# A Hasse diagram for rational toral ranks

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**Abstract**

Let $X$ be a simply connected CW complex with finite rational cohomology. For the finite quotient set of rationalized orbit spaces of $X$ obtained by almost free toral actions, $\mathcal{T}_0(X) = \{[Y_i]\}$, induced by an equivalence relation based on rational toral ranks, we order as $[Y_i] < [Y_j]$ if there is a rationalized Borel fibration $Y_i \to Y_j \to BT^n_{\mathbb{Q}}$ for some $n > 0$. It presents a variation of almost free toral actions on $X$. We consider about the Hasse diagram $\mathcal{H}(X)$ of the poset $\mathcal{T}_0(X)$, which makes a based graph $G\mathcal{H}(X)$, with some examples. Finally we will try to regard $G\mathcal{H}(X)$ as the 1-skeleton of a finite CW complex $\mathcal{T}(X)$ with base point $X_{\mathbb{Q}}$.

## 1 Introduction

Let $r_0(X)$ be the *rational toral rank* of a simply connected CW complex $X$ of $\dim H^*(X;\mathbb{Q}) < \infty$, i.e., the largest integer $r$ such that an $r$-torus $T^r = S^1 \times \cdots \times S^1$($r$-factors) can act continuously on a CW-complex $Y$ in the rational homotopy type of $X$ with all its isotropy subgroups finite (almost free action) [2], [4], [6]. Recall that the rationalized Borel space of almost free toral action $(ET^n \times^{\mu}_{T^n} Y)_{\mathbb{Q}}$ is homotopy equivalent to the rationalization of the orbit space of $Y$ obtained by the action $\mu$. In a work of V.Puppe (for example see [8]), we can see a Hasse diagram of the cohomology algebras of the fixed point sets of circle actions on $X$, which are correspond to the rationalized Borel spaces, from a point of view of a deformation. We are interested in a rational variation of Borel spaces of toral actions with no-fixed point and the aim of this note is giving a framework for such an approach based on rational toral ranks.

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Due to the rational homotopy theory of D.Sullivan, rationalized fibrations are equivalent to Koszul-Sullivan(KS) extensions. Remark that, when we give certain KS-extensions of the Sullivan minimal model $M(X)$ of $X$ [3], the existences of the free toral actions on complexes in the rational homotopy type of $X$, whose Borel fibrations induce the KS-extensions, are rationally guaranteed by a result of S.Halperin [6, Proposition 4.2] (see Proposition 2.1 below). We denote the homotopy set of rationalized Borel spaces of almost free toral actions $\mu$ on complexes $X_\mu$ in the rational homotopy type of $X$, which are given by certain KS-extensions (see §2), by $\mathcal{X} = \coprod_{n=0}^{r_0(X)} \mathcal{X}_n$ where $\mathcal{X}_n := \{(ET^n \times^{\mu}_{T^n} X_\mu)_{\mathbb{Q}}\}$ for $n > 0$ and $\mathcal{X}_0 := \{X_{\mathbb{Q}}\}$. For two elements $Y_1 := (ET^m \times^{\mu_1}_{T^m} X_1)_{\mathbb{Q}}$ and $Y_2 := (ET^n \times^{\mu_2}_{T^n} X_2)_{\mathbb{Q}}$ of $\mathcal{X}$ for $m < n$, we denote $Y_1 < Y_2$ if there is a rationalized Borel fibration $Y_1 \to Y_2 \to BT^{n-m}_{\mathbb{Q}}$, which satisfies the homotopy commutative diagram

$$\begin{tikzcd}
X_{\mathbb{Q}} \arrow[r, equal] \arrow[d] & X_{\mathbb{Q}} \arrow[d] & \\
Y_1 \arrow[r] \arrow[d] & Y_2 \arrow[r] \arrow[d] & BT^{n-m}_{\mathbb{Q}} \arrow[d, equal] \\
BT^m_{\mathbb{Q}} \arrow[r, "Bi^m_{n\,\mathbb{Q}}"] & BT^n_{\mathbb{Q}} \arrow[r] & BT^{n-m}_{\mathbb{Q}}.
\end{tikzcd}$$

Here we put $Y_1 := X_{\mathbb{Q}}$ if $m = 0$. Then $(\mathcal{X}, <)$ is a strict partially ordered set (poset).

