\mu}\sum_{j=1}^{n}\tilde{s}_j^2 h_j \beta_{j.\mu}(0,x^{(\tilde{m})}) \geq A_{0.\mu}\tilde{b}_\mu \sum_{j=1}^{n}\tilde{s}_j^2 h_j$$

and we obtain a contradiction to the condition (15). If we assume that $\zeta_{h.\mu}^{(0)} = -u_{h.\mu}^{(0,\tilde{m}+\tilde{s})}$ then we get the same contradiction.

Suppose that the estimate (20) is satisfied for fixed r, $0 \leq r < N_0$. It follows that

$$\delta_0 u_{h.\mu}^{(r,m)} = \sum_{i,j=1}^{n} \varrho_{ij}^{(\mu)}(\, t^{(r)}, x^{(m)}, \mathcal{T}_h u_h\,)\, \delta_{ij} u_{h.\mu}^{(r,m)} +$$

$$+\sum_{i=1}^{n}\partial_{q_i} f^{(\mu)}(\, P_\mu^{(r,m)}[u_h]\,)\,\delta_i u_{h.\mu}^{(r,m)} + f^{(\mu)}(\, t^{(r)}, x^{(m)}, \mathcal{T}_h u_h, 0\,),\ \mu \in \mathbf{N}, \tag{21}$$

where $(t^{(r)}, x^{(m)}) \in E_h'$ and $P_\mu^{(r,m)}[u_h] = (\, t^{(r)}, x^{(m)}, \mathcal{T}_h u_h,\ \xi \delta u_{h.\mu}^{(r,m)}\,)$ with some $\xi \in (0,1)$ is an intermediate point. Set

$$J_{\mu.+}^{(r,m)}[u_h] = \Big\{(i,j) \in J : \varrho_{ij}^{(\mu)}(t^{(r)}, x^{(m)}, \mathcal{T}_h u_h) \geq 0\Big\},\ \ J_{\mu.-}^{(r,m)}[u_h] = J \setminus J_{\mu.+}^{(r,m)}[u_h].$$

We conclude from (21) that

$$u_{h.\mu}^{(r+1,m)} = u_{h.\mu}^{(r,m)}\mathcal{A}_\mu^{(r,m)} + \sum_{i=1}^{n} u_{h.\mu}^{(r,m+e_i)}\mathcal{B}_{\mu.i}^{(r,m)} + \sum_{i=1}^{n} u_{h.\mu}^{(r,m-e_i)}\mathcal{C}_{\mu.i}^{(r,m)} +$$

$$+\sum_{(i,j)\in J_{\mu.+}^{(r,m)}[u_h]}\Big(u_{h.\mu}^{(r,m+e_i+e_j)} + u_{h.\mu}^{(r,m-e_i-e_j)}\Big)\mathcal{D}_{\mu.ij}^{(r,m)} + \tag{22}$$

$$+\sum_{(i,j)\in J_{\mu.-}^{(r,m)}[u_h]}\Big(u_{h.\mu}^{(r,m+e_i-e_j)} + u_{h.\mu}^{(r,m-e_i+e_j)}\Big)\mathcal{D}_{\mu.ij}^{(r,m)} + h_0 f^{(\mu)}(\, t^{(r)}, x^{(m)}, \mathcal{T}_h u_h, 0\,)$$

where

$$\mathcal{A}_\mu^{(r,m)} = 1 - 2\sum_{i=1}^{n}\frac{h_0}{h_i^2}\varrho_{ii}^{(\mu)}(Q) + \sum_{(i,j)\in J}\frac{h_0}{h_i h_j}|\varrho_{ij}^{(\mu)}(Q)|,$$

$$\mathcal{B}_{\mu.i}^{(r,m)} = \frac{h_0}{h_i^2}\varrho_{ii}^{(\mu)}(Q) - \sum_{j=1, j\neq i}^{n}\frac{h_0}{h_i h_j}|\varrho_{ij}^{(\mu)}(Q)| + \frac{h_0}{2h_i}\partial_{q_i} f^{(\mu)}(P),\ \ 1 \leq i \leq n,$$

$$\mathcal{C}_{\mu.i}^{(r,m)} = \frac{h_0}{h_i^2}\varrho_{ii}^{(\mu)}(Q) - \sum_{j=1, j\neq i}^{n}\frac{h_0}{h_i h_j}|\varrho_{ij}^{(\mu)}(Q)| - \frac{h_0}{2h_i}\partial_{q_i} f^{(\mu)}(P),\ \ 1 \leq i \leq n,$$

$$\mathcal{D}^{(r,m)}_{\mu.ij} = \frac{h_0}{2h_i h_j} |\varrho^{(\mu)}_{ij}(Q)|, \quad (i,j) \in J,$$

and $Q = (\, t^{(r)}, x^{(m)}, \mathcal{T}_h u_h \,)$, $P = P^{(r,m)}_{\mu}[u_h]$. Since $\mathcal{A}^{(r,m)}_{\mu}$, $\mathcal{B}^{(r,m)}_{\mu.i}$, $\mathcal{C}^{(r,m)}_{\mu.i}$, $\mathcal{D}^{(r,m)}_{\mu.ij}$ are nonnegative and

$$\mathcal{A}^{(r,m)}_{\mu} + \sum_{i=1}^{n} \left(\mathcal{B}^{(r,m)}_{\mu.i} + \mathcal{C}^{(r,m)}_{\mu.i} \right) + \sum_{i,j=1}^{n} 2\mathcal{D}^{(r,m)}_{\mu.ij} = 1, \;\; \mu \in \mathbf{N},$$

we obtain

$$|u^{(r+1,m)}_{h.\mu}| \le \zeta^{(r)}_h + h_0 \sigma_0(\, t^{(r)}, \zeta^{(r)}_h \,).$$

The components of ω_0 are convex on $[0,a]$ so we have the following inequality

$$\omega_0(t^{(r+1)}) \ge \omega_0(t^{(r)}) + h_0\, \sigma_0(\, t^{(r)}, \omega_0(t^{(r)}) \,).$$

The above inequalities and the inductive assumption yield

$$|u^{(r+1,m)}_{h.\mu}| \le \omega_0(t^{(r+1)}), \;\; (t^{(r+1)}, x^{(m)}) \in E_h. \tag{23}$$