**Definition 1.1.** *We give an equivalence relation of $\mathcal{X}$ by $Y_1 \sim Y_2$ when $Y_1, Y_2 \in \mathcal{X}_n$ for some $n$ and $r_0(Y_1) = r_0(Y_2)$. For the quotient set $\mathcal{T}_0(X) := \mathcal{X}/_\sim = \{P_i\}_i$, we put $P_i < P_j$ if there are elements $Y_i, Y_j \in \mathcal{X}$ such that $[Y_i] = P_i$, $[Y_j] = P_j$ and $Y_i < Y_j$ or if there is an element $P_k \in \mathcal{T}_0(X)$ with $P_i < P_k$ and $P_k < P_j$.*

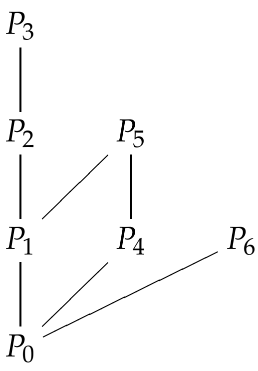
Notice that even if $T^m$ acts almost freely on $X$ and $r_0(X) = n(> m)$, then there does not always exist an almost free action of $T^{n-m}$ on a complex in the rational homotopy type of the Borel space $ET^m \times_{T^m} X$. For example, when $X = S^3 \times S^3 \times S^7$, we obtain $r_0(X) = 3$ by standard $T^3$-action $(s_1, s_2, s_3) \cdot (z_1, z_2, z_3) = (s_1 z_1, s_2 z_2, s_3 z_3)$. But there exists a free $S^1$-action $\mu : S^1 \times Y \to Y$ for a finite complex $Y$ with $Y_{\mathbb{Q}} \simeq X_{\mathbb{Q}}$ and $r_0(ES^1 \times^{\mu}_{S^1} Y) = 0$. It is also rationally given as the total space of a non-trivial fibration with fiber $\mathbb{C}P^3$ and base $S^3 \times S^3$. See Example 3.5 below for detail. Thus we stand on our starting point.

**Claim 1.2.** *The poset $\mathcal{T}_0(X) = (\{P_i\}_i, <)$ is not totally ordered in general.*

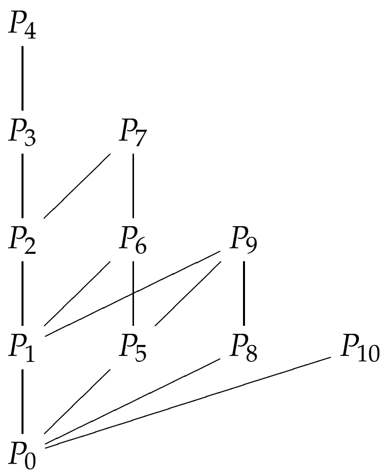
The poset $\mathcal{T}_0(X)$ makes a Hasse diagram of the sets $\{P_i\}_i$. We denote it as $\mathcal{H}(X)$. It is not a numerical but is a graphical (rational) homotopy invariant of spaces. Here we can put $i < j$ if $P_i < P_j$ and fix $P_0 = [X_{\mathbb{Q}}]$, $P_1 = [(ES^1 \times_{S^1} Y)_{\mathbb{Q}}]$ with $r_0(ES^1 \times_{S^1} Y) = r_0(X) - 1, \cdots, P_{r_0(X)} = [(ET^{r_0(X)} \times_{T^{r_0(X)}} Y)_{\mathbb{Q}}]$ with $r_0(ET^{r_0(X)} \times_{T^{r_0(X)}} Y) = 0$ for a complex $Y$ in the rational homotopy of $X$. The subset $\{P_1, .., P_{r_0(X)}\}$ always exists by the restrictions to $i^m_{r_0(X)}(T^m) = \{(s_1, .., s_m, 1, .., 1) | s_i \in S^1\}$ of an almost free $T^{r_0(X)}$-action on $Y$ for $m = 1, .., r_0(X)$. We observe from the above definition