If we assume that the inequality $\zeta^{(r+1)}_h \le \omega_0(t^{(r+1)})$ does not hold and $\zeta^{(r+1)}_{h.\mu} > \omega_{0.\mu}(t^{(r+1)})$ for some $\mu \in \mathbf{N}$ then there is $(t^{(\bar{v})}, x^{(\bar{m}+\bar{s})}) \in \partial_0^+ E_h$ with $\bar{v} \le r+1$, $\bar{s} \in S^{(\bar{m})}$ such that $\zeta^{(r+1)}_{h.\mu} = |u^{(\bar{v},\bar{m}+\bar{s})}_{h.\mu}|$. Consider the case $\zeta^{(r+1)}_{h.\mu} = u^{(\bar{v},\bar{m}+\bar{s})}_{h.\mu}$. It follows from (13) and (23) that

$$\zeta^{(r+1)}_{h.\mu} \Big(1 + \sum_{j=1}^{n} \bar{s}_j^2 h_j \beta_{j.\mu}(t^{(\bar{v})}, x^{(\bar{m})}) \Big) \le$$

$$\le \omega_{0.\mu}(t^{(r+1)}) \Big(1 - \sum_{j=1}^{n} \bar{s}_j^2 h_j \beta_{j.\mu}(t^{(\bar{v})}, x^{(\bar{m})}) \Big) + 2 \sum_{j=1}^{n} \bar{s}_j^2 h_j |\psi_{j.\mu}(t^{(\bar{v})}, x^{(\bar{m})})|$$

and thus

$$\sum_{j=1}^{n} \bar{s}_j^2 h_j |\psi_{j.\mu}(t^{(\bar{v})}, x^{(\bar{m})})| > \tilde{b}_\mu \omega_{0.\mu}(t^{(r+1)}) \sum_{j=1}^{n} \bar{s}_j^2 h_j.$$

This is a contradiction to the assumption (15). Analogously we obtain in the case $\zeta^{(r+1)}_{h.\mu} = -u^{(\bar{v},\bar{m}+\bar{s})}_{h.\mu}$. Therefore

$$\zeta^{(r+1)}_h \le \omega_0(t^{(r+1)})$$

and this completes the proof of Lemma 1.

3 Convergence of difference methods

For $p \in l^\infty_+$ we define $C_p(E, l^\infty) = \{w \in C(E, l^\infty) : |w|_a \le p\}$.

Assumption H[σ]. The functions $\varrho : \Xi \to \mathcal{M}^\infty_{n\times n}$, $f : \Sigma \to l^\infty$ are continuous, Assumption H[σ_0] is satisfied and

1) the sequence $A \in l^\infty_+$ is such that $A > \mathbf{0}$ and $A \ge \omega_0(a)$,

2) there exist $\sigma \in C([0, a] \times l^\infty_+, l^\infty_+)$, $\sigma = \{\sigma_\mu\}$, and $L \in l^\infty_+$ such that

   (i) σ is nondecreasing with respect to both variables, $\sigma(t, \mathbf{0}) = \mathbf{0}$, $t \in [0, a]$, and $\sigma(t, p) \le L$ on $[0, a] \times l^\infty_+$,

   (ii) for each $C \in l^\infty_+$, $C \ge \mathbf{1}$, a function $\tilde{\omega}(t) = \mathbf{0}$, $t \in [0, a]$, is the unique solution of the problem

$$\omega'(t) = C * \sigma(\, t, \omega(t)\,), \;\; \omega(0) = \mathbf{0},$$

3) the estimates

$$\sum_{i,j=1}^{n} |\, \varrho_{ij}(t, x, w) - \varrho_{ij}(t, x, \bar{w})\,| \le \sigma(\, t, |w - \bar{w}|_t\,),$$

$$|\, f(t, x, w, q) - f(t, x, \bar{w}, q)\,| \le \sigma(\, t, |w - \bar{w}|_t\,)$$

   are satisfied for $(t, x) \in E$, $q \in R^n$ and $w, \bar{w} \in C_A(E, l^\infty)$.

Assumption H$_1$[Δ]. The functions $\varrho : \Xi \to \mathcal{M}^\infty_{n\times n}$, $f : \Sigma \to l^\infty$, $\beta : \partial_0 E \to \mathcal{R}^\infty_n$, $\varphi_h : E_{0.h} \to l^\infty$ and $h \in \Delta$ satisfy Assumption H$_0$[Δ] and

1) there is a sequence $B \in l^\infty_+$ such that $\beta_j(t, x) \le B$ on $\partial_0 E$, $1 \le j \le n$,

2) there is a constant $\tilde{C} > 0$ such that $\|h'\|^2 \le \tilde{C} h_0$.

Theorem 1. *Suppose that Assumptions H[$\mathcal{T}_h$], H$_1$[Δ] and H[σ] are satisfied and*

1) *the function $v : E^+ \to l^\infty$, $v = \{v_\mu\}$, is such that $v(\cdot, x) : [0, a] \to l^\infty$, $x \in (-b^+, b^+)$, is of class C^1, $v(t, \cdot) : (-b^+, b^+) \to l^\infty$, $t \in [0, a]$, is of class C^3 and there are $c_0, c_1 \in l^\infty_+$ such that*

$$|\partial_{x_i x_j} v(t, x)| \le c_0, \;\; |\partial_{x_i x_j x_k} v(t, x)| \le c_1 \;\text{ on }\; E^+, \;\; 1 \le i, j, k \le n,$$

   and v is $\mathcal{P}$-solution of (1)-(4) on E,

2) *the function $u_h : E_h^+ \to l^\infty$, $u_h = \{u_{h.\mu}\}$, is the solution of problem (11)-(13),*

3) *there exists a function $\gamma_\varphi : \Delta \to l^\infty_+$ such that $\lim_{h\to 0} \gamma_\varphi(h) = \mathbf{0}$ and*

$$|\, \varphi_h^{(r,m)} - \varphi(t^{(r)}, x^{(m)})\,| \le \gamma_\varphi(h) \;\text{ on }\; E_{0.h}.$$

Then there is $\gamma : \Delta \to l^{\infty}_{+}$ *such that* $\lim_{h\to 0} \gamma(h) = \mathbf{0}$ *and*

$$| u_h^{(r,m)} - v(t^{(r)}, x^{(m)}) | \le \gamma(h) \quad \text{on} \quad E_h^{+}. \tag{24}$$