**Lemma 1.3.** *(1) For $Y \in \mathcal{X}$, $r_0(Y) = n$ if and only if $n = max\{k \mid [Y] = P_i < P_{i_1} < \cdots < P_{i_k},\ r_0(P_{i_k}) = 0\}$. Then the path of length $n$, $P_i \to P_{i_1} \to \cdots \to P_{i_n}$, is unique in the graph. In particular, if $Y = X_{\mathbb{Q}}$, $P_i = P_0$, $P_{i_j} = P_j$ for $j = 1, .., n$.*
*(2) For $Y \in \mathcal{X}$, $Y \in \mathcal{X}_n$ if and only if $n = max\{k \mid P_0 < P_{i_1} < \cdots < P_{i_k} = [Y]\}$. Then $n = d(P_0, [Y])$, the distance between $P_0$ and $[Y]$ in the graph.*

Our Hasse diagrams are restricted to certain forms. Of course, $\mathcal{T}_0(X)$ is a finite set. Especially, when $r_0(X) = n$, $n < \sharp\mathcal{T}_0(X) \leq (n + (n-1) + \cdots + 2 + 1) + 1 = (n^2 + n)/2 + 1$ ($0 < \sharp\, \mathcal{X}_m/_\sim\ \leq n - m + 1$ for $m \leq n$). In particular, $\sharp\mathcal{T}_0(X) = r_0(X) + 1$ if and only if $\mathcal{T}_0(X)$ is totally ordered. For example, if $r_0(X) = 3$ and $r_0(X') = 4$ for some spaces $X$ and $X'$, $4 \leq \sharp\mathcal{T}_0(X) \leq 7$ and $5 \leq \sharp\mathcal{T}_0(X') \leq 11$ and $\mathcal{H}(X)$ and $\mathcal{H}(X')$ are certain sub-diagrams (see Remark 3.7 below) of the Hasse diagrams:




and




, respectively. If there exist such spaces, $r_0(P_0) = 3$, $r_0(P_1) = 2$, $r_0(P_2) = r_0(P_4) = 1$, $r_0(P_3) = r_0(P_5) = r_0(P_6) = 0$ in the left hand and $r_0(P_0) = 4$, $r_0(P_1) = 3$, $r_0(P_2) = r_0(P_5) = 2$, $r_0(P_3) = r_0(P_6) = r_0(P_8) = 1$, $r_0(P_4) = r_0(P_7) = r_0(P_9) = r_0(P_{10}) = 0$ in the right hand. Here $r_0(P_i)$ means $r_0(Y)$ for some space $Y$ with $P_i = [Y]$.

We can describe a point $P_i = [Y]$ of $\mathcal{T}_0(X)$ by the double index (lattice point)

$$d.i.(P_i) := (s, t) \quad ; \ s + t \leq r_0(X)$$

when

$$Y \in \mathcal{X}_t \ \text{ and } \ r_0(Y) = r_0(X) - s - t$$

by Definition 1.1. If $P_i \neq P_j$ in $\mathcal{T}_0(X)$, $d.i.(P_i) \neq d.i.(P_j)$. For example, in the above right diagram of $r_0(X) = 4$, we see $d.i.(P_0) = (0,0)$, $d.i.(P_1) = (0,1)$, $d.i.(P_2) = (0,2)$, $d.i.(P_3) = (0,3)$, $d.i.(P_4) = (0,4)$, $d.i.(P_5) = (1,1)$, $d.i.(P_6) = (1,2)$, $d.i.(P_7) = (1,3)$, $d.i.(P_8) = (2,1)$, $d.i.(P_9) = (2,2)$ and $d.i.(P_{10}) = (3,1)$. In general, when $r_0(X) > 1$, if there is a circle action on $X$ that represents $P$ with $d.i.(P) = (r_0(X) - 1, 1)$, then it is a "bad" action in a meaning since the orbit space permits no almost free circle action.

**Claim 1.4.** *(1) If $P_i < P_j$ for $d.i.(P_i) = (s,t)$ and $d.i.(P_j) = (s',t')$, then $s \le s'$ and $t < t'$.*
*(2) If there is a point $P_i$ with $d.i.(P_i) = (s,t)$, then there are points $\{P_j\}$ with double indexes $(s, t+1), .., (s, r_0(X) - s)$, too.*

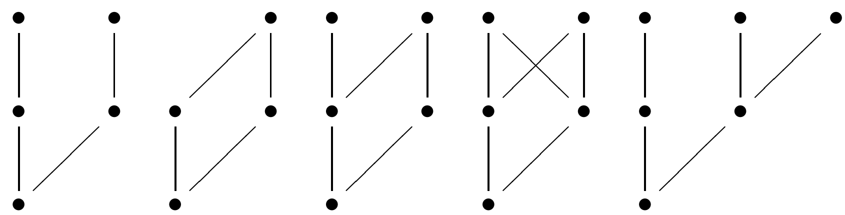
Notice that a Hasse diagram $\mathcal{H}$ can be seen as a connected, finite, non-directed, simple graph $G\mathcal{H}$ with base point corresponding to the minimal element in general. We say a graph with a base point as a *based graph* in this paper. Define $\phi : \mathcal{T}_0(X) \to \mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\ge 0}$ by $\phi(P) := d.i.(P)$ and extend $\tilde{\phi} : G\mathcal{H}(X) \to \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0}$ by $\tilde{\phi}(P_iP_j) = d.i.(P_i) - d.i.(P_j)$, the line segment with extremal points $d.i.(P_i)$ and $d.i.(P_j)$. Then there is a commutative diagram

$$\begin{tikzcd}
\mathcal{T}_0(X) \arrow[r, "\phi"] \arrow[d, "\cap"'] & \mathbb{Z}_{\ge 0} \times \mathbb{Z}_{\ge 0} \arrow[d, "\cap"] \\
G\mathcal{H}(X) \arrow[r, "\tilde{\phi}"] & \mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0}.
\end{tikzcd}$$

Note that $\tilde{\phi}$ is injective, that is, $\tilde{\phi}$ gives the realization of $\mathcal{H}(X)$ into $\mathbb{R}_{\ge 0} \times \mathbb{R}_{\ge 0}$ induced by the above double indexes. We see $\mathcal{H}(X) = \mathcal{H}(Y)$ if and only if $\tilde{\phi}G\mathcal{H}(X) = \tilde{\phi}G\mathcal{H}(Y)$. On the other hand, we can reconstruct $\tilde{\phi}G\mathcal{H}(X)$ from $G\mathcal{H}(X)$ graphically (see §4). Thus

**Theorem 1.5.** *For some spaces X and Y, $\mathcal{H}(X) = \mathcal{H}(Y)$ if and only if $G\mathcal{H}(X)$ and $G\mathcal{H}(Y)$ are isomorphic as based graphs.*

There do not exist the following Hasse diagrams in our ones:



$\cdots$. For example, the graph

$$A \text{---} B \text{---} C \text{---} D \text{---} E$$

represents the totally ordered Hasse diagram of a space with rational toral rank 4 if we choose the base point as $A$ or $E$. Also it represents (2) of Example 3.5 if we choose the base point as $B$ or $D$. But if we choose the base point as $C$, the graph corresponds to non of our Hasse diagrams. Also the graph

$$\begin{tikzcd}[arrows=dash]
 & F \arrow[r] \arrow[d] & G \arrow[d] & & \\
A \arrow[r] & B \arrow[r] & C \arrow[r] & D \arrow[r] & E
\end{tikzcd}$$

represents our Hasse diagrams $(a)$ or $(b)$ below if we choose the base point as $A$ or $B$, respectively.