Proof. Denote by $\tilde{v}_h = \{\tilde{v}_{h.\mu}\}$ the restriction of v to the set E_h^{+}. Let $\Gamma_h^{(r,m)} = \{\Gamma_{h.\mu}^{(r,m)}\}$ be defined on E_h' by the relation

$$\Gamma_h^{(r,m)} = \delta_0 \tilde{v}_h^{(r,m)} - F_h[\tilde{v}_h]^{(r,m)}.$$

It follows from the regularity of v and from Assumption H[σ] that there is γ_Γ : $\Delta \to l^{\infty}_{+}$ such that

$$| \Gamma_h^{(r,m)} | \le \gamma_\Gamma(h) \;\; \text{on} \;\; E_h'$$

and $\lim_{h\to 0} \gamma_\Gamma(h) = \mathbf{0}$. For $(t^{(r)}, x^{(m)}) \in \partial_0 E_h$ and $s \in S^{(m)}$ we define

$$\tau_h^{(r,m,s)} = \tilde{v}_h^{(r,m+s)} - \tilde{v}_h^{(r,m-s)} - g_h[\tilde{v}_h]^{(r,m,s)}.$$

Using the Taylor formula for v with the third order derivatives with respect to x and by the relation

$$s_j \partial_{x_j} v(t^{(r)}, x^{(m)}) = s_j^2 \Big(\psi_j(t^{(r)}, x^{(m)}) - \beta_j(t^{(r)}, x^{(m)}) * v(t^{(r)}, x^{(m)}) \Big), \;\; 1 \le j \le n,$$

we obtain

$$|\tau_h^{(r,m,s)}| \le \hat{c}\, \|h'\|^3 \quad \text{where} \quad \hat{c} = \frac{1}{3} c_1 + B * c_0. \tag{25}$$

Let the function $\varepsilon_h : E_h^{+} \to l^{\infty}$, $\varepsilon_h = \{\varepsilon_{h.\mu}\}$, be given by $\varepsilon_h = u_h - \tilde{v}_h$. Then ε_h satisfies the difference functional system

$$\delta_0 \varepsilon_{h.\mu}^{(r,m)} = \sum_{i,j=1}^{n} \varrho_{ij}^{(\mu)}(\, t^{(r)}, x^{(m)}, \mathcal{T}_h u_h \,)\, \delta_{ij} \varepsilon_{h.\mu}^{(r,m)} +$$

$$+ \sum_{i=1}^{n} \partial_{q_i} f^{(\mu)}(\, P_\mu^{(r,m)}[u_h, \tilde{v}_h] \,)\, \delta_i \varepsilon_{h.\mu}^{(r,m)} + \Lambda_{h.\mu}^{(r,m)} \tag{26}$$

where $(t^{(r)}, x^{(m)}) \in E_h'$, $\mu \in \mathbf{N}$,

$$\Lambda_{h.\mu}^{(r,m)} = \sum_{i,j=1}^{n} \Big(\varrho_{ij}^{(\mu)}(\, t^{(r)}, x^{(m)}, \mathcal{T}_h u_h \,) - \varrho_{ij}^{(\mu)}(\, t^{(r)}, x^{(m)}, \mathcal{T}_h \tilde{v}_h \,) \Big)\, \delta_{ij} \tilde{v}_{h.\mu}^{(r,m)} +$$

$$+ f^{(\mu)}(\, t^{(r)}, x^{(m)}, \mathcal{T}_h u_h,\, \delta \tilde{v}_{h.\mu}^{(r,m)} \,) - f^{(\mu)}(\, t^{(r)}, x^{(m)}, \mathcal{T}_h \tilde{v}_h,\, \delta \tilde{v}_{h.\mu}^{(r,m)} \,) - \Gamma_{h.\mu}^{(r,m)}$$

and

$$P_\mu^{(r,m)}[u_h, \tilde{v}_h] = (\, t^{(r)}, x^{(m)}, \mathcal{T}_h u_h, \delta \tilde{v}_{h.\mu}^{(r,m)} + \xi(\delta u_{h.\mu}^{(r,m)} - \delta \tilde{v}_{h.\mu}^{(r,m)}) \,)$$

with some $\xi \in (0,1)$. Define $\eta_h^{(r)} = \{\eta_{h.\mu}^{(r)}\}$, $0 \le r \le N_0$, by

$$\eta_{h.\mu}^{(r)} = \max\left\{ |\varepsilon_{h.\mu}^{(\nu,m)}| : (t^{(\nu)}, x^{(m)}) \in E_h^+,\ \nu \le r \right\},\quad \mu \in \mathbf{N}.$$

Observe that for $\Lambda_h^{(r,m)} = \{\Lambda_{h.\mu}^{(r,m)}\}$ we obtain the estimate

$$|\Lambda_h^{(r,m)}| \le (\mathbf{1} + c_0) * \sigma(\, t^{(r)}, \eta_h^{(r)}\,) + \gamma_\Gamma(h).$$

We can rewrite (26) into the form which is analogous to (22) and we conclude that

$$|\varepsilon_h^{(r+1,m)}| \le \eta_h^{(r)} + h_0(\mathbf{1} + c_0) * \sigma(\, t^{(r)}, \eta_h^{(r)}\,) + h_0\gamma_\Gamma(h) \tag{27}$$

where $(t^{(r+1)}, x^{(m)}) \in E_h$.

Now we take $(t^{(r+1)}, x^{(m)}) \in \partial_0 E_h$, $s \in S^{(m)}$. Then $(t^{(r+1)}, x^{(m+s)}) \in \partial_0^+ E_h$ and

$$\varepsilon_h^{(r+1,m+s)} * \Big(\mathbf{1} + \sum_{j=1}^n s_j^2 h_j \beta_j(t^{(r+1)}, x^{(m)})\Big) =$$

$$= \varepsilon_h^{(r+1,m-s)} * \Big(\mathbf{1} - \sum_{j=1}^n s_j^2 h_j \beta_j(t^{(r+1)}, x^{(m)})\Big) - \tau_h^{(r+1,m,s)}.$$

Thus

$$|\varepsilon_h^{(r+1,m+s)}| \le |\varepsilon_h^{(r+1,m-s)}| + |\tau_h^{(r+1,m,s)}|.$$