$$(a)\quad \begin{tikzcd}[column sep=small] E \arrow[d, dash] & \\ D \arrow[d, dash] & G \arrow[d, dash] \\ C \arrow[d, dash] \arrow[ur, dash] & F \\ B \arrow[d, dash] \arrow[ur, dash] & \\ A & \end{tikzcd} \qquad (b)\quad \begin{tikzcd}[column sep=small] E \arrow[d, dash] & & \\ D \arrow[d, dash] & G \arrow[d, dash] & \\ C \arrow[d, dash] \arrow[ur, dash] & F & A \\ B \arrow[ur, dash] \arrow[urr, dash] & & \end{tikzcd} \qquad (c)\quad \begin{tikzcd}[column sep=small] E \arrow[d, dash] & \\ D \arrow[d, dash] & \\ C \arrow[d, dash] \arrow[dr, dash] & A \arrow[d, dash] \\ G \arrow[d, dash] & B \\ F \arrow[ur, dash] & \end{tikzcd}$$

If we choose the base point as $F$, the Hasse diagram is given as $(c)$, which is not ours since the points $G$ and $B$ must be a same one from Definition 1.1. Also we can check that the other points are not impossible to be realized as the minimal elements of our Hasse diagrams (the base point of $G\mathcal{H}$). Note that the author does not know whether or not exists a space (rational model) $X$ with $\mathcal{H}(X) = (a)$. The following question is essential.

**Question 1.6.** *Find an example of two spaces (rational models) $X$ and $Y$ such that $\phi\mathcal{T}_0(X) = \phi\mathcal{T}_0(Y)$ in $\mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ but $\mathcal{H}(X) \neq \mathcal{H}(Y)$.*

**Remark 1.7.** *Our definition of $(\mathcal{X}/\sim, >)$ in Definition 1.1 may be rough. But if we do not take the quotient, the poset $(\mathcal{X}, >)$ seems very complicated. For example, even when $X = S^3 \times S^3$, the Hasse diagram is*

$$\begin{array}{c} \cdots P_i \cdots \\ \cdots \big| \cdots \\ (S^2 \times S^3)_{\mathbb{Q}} \\ \big| \\ (S^3 \times S^3)_{\mathbb{Q}} \end{array}$$

*(it seems as a broom) where $H^*(S^2 \times S^3; \mathbb{Q}) \cong \mathbb{Q}[t_1]/(t_1^2) \otimes \Lambda(v)$ with $|v| = 3$ and*

$$\{P_i\}_i \cong \{H^*(P_i; \mathbb{Q})\}_i \cong \{\frac{\mathbb{Q}[t_1, t_2]}{(t_1^2 + a_i t_2^2, t_1 t_2)} \mid a_i \in \mathbb{Q}^*\} \cong \mathbb{Q}^*/\mathbb{Q}^{*2},$$

*which is an infinite set ($\mathbb{Q}^* = \mathbb{Q} - 0$ is the unite group of $\mathbb{Q}$). Note that $D_1 u = t_1^2$, $D_1 v = 0$, $D_2 u = t_1^2 + a_i t_2^2$ and $D_2 v = t_1 t_2$ for $M(X) = (\Lambda(u, v), 0)$ (see §2).*

**Remark 1.8.** *For $Y \in \mathcal{X}$, $\mathcal{T}_0(Y) \equiv \{P_i \in \mathcal{T}_0(X) \mid [Y] = P_i$ or $[Y] < P_i\}$ as ordered sets. Thus $\mathcal{H}(Y)$ is a sub-Hasse diagram of $\mathcal{H}(X)$. Also for two spaces $X$ and $X'$, $G\mathcal{H}(X \times X') \supset G\mathcal{H}(X) \vee G\mathcal{H}(X')$ as a subgraph with vertexes $\{P_i\}_i$ and edges $\{P_i P_j\} = \{P_i < P_j \mid$ there is no $P_k$ with $P_i < P_k < P_j\}_{i,j}$. Here the right hand is the one point union $G\mathcal{H}(X) \coprod G\mathcal{H}(X')/\sim$ where $P_{r_0(X)} \sim P'_0$ for $\mathcal{T}_0(X) = \{P_i\}_i$ and $\mathcal{T}_0(X') = \{P'_i\}_i$. It is a grafting of one on the other. By using a Sullivan model, S.Halperin indicates that* *rational toral rank does not preserve the product formula $r_0(X \times X') = r_0(X) + r_0(X')$ in general [7]([4, Ex.7.19]). Thus this embedding may be complicated in general (see Example 3.9 below).*

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## 2 A Halperin's result

Let $X$ be a simply connected CW complex of finite type and the Sullivan minimal model $M(X) = (\Lambda V, d)$. It is a free $\mathbb{Q}$-commutative differential graded algebra with a $\mathbb{Q}$-graded vector space $V = \bigoplus_{i\geq 2} V^i$ where $\dim V^i < \infty$ and a decomposable differential; i..e., $d(V^i) \subset (\Lambda^+ V \cdot \Lambda^+ V)^{i+1}$ and $d \circ d = 0$. Here $\Lambda^+ V$ is the ideal of $\Lambda V$ generated by elements of positive degree. Denote the degree of a homogeneous element $x$ of a graded algebra as $|x|$. Then $xy = (-1)^{|x||y|} yx$ and $d(xy) = d(x)y + (-1)^{|x|} x d(y)$. Note that $M(X)$ determines the rational homotopy type of $X$, $X_{\mathbb{Q}}$. In particular, $H^*(\Lambda V, d) \cong H^*(X; \mathbb{Q})$. Refer [3] for detail.

If an $r$-torus $T^r$ acts on $X$ by $\mu : T^r \times X \to X$, there is the Borel fibration

$$X \to ET^r \times^{\mu}_{T^r} X \to BT^r,$$

where $ET^r \times^{\mu}_{T^r} X$ is the orbit space of the action $g(e, x) = (eg^{-1}, gx)$ on the product $ET^r \times X$. It is rationally given by the KS extension (model)

$$(\mathbb{Q}[t_1, \ldots, t_r], 0) \to (\mathbb{Q}[t_1, \ldots, t_r] \otimes \Lambda V, D) \to (\Lambda V, d) \quad (*)$$

where with $|t_i| = 2$ for $i = 1, \ldots, r$, $Dt_i = 0$ and $Dv \equiv dv$ modulo the ideal $(t_1, \ldots, t_r)$ for $v \in V$.

**Proposition 2.1.** *[6, Proposition 4.2] Suppose that $X$ is a simply connected CW-complex with $\dim H^*(X; \mathbb{Q}) < \infty$. Put $M(X) = (\Lambda V, d)$. Then $r_0(X) \geq r$ if and only if there is a KS extension $(*)$ satisfying $\dim H^*(\mathbb{Q}[t_1, \ldots, t_r] \otimes \wedge V, D) < \infty$. Moreover, if $r_0(X) \geq r$, then $T^r$ acts freely on a finite complex $Y$ that has the same rational homotopy type as $X$ and $M(ET^r \times_{T^r} Y) \cong (\mathbb{Q}[t_1, \ldots, t_r] \otimes \wedge V, D)$.*

Thus we can put

$$\mathcal{X}_n = \{(\mathbb{Q}[t_1, \ldots, t_n] \otimes \Lambda V, D) \mid \dim H^*(\mathbb{Q}[t_1, \ldots, t_n] \otimes \wedge V, D) < \infty\}/_{\cong}$$

for $M(X) = (\Lambda V, d)$. The KS extension of the fibration $Y_1 \to Y_2 \to BT^{n-m}_{\mathbb{Q}}$ in §1 is given by the homotopy commutative diagram