It follows from (25), (27) and from the assumption 2) of $\mathrm{H}_1[\,\Delta\,]$ that

$$|\varepsilon_h^{(r+1,m+s)}| \le \eta_h^{(r)} + h_0(\mathbf{1} + c_0) * \sigma(\, t^{(r)}, \eta_h^{(r)}\,) + h_0\overset{+}{\gamma}_\Gamma(h)$$

where $\overset{+}{\gamma}_\Gamma(h) = \gamma_\Gamma(h) + \hat{c}\,\tilde{C}\sqrt{\tilde{C}h_0}$. Thus the following recursive inequality

$$\eta_h^{(r+1)} \le \eta_h^{(r)} + h_0(\mathbf{1} + c_0) * \sigma(\, t^{(r)}, \eta_h^{(r)}\,) + h_0\overset{+}{\gamma}_\Gamma(h) \tag{28}$$

holds with $0 \le r < N_0$. Let ω_h be the maximal solution of the problem

$$\omega'(t) = (\mathbf{1} + c_0) * \sigma(\, t, \omega(t)\,) + \overset{+}{\gamma}_\Gamma(h), \quad \omega(0) = \overset{+}{\gamma}_\varphi(h),$$

where $\overset{+}{\gamma}_\varphi(h) = \gamma_\varphi(h) + \hat{c}\,\|h'\|^3$. The solution ω_h is defined on $[0,a]$ and $\lim_{h\to 0} \omega_h(t) = \mathbf{0}$ uniformly on $[0,a]$. Since

$$\omega_h(t^{(r+1)}) \ge \omega_h(t^{(r)}) + h_0\,(\mathbf{1} + c_0) * \sigma(\, t^{(r)}, \omega_h(t^{(r)})\,) + h_0\overset{+}{\gamma}_\Gamma(h),\quad 0 \le r < N_0,$$

and $\eta_h^{(0)} \le \overset{+}{\gamma}_\varphi(h)$, we have

$$\eta_h^{(r)} \le \omega_h(t^{(r)}),\quad 0 \le r \le N_0.$$

We obtain the assertion (24) with $\gamma(h) = \omega_h(a)$.

4 Finite systems of difference equations

The main task in investigations presented in this part of the paper is to find a finite difference scheme corresponding to the original infinite problem (1)-(4). We will apply the truncation method.

Fix $k \in \mathbf{N}$. Let $\tilde{\varphi} \in C(E, l^\infty)$, $\tilde{\varphi} = \{\tilde{\varphi}_\mu\}$, be such that $\tilde{\varphi} = \varphi$ on E_0. For $w : E \to l^\infty$, $w = \{w_\mu\}$, we put

$$[w]_{k.\tilde{\varphi}} = \{\bar{w}_\mu\} \text{ where } \bar{w}_\mu = w_\mu \text{ for } 1 \le \mu \le k \text{ and } \bar{w}_\mu = \tilde{\varphi}_\mu \text{ for } \mu > k.$$

If $D \subset E$ and $w : D \to l^\infty$, $w = \{w_\mu\}$, then the symbol $w^{[k]}$ denotes the function $w^{[k]} : D \to R^k$ given by $w^{[k]} = (w_1, \ldots, w_k)$. We will treat an element $p \in R^k$, $p = (p_1, \ldots, p_k)$, also as the sequence $p = \{p_\mu\}$ with $p_\mu = 0$ for $\mu > k$. Write

$$F^{[k]}[z] = (\, F_1^{[k]}[z], \ldots, F_k^{[k]}[z] \,),$$

$$F_\mu^{[k]}[z](t,x) = \sum_{i,j=1}^{n} \varrho_{ij}^{(\mu)}(\, t, x, [z]_{k.\tilde{\varphi}} \,)\, \partial_{x_i x_j} z_\mu(t,x) + f^{(\mu)}(\, t, x, [z]_{k.\tilde{\varphi}}, \partial_x z_\mu(t,x) \,),$$

where $z : E \to R^k$, $z = (z_1, \ldots, z_k)$, $1 \le \mu \le k$.

Consider the finite differential functional system

$$\partial_t z(t,x) = F^{[k]}[z](t,x) \tag{29}$$

with the initial boundary conditions

$$z(t,x) = \varphi^{[k]}(t,x), \quad (t,x) \in E_0, \tag{30}$$

$$\beta^{[k]}(t,x) * z(t,x) + \partial_{x_j} z(t,x) = \psi^{[k]}(t,x), \quad (t,x) \in \partial_{j.+}E, \tag{31}$$

$$\beta^{[k]}(t,x) * z(t,x) - \partial_{x_j} z(t,x) = \psi^{[k]}(t,x), \quad (t,x) \in \partial_{j.-}E, \tag{32}$$

where $1 \le j \le n$.

To estimate the difference between the solution of infinite problem (1)-(4) and the solution of truncated problem (29)-(32) we formulate additional assumptions.

Assumption H[σ, φ]. The functions $\varrho : \Xi \to \mathcal{M}^\infty_{n \times n}$, $f : \Sigma \to l^\infty$, $\beta : \partial_0 E \to \mathcal{R}^\infty_n$ satisfy Assumption H[σ] and the function $\varphi \in C(E_0, l^\infty)$ is such that there exists $\tilde{\varphi} \in C(E, l^\infty)$, $\tilde{\varphi} = \{\tilde{\varphi}_\mu\}$, with the properties:

1) $\tilde{\varphi}(t,x) = \varphi(t,x)$ for $(t,x) \in E_0$ and $|\tilde{\varphi}|_a \le \tilde{A}$ with $\tilde{A} = \frac{1}{2}A$,

2) the function $\tilde{\varphi}(\cdot, x) : [0,a] \to l^\infty$, $x \in [-b,b]$, is of class C^1, the function $\tilde{\varphi}(t, \cdot) : [-b,b] \to l^\infty$, $t \in [0,a]$, is of class C^2 and there is $d \in l^\infty_+$, $d = \{d_\mu\}$, such that
$$|\partial_{x_i x_j} \tilde{\varphi}(t,x)| \le d, \quad (t,x) \in E, \quad 1 \le i,j \le n,$$

3) there is $c \in l^\infty_0$, $c = \{c_\mu\}$, such that
$$|\, \partial_t \tilde{\varphi}(t,x) - F[\tilde{\varphi}](t,x) \,| \le c \text{ for } (t,x) \in E, \tag{33}$$
and the maximal solution $\tilde{\omega} = \{\tilde{\omega}_\mu\}$ of the problem
$$\omega'(t) = (\mathbf{1} + d) * \sigma(\, t, \omega(t) \,) + c, \quad \omega(0) = \mathbf{0}, \tag{34}$$
exists on $[0,a]$ and $\lim_{\mu \to \infty} \tilde{\omega}_\mu(t) = 0$ uniformly on $[0,a]$,

4) the estimates

$$|\,\beta_j(t,x) * \tilde{\varphi}(t,x) + \partial_{x_j}\tilde{\varphi}(t,x) - \psi_j(t,x)\,| \le \tilde{b} * \tilde{\omega}(t), \;\; (t,x) \in \partial_{j.+}E,$$

$$|\,\beta_j(t,x) * \tilde{\varphi}(t,x) - \partial_{x_j}\tilde{\varphi}(t,x) - \psi_j(t,x)\,| \le \tilde{b} * \tilde{\omega}(t), \;\; (t,x) \in \partial_{j.-}E,$$

are satisfied for $1 \le j \le n$.

Remark 3. *Let $a_{\mu j} \in R_+$, $\mu, j \in \mathbf{N}$, be such that the series $S_\mu = \sum_{j=1}^{\infty} a_{\mu j}$, $\mu \in \mathbf{N}$, are convergent and the sequence $S = \{S_\mu\}$ tends to zero. Fix the sequence $\tilde{p} \in l^\infty_+$, $\tilde{p} = \{\tilde{p}_\mu\}$, such that $\tilde{p}_\mu > 0$ for $\mu \in \mathbf{N}$. Put $I[\tilde{p}] = \Big\{p \in l^\infty_+ \,:\, p \le \tilde{p}\Big\}$. Then the function $\sigma : [0,a] \times l^\infty_+ \to l^\infty_+$, $\sigma = \{\sigma_\mu\}$, given by*

$$\sigma_\mu(t,p) = \sum_{j=1}^{\infty} a_{\mu j}\, p_j, \;\; p \in I[\tilde{p}], \quad \text{and} \quad \sigma_\mu(t,p) = \sum_{j=1}^{\infty} a_{\mu j}\, \tilde{p}_j, \;\; p \in l^\infty_+ \setminus I[\tilde{p}],$$

where $t \in [0,a]$, $\mu \in \mathbf{N}$, satisfies the required conditions.

Lemma 2. *If Assumption $H[\,\sigma, \varphi\,]$ is satisfied and the function $v : E \to l^\infty$ is $\mathcal{P}$-solution of (1)-(4) then*

$$|v(t,x) - \tilde{\varphi}(t,x)| \le \tilde{\omega}(t), \;\; (t,x) \in E,$$

where $\tilde{\omega}$ is the maximal solution of the problem (34).

Proof. Define $\tilde{v} : E \to l^\infty$, $\tilde{v} = \{\tilde{v}_\mu\}$, by $\tilde{v} = v - \tilde{\varphi}$. Let the function $G = \{G_\mu\}$ be defined on $E \times C_{\tilde{A}}(E, l^\infty) \times R^n \times M_{n\times n}$ in the following way

$$G_\mu(t,x,w,q,r) = \sum_{i,j=1}^{n} \varrho^{(\mu)}_{ij}(t,x,w+\tilde{\varphi})\Big(r_{ij} + \partial_{x_i x_j}\tilde{\varphi}_\mu(t,x)\Big) +$$

$$+ f^{(\mu)}(\,t,x,w+\tilde{\varphi}, q + \partial_x \tilde{\varphi}_\mu(t,x)\,) - \partial_t \tilde{\varphi}_\mu(t,x)$$

where $\mu \in \mathbf{N}$ and $r = [r_{ij}]_{i,j=1,\ldots,n}$. Consider the infinite differential functional system

$$\partial_t z_\mu(t,x) = G_\mu(t,x,z,\partial_x z_\mu(t,x), \partial_{xx} z_\mu(t,x)), \;\; \mu \in \mathbf{N}, \tag{35}$$

where $z = \{z_\mu\}$, $\partial_{xx} z_\mu = [\partial_{x_i x_j} z_\mu]_{i,j=1,\ldots,n}$. It follows that the function $\tilde{v}$ is a parabolic solution of (35) such that $\tilde{v}(t,x) = 0$ on E_0 and

$$|\,\beta_j(t,x) * \tilde{v}(t,x) + \partial_{x_j}\tilde{v}(t,x)\,| \le \tilde{b} * \tilde{\omega}(t) \;\; \text{on} \;\; \partial_{j.+}E,$$

$$|\,\beta_j(t,x) * \tilde{v}(t,x) - \partial_{x_j}\tilde{v}(t,x)\,| \le \tilde{b} * \tilde{\omega}(t) \;\; \text{on} \;\; \partial_{j.-}E$$

where $1 \le j \le n$. The following estimates

$$|G_\mu(t,x,w,0,0)| \le \sum_{i,j=1}^{n} |\varrho^{(\mu)}_{ij}(\,t,x,w+\tilde{\varphi}\,) - \varrho^{(\mu)}_{ij}(\,t,x,\tilde{\varphi}\,)| \cdot |\partial_{x_i x_j}\tilde{\varphi}_\mu(t,x)| +$$

$$+ |f^{(\mu)}(\,t,x,w+\tilde{\varphi}, \partial_x\tilde{\varphi}_\mu(t,x)\,) - f^{(\mu)}(\,t,x,\tilde{\varphi}, \partial_x\tilde{\varphi}_\mu(t,x)\,)| +$$

$$+ |F^{(\mu)}[\tilde{\varphi}] - \partial_t\tilde{\varphi}_\mu(t,x)| \le (1+d_\mu)\sigma_\mu(t, |w|_t) + c_\mu$$

are satisfied for $(t,x) \in E$, $w \in C_{\tilde{A}}(E, l^\infty)$ and $\mu \in \mathbf{N}$. It follows from the comparison theorem (see [3]) that

$$|\tilde{v}(t,x)| \leq \tilde{\omega}(t) \text{ on } E.$$

The proof is complete.

For a function $w \in C(E, R^k)$, $w = (w_1, \dots, w_k)$, we write $|w|_t = (\,|w_1|_t, \dots, |w_k|_t\,)$, $t \in [0,a]$. Put $C_{\tilde{A}}(E, R^k) = \{w \in C(E,R^k) : |w|_a \leq \tilde{A}\}$ where $\tilde{A} \in l^\infty$ is given in Assumption H[σ, φ].