$$\begin{tikzcd}
 & (\Lambda V, d) \arrow[r, equal] & (\Lambda V, d) \\
(\mathbb{Q}[t_{m+1}, .., t_n], 0) \arrow[r] & (\mathbb{Q}[t_1, .., t_n] \otimes \Lambda V, D_2) \arrow[r] \arrow[u] & (\mathbb{Q}[t_1, .., t_m] \otimes \Lambda V, D_1) \arrow[u] \\
(\mathbb{Q}[t_{m+1}, .., t_n], 0) \arrow[r] \arrow[u, equal] & (\mathbb{Q}[t_1, .., t_n], 0) \arrow[r] \arrow[u] & (\mathbb{Q}[t_1, .., t_m], 0) \arrow[u]
\end{tikzcd}$$

for $M(BT^{n-m}) = (\mathbb{Q}[t_{m+1}, .., t_n], 0)$, $M(Y_1) = (\mathbb{Q}[t_1, .., t_m] \otimes \Lambda V, D_1)$ and $M(Y_2) = (\mathbb{Q}[t_1, .., t_n] \otimes \Lambda V, D_2)$. Then we simply write $[D_1] < [D_2]$.

Even if $r_0(X) > i$ and $\dim H^*(\mathbb{Q}[t_1, \ldots, t_{i-1}] \otimes \wedge V, D) < \infty$, we may not be able to construct the KS extension

$$(\mathbb{Q}[t_i], 0) \to (\mathbb{Q}[t_1, \ldots, t_i] \otimes \wedge V, D') \to (\mathbb{Q}[t_1, \ldots, t_{i-1}] \otimes \wedge V, D)$$

satisfying $\dim H^*(\mathbb{Q}[t_1, \ldots, t_i] \otimes \wedge V, D') < \infty$ in general (see Claim 1.2).

## 3 Examples of $r_0(X) \leq 4$

Refer the arguments of [4, 7.3.2] or [7] for the computations of toral ranks with minimal models. We put $M(X) = (\Lambda V, d)$. A manner to draw $\tilde{\phi}G\mathcal{H}(X)$ often is the following steps.

i) Estimate $r_0(X)$ by Proposition 2.1.

ii) Dot $V = \{(s, t) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} | s \geq 0, t > 0, s + t \leq r_0(X)\}$.

iii) Check whether or not a point '$P$' exists so that $d.i.(P) = (s, r_0(X) - s) \in V$ for $s = 1, .., r_0(X) - 1$. If exists (then we say it a *bud*), next check the below $\cdots$. See Claim 1.4 (2).

iv) Check whether or not an edge '$<$' exists between $P$ and $P'$ with $d.i.(P) = (s, t)$ and $d.i.(P') = (s', t + 1)$ for $s < s'$. See Claim 1.4 (1). In particular, the *trunk* $P_0 - P_1 - \cdots - P_{r_0(X)}$ always exists.

**Example 3.1.** When $X = S^{2m+1} \times S^{2n+1}$, the Hasse diagram of $\mathcal{T}_0(X)$ is totally ordered as (1) for any $m$ and $n$. Next put $M(X) = (\Lambda(v_1, v_2, v_3, v_4), d)$ with $dv_1 = dv_2 = dv_4 = 0$, $dv_3 = v_1 v_2$ and $|v_1| = |v_2| = 3$, $|v_3| = 5$, $|v_4| = 9$. It is given by the total space of a non-trivial fibration $S^5 \to X \to S^3 \times S^3 \times S^9$. Then $\mathcal{H}(X)$ is given as (2):

$$(1)\quad \begin{tikzcd} P_2 \arrow[d, no head] \\ P_1 \arrow[d, no head] \\ P_0 \end{tikzcd} \qquad\qquad (2)\quad \begin{tikzcd} P_2 \arrow[d, no head] & \\ P_1 \arrow[d, no head] & P_3 \arrow[dl, no head] \\ P_0 & \end{tikzcd}$$