If $v : E \to l^\infty$ is $\mathcal{P}$-solution of (1)-(4) and there is $c_0 \in l^\infty$, $c_0 = \{c_{0.\mu}\}$, such that $|\partial_{x_i x_j} v(t,x)| \leq c_0$ on E, $1 \leq i,j \leq n$, then v is said to be $\mathcal{P}[c_0]$-*solution*.

Lemma 3. *Suppose that Assumption H[σ, φ] is satisfied and*

1) *the function $v : E \to l^\infty$, $v = \{v_\mu\}$, is $\mathcal{P}[c_0]$-solution of (1)-(4),*
2) *for each $k \in \mathbf{N}$ the function $u^{[k]} : E \to R^k$, $u^{[k]} = (u_1^{[k]}, \dots, u_k^{[k]})$, is the parabolic solution of (29)-(32).*

Then there exists $\omega^{[k]} \in C([0,a], R^k_+)$ such that

$$|v^{[k]}(t,x) - u^{[k]}(t,x)| \leq \omega^{[k]}(t), \quad (t,x) \in E,$$

and $\lim_{k\to\infty} \|\omega^{[k]}(t)\|_\infty = 0$ *uniformly on* $[0,a]$.

Proof. Let $k \in \mathbf{N}$ be fixed and let the function $\tilde{v}^{[k]} : E \to R^k$ be given by $\tilde{v}^{[k]} = u^{[k]} - v^{[k]}$. We define the function $H : E \times C_{\tilde{A}}(E, R^k) \times R^n \times M_{n\times n} \to R^k$, $H = (H_1, \dots, H_k)$, as follows:

$$H_\mu(t,x,w,q,r) = \sum_{i,j=1}^{n} \varrho_{ij}^{(\mu)}(\,t,x,[w+v]_{k.\tilde{\varphi}}\,) r_{ij} +$$

$$+ \sum_{i,j=1}^{n} \Big(\varrho_{ij}^{(\mu)}(\,t,x,[w+v]_{k.\tilde{\varphi}}\,) - \varrho_{ij}^{(\mu)}(t,x,v) \Big) \partial_{x_i x_j} v_\mu(t,x) +$$

$$+ f^{(\mu)}(\,t,x,[w+v]_{k.\tilde{\varphi}}, q + \partial_x v_\mu(t,x)\,) - f^{(\mu)}(\,t,x,v,\partial_x v_\mu(t,x)\,).$$

Consider the differential functional system

$$\partial_t z_\mu(t,x) = H_\mu(\,t,x,z,\partial_x z_\mu(t,x), \partial_{xx} z_\mu(t,x)\,), \quad 1 \leq \mu \leq k, \tag{36}$$

where $z = (z_1, \dots, z_k)$, with the homogeneous initial boundary conditions

$$z(t,x) = 0 \text{ on } E_0, \tag{37}$$

$$\beta_j^{[k]}(t,x) * z(t,x) + \partial_{x_j} z(t,x) = 0 \text{ on } \partial_{j.+}E, \tag{38}$$

$$\beta_j^{[k]}(t,x) * z(t,x) - \partial_{x_j} z(t,x) = 0 \text{ on } \partial_{j.-}E \tag{39}$$

where $1 \le j \le n$. The function $\tilde{v}^{[k]}$ is the parabolic solution of the problem (36)-(39). We use the comparison theorem for systems of differential functional equations to estimate the values of $\tilde{v}^{[k]}$.

We need the following additional notation. For $p \in l^\infty$, $p = \{p_\mu\}$, we denote by $\mathbf{0}_{k.p}$ the sequence $\{\bar{p}_\mu\}$ such that $\bar{p}_\mu = 0$ for $1 \le \mu \le k$ and $\bar{p}_\mu = p_\mu$ for $\mu > k$. Let $\sigma^{[k]} : [0,a] \times R^k_+ \to R^k_+$, $\sigma^{[k]} = (\sigma^{[k]}_1, \ldots, \sigma^{[k]}_k)$, be given by

$$\sigma^{[k]}_\mu(t,p) = \sigma_\mu(\,t,p\,),\ \ 1 \le \mu \le k. \tag{40}$$

We observe that the terms

$$\sum_{i,j=1}^{n} |\varrho^{(\mu)}_{ij}(\,t,x,[w+v]_{k.\tilde{\varphi}}\,) - \varrho^{(\mu)}_{ij}(\,t,x,v\,)|,$$

$$|f^{(\mu)}(\,t,x,[w+v]_{k.\tilde{\varphi}}, \partial_x v_\mu(t,x)\,) - f^{(\mu)}(\,t,x,v,\partial_x v_\mu(t,x)\,)|,$$

with $(t,x) \in E$, $w \in C_{\tilde{A}}(E,R^k)$, $1 \le \mu \le k$, are bounded from above by

$$\sigma^{[k]}_\mu(\,t,|w|_t\,) + \sigma_\mu(\,t,\mathbf{0}_{k.\tilde{\omega}(t)}\,)$$

where $\tilde{\omega}$ is the maximal solution of (34). Then

$$|H_\mu(t,x,w,0,0)| \le (1+c_{0.\mu})\sigma^{[k]}_\mu(\,t,|w|_t\,) + \alpha^{[k]}_\mu$$

with $\alpha^{[k]}_\mu = (1+c_{0.\mu})\sigma_\mu(\,a,\mathbf{0}_{k.\tilde{\omega}(a)}\,)$, $1 \le \mu \le k$. Write $\alpha^{[k]} = (\alpha^{[k]}_1, \ldots, \alpha^{[k]}_k)$. It follows that

$$|\tilde{v}^{[k]}(t,x)| \le \omega^{[k]}(t) \ \text{ on } \ E,$$

where $\omega^{[k]}$ is the maximal solution of the problem

$$\omega'(t) = (\mathbf{1}+c_0) * \sigma^{[k]}(\,t,\omega(t)\,) + \alpha^{[k]},\ \ \omega(0) = 0. \tag{41}$$

Since $\lim\limits_{k\to\infty} \|\alpha^{[k]}\|_\infty = 0$, we have that $\lim\limits_{k\to\infty} \|\omega^{[k]}(t)\|_\infty = 0$ uniformly on $[0,a]$. This finishes the proof of Lemma 3.

Now we construct the difference problem related to (29)-(32). For $z : E^+_h \to R^k$, $z = (z_1, \ldots, z_k)$, we write

$$F^{[k]}_h[z] = (\,F^{[k]}_{h.1}[z], \ldots, F^{[k]}_{h.k}[z]\,),$$

$$F^{[k]}_{h.\mu}[z]^{(r,m)} = \sum_{i,j=1}^{n} \varrho^{(\mu)}_{ij}(\,t^{(r)},x^{(m)},[\mathcal{T}_h z]_{k.\tilde{\varphi}}\,)\,\delta_{ij} z^{(r,m)}_\mu + f^{(\mu)}(\,t^{(r)},x^{(m)},[\mathcal{T}_h z]_{k.\tilde{\varphi}},\delta z^{(r,m)}_\mu\,)$$

on E'_h, $1 \le \mu \le k$. Difference operators δ_0, $\delta = (\delta_1, \ldots, \delta_n)$ and δ_{ii}, $1 \le i \le n$, are given by (5), (6) and (8). The difference expressions $\delta_{ij} z_\mu$ for $(i,j) \in J$ are defined by (9) and (10) with $(\,t^{(r)},x^{(m)},[\mathcal{T}_h z]_{k.\tilde{\varphi}}\,)$ instead of $(\,t^{(r)},x^{(m)},\mathcal{T}_h z\,)$.

For $(t^{(r)},x^{(m)}) \in \partial_0 E_h$, $s \in S^{(m)}$ we put

$$g^{[k]}_h[z]^{(r,m,s)} = 2\sum_{j=1}^{n} s^2_j h_j \psi^{[k]}_j(t^{(r)},x^{(m)}) - (z^{(r,m+s)} + z^{(r,m-s)}) * \sum_{j=1}^{n} s^2_j h_j \beta^{[k]}_j(t^{(r)},x^{(m)}).$$

Consider the difference functional problem

$$\delta_0 z^{(r,m)} = F_h^{[k]}[z]^{(r,m)} \text{ on } E'_h, \tag{42}$$

$$z^{(r,m)} = (\varphi_h^{[k]})^{(r,m)} \text{ on } E_{0.h}, \tag{43}$$

$$z^{(r,m+s)} - z^{(r,m-s)} = g_h^{[k]}[z]^{(r,m,s)} \text{ on } \partial_0 E_h, \ s \in S^{(m)}. \tag{44}$$

We formulate the main theorem in this part of the paper.

Theorem 2. *Suppose that Assumptions $H[\,\sigma,\varphi\,]$, $H[\,\mathcal{T}_h\,]$, $H_1[\,\Delta\,]$ are satisfied, the function $v : E \to l^\infty$ is $\mathcal{P}[c_0]$-solution of (1)-(4) and for each $k \in \mathbf{N}$*

1) *the function $u^{[k]} : E^+ \to R^k$ is such that $u^{[k]}(\cdot,x) : [0,a] \to R^k$, $x \in (-b^+, b^+)$, is of class C^1, $u^{[k]}(t,\cdot) : (-b^+, b^+) \to R^k$, $t \in [0,a]$, is of class C^3 and there are $c_0^{[k]}, c_1^{[k]} \in R_+^k$ such that*

$$|\partial_{x_i x_j} u^{[k]}(t,x)| \le c_0^{[k]}, \ \ |\partial_{x_i x_j x_k} u^{[k]}(t,x)| \le c_1^{[k]}, \ \ (t,x) \in E^+, \ \ 1 \le i,j,k \le n,$$

and $u^{[k]}$ is the parabolic solution of (29)-(32) on E,

2) *the function $u_h^{[k]} : E_h^+ \to R^k$ is the solution of (42)-(44),*

3) *there is $\gamma_\varphi^{[k]} : \Delta \to R_+^k$ such that $\lim_{h\to 0} \gamma_\varphi^{[k]}(h) = 0$ and*

$$|(\varphi_h^{[k]})^{(r,m)} - \varphi^{[k]}(t^{(r)}, x^{(m)})| \le \gamma_\varphi^{[k]}(h) \text{ on } E_{0.h}.$$

Then there exist $\gamma^{[k]} : \Delta \to R_+^k$ and $\varepsilon^{[k]} \in R_+^k$ such that

$$|(u_h^{[k]})^{(r,m)} - v^{[k]}(t^{(r)}, x^{(m)})| \le \gamma^{[k]}(h) + \varepsilon^{[k]} \text{ on } E_h \tag{45}$$

and $\lim_{h\to 0} \gamma^{[k]}(h) = 0$, $\lim_{k\to\infty} \|\varepsilon^{[k]}\|_\infty = 0$.

Proof. Let us fix $k \in \mathbf{N}$. Using the methods from the proof of Theorem 1 we can prove that

$$|(u_h^{[k]})^{(r,m)} - u^{[k]}(t^{(r)}, x^{(m)})| \le \hat{\omega}_h^{[k]}(t^{(r)}) \text{ on } E_h^+$$

where $\hat{\omega}_h^{[k]}$ is the maximal solution of the problem

$$\omega'(t) = (\mathbf{1} + c_0^{[k]}) * \sigma^{[k]}(\,t, \omega(t)\,) + \tilde{\gamma}^{[k]}(h), \ \ \omega(0) = \gamma_0^{[k]}(h),$$

with $\tilde{\gamma}^{[k]}, \gamma_0^{[k]} : \Delta \to R_+^k$ satisfying condition $\lim_{h\to 0} \tilde{\gamma}^{[k]}(h) = \lim_{h\to 0} \gamma_0^{[k]}(h) = 0$ and with $\sigma^{[k]}$ given by (40). It follows from Lemma 3 that

$$|u^{[k]}(t^{(r)}, x^{(m)}) - v^{[k]}(t^{(r)}, x^{(m)})| \le \omega^{[k]}(t^{(r)}) \text{ on } E_h$$

where $\omega^{[k]}$ is the maximal solution of (41). Thus we obtain the assertion (45) with $\gamma^{[k]}(h) = \omega_h^{[k]}(a)$ and $\varepsilon^{[k]} = \omega^{[k]}(a)$.

5 Numerical examples

We take $n = 2$ and $E = [0, a] \times [-1, 1]^2$ with $a = 0.25$.