where $P_1 = [(\mathbb{Q}[t] \otimes \Lambda V, D)]$ with $Dv_1 = Dv_2 = Dv_4 = 0$ $Dv_3 = v_1 v_2 + t^3$, $P_2 = [(\mathbb{Q}[t_1, t_2] \otimes \Lambda V, D)]$ with $Dv_1 = Dv_2 = 0$ $Dv_3 = v_1 v_2 + t_1^3$, $Dv_4 = t_2^5$, $P_3 = [(\mathbb{Q}[t] \otimes \Lambda V, D)]$ with $Dv_1 = Dv_2 = 0$, $Dv_3 = v_1 v_2$, $Dv_4 = v_1 v_3 t + t^5$. Note that $\dim H^*(\mathbb{Q}[t, t'] \otimes \Lambda V, D') = \infty$ for any KS extension $(\mathbb{Q}[t, t'] \otimes \Lambda V, D')$ of it.

In general, $\mathcal{H}(X)$ is given as only (1) or (2) if $r_0(X) = 2$. Next we will consider the cases of $r_0(X) > 2$.

A minimal model $(\Lambda V, d)$ is said to be *pure* if $dV^{even} = 0$ and $dV^{odd} \subset \Lambda V^{even}$.

**Lemma 3.2.** *For $m < n$, if $M(Y) = (\Lambda(u_1,..,u_m,v_1,..,v_n),d)$ with $|u_i|$ even and $|v_1| = \cdots = |v_n|$ odd is a pure model, then $r_0(Y) = n - m$.*

*Proof.* From [1, Theorem 1], $r_0(Y) \leq n - m$. From [5, Lemma 8], there is a sub-basis $v_1',..,v_m'$ of $\mathbb{Q}(v_1,..,v_n)$ such that $dv_1',..,dv_m'$ is a regular sequence, i.e., $\dim H^*(\Lambda(u_1,..,u_m,v_1',..,v_m'),d) < \infty$. Then there is a sub-basis $v_{i_1},..,v_{i_{n-m}}$ with $\mathbb{Q}(v_{i_1},..,v_{i_{n-m}}) \oplus \mathbb{Q}(v_1',..,v_m') = \mathbb{Q}(v_1,..,v_n)$. For $j = 1,..,n-m$, put $Dv_{i_j} = dv_{i_j} + t_j^{a_j}$ with $a_j = (|v_{i_j}|+1)/2$. Then $[t_j^{a_j}] \in \mathbb{Q}[u_1,..,u_m]/(dv_1',..,dv_m')$ and especially $\dim H^*(\mathbb{Q}[t_{i_1},..,t_{i_{n-m}}] \otimes \Lambda(u_1,..,u_m,v_1,..,v_n),D) < \infty$. From Proposition 2.1, $r_0(Y) \geq n - m$. ■

**Theorem 3.3.** *If $X$ has the rational homotopy type of product of odd spheres with same dimensions, $X_{\mathbb{Q}} \simeq (S^k \times \cdots \times S^k)_{\mathbb{Q}}$ for some $k > 1$, then $\mathcal{T}_0(X)$ is totally ordered.*

*Proof.* Put $r_0(X) = n$. Suppose $A = (\mathbb{Q}[t_1,..,t_{n-s}] \otimes \Lambda(v_1,..,v_n),D)$ satisfies $\dim H^*(A) < \infty$. It is easy to check that $A$ is pure. From the above lemma, $r_0(A) = s$. Thus there is no point $P$ in $\mathcal{H}(X)$ such that $d.i.(P) = (s, n-s)$ for $s > 0$. We have done from Claim 1.4 (2). ■

**Theorem 3.4.** *Suppose that $1 < n_1 \leq n_2 \leq n_3 \leq n_4$ are odd.*

*(1) For $X = S^{n_1} \times S^{n_2} \times S^{n_3}$, there exists an element $P$ in $\mathcal{T}_0(X)$ with $d.i.(P) = (2,1)$ if and only if $n_1 + n_2 < n_3$.*

*(2) For $X = S^{n_1} \times S^{n_2} \times S^{n_3} \times S^{n_4}$, there exists an element $P$ in