Example 1. Suppose that $\tilde{f} : E \times C(E, R) \to R$ is defined by

$$\tilde{f}(t, x, y, z) = x^2 y^2 (x^2 - 22t) - 4tx^4 (1 + \frac{4}{3} tx^4 y) - 5x^2 \int_{-1}^{1} z(t, \tau, y) d\tau.$$

Consider the differential integral equation

$$\partial_t z(t, x, y) =$$

$$= 2\partial_{xx} z(t, x, y) + x\partial_{xy} z(t, x, y) \int_{-1}^{1} z(t, x, \tau) d\tau + 2\partial_{yy} z(t, x, y) + \tilde{f}(t, x, y, z) \text{ on } E,$$

with the initial boundary conditions

$$z(0, x, y) = 0, \ \ (x, y) \in [-1, 1]^2,$$

$$z(t, s, y) + s\partial_x z(t, s, y) = 5ty^2, \ \ (t, y) \in [0, a] \times [-1, 1], \ \ s = 1 \text{ or } s = -1,$$

$$z(t, x, s) + s\partial_y z(t, x, s) = 3tx^4, \ \ (t, x) \in [0, a] \times [-1, 1], \ \ s = 1 \text{ or } s = -1.$$

The exact solution of the above problem is known. It is $v(t, x, y) = tx^4 y^2$, $(t, x, y) \in E$. To approximate v we use the solution u_h of problem (11)-(13) reduced to the scalar case with $\varphi_h = \varphi$. We apply the interpolating operator $\mathcal{T}_h : \mathcal{F}(E_h^+, R) \to C(E, R)$ given in [6]. Then for $z : E_h^+ \to R$ the function $\mathcal{T}_h z$ is obtained by interpolation by splines of z and the integrals in (11) are calculated as follows:

$$\int_{-1}^{1} (\mathcal{T}_h z)(t^{(r)}, \tau, y^{(m_2)}) d\tau = \frac{h_1}{2} (z^{(r, -N_1, m_2)} + z^{(r, N_1, m_2)}) + h_1 \sum_{j=-N_1+1}^{N_1-1} z^{(r, j, m_2)},$$

$$\int_{-1}^{1} (\mathcal{T}_h z)(t^{(r)}, x^{(m_1)}, \tau) d\tau = \frac{h_2}{2} (z^{(r, m_1, -N_2)} + z^{(r, m_1, N_2)}) + h_2 \sum_{j=-N_2+1}^{N_2-1} z^{(r, m_1, j)}$$

where $0 \le r < N_0$, $-N_1 \le m_1 \le N_1$, $-N_2 \le m_2 \le N_2$. Maximal error values $e_h = |\, u_h - v \,|_{(N_0)}$ for several steps $h = (h_0, h_1, h_2)$ are listed in the following table.

h_0	$h_1 = h_2$	e_h	$-\log_2 e_h$
2^{-11}	2^{-4}	$1.474708 \cdot 10^{-3}$	9.405354
2^{-13}	2^{-5}	$3.636181 \cdot 10^{-4}$	11.425288
2^{-15}	2^{-6}	$9.010732 \cdot 10^{-5}$	13.437997
2^{-17}	2^{-7}	$2.245604 \cdot 10^{-5}$	15.442536

All the assumptions of Theorem 1 are satisfied and calculated error estimates e_h are consistent with Theorem 1.

Example 2. Let $g_\mu : E \times C(E, l^\infty) \to R$, $\mu \in \mathbf{N}$, be given by

$$g_\mu(t, x, y, z) = \mu^{-1} g(t, x, y) z_\mu(\beta(t, x, y)) +$$

$$+\mu^{-1}g(t,y,x)z_{\mu+1}(t,x,y)-\mu^{-1}(x^2-1)^2(y^2-1)^2\exp[\,t(x^2-1)^2(y^2-1)^2\,]$$

where $z=\{z_\mu\}$, $\beta(t,x,y)=(t,y,x)$ and

$$g(t,x,y)=4t(3x^2-1)(y^2-1)^2+16t^2x^2(x^2-1)^2(y^2-1)^4.$$

Consider the infinite functional differential problem

$$\partial_t z_\mu(t,x,y)=\mu^{-1}\partial_{xx}z_\mu(t,x,y)+(\mu+1)^{-1}\partial_{yy}z_\mu(t,x,y)-g_\mu(t,x,y,z)\ \text{ on } E,$$

$$z_\mu(0,x,y)=\mu^{-1},\ \ (x,y)\in[-1,1]^2,$$

$$z_\mu(t,s,y)+s\partial_x z_\mu(t,s,y)=\mu^{-1},\ \ (t,y)\in[0,a]\times[-1,1],\ \ s=1\ \text{ or }\ s=-1,$$

$$z_\mu(t,x,s)+s\partial_y z_\mu(t,x,s)=\mu^{-1},\ \ (t,x)\in[0,a]\times[-1,1],\ \ s=1\ \text{ or }\ s=-1,$$

where $\mu\in\mathbf{N}$. The exact solution is $v_\mu(t,x,y)=\mu^{-1}\exp[\,t(x^2-1)^2(y^2-1)^2\,]$, $(t,x,y)\in E$, $\mu\in\mathbf{N}$. We take $\tilde\varphi_\mu(t,x,y)=\mu^{-1}$ on E, $\mu\in\mathbf{N}$. We use the difference method (42)-(44) with $\varphi_h=\varphi$. Let $u_h^{[k]}$ be its solution. The following table shows maximal error values $\|\,e_h^{[k]}\,\|_\infty$ where $e_h^{[k]}=|\,u_h^{[k]}-v^{[k]}\,|_{(N_0)}$ for several steps $h=(h_0,h_1,h_2)$ and system sizes k.

k	h_0	$h_1=h_2$	$\|\,e_h^{[k]}\,\|_\infty$	$-\log_2\|\,e_h^{[k]}\,\|_\infty$
2	2^{-4}	2^{-1}	$7.250851\cdot10^{-2}$	3.785706
2	2^{-6}	2^{-2}	$2.536511\cdot10^{-2}$	5.301011
2	2^{-8}	2^{-3}	$6.994665\cdot10^{-3}$	7.159529
2	2^{-10}	2^{-4}	$2.619863\cdot10^{-3}$	8.576293
4	2^{-10}	2^{-4}	$1.793981\cdot10^{-3}$	9.122621
4	2^{-12}	2^{-5}	$9.194016\cdot10^{-4}$	10.087017
8	2^{-12}	2^{-5}	$4.511476\cdot10^{-4}$	11.114113
8	2^{-14}	2^{-6}	$2.721995\cdot10^{-4}$	11.843047
16	2^{-14}	2^{-6}	$1.133084\cdot10^{-4}$	13.107457
16	2^{-16}	2^{-7}	$6.374717\cdot10^{-5}$	13.937279

The results shown in the table are consistent with Theorem 2.

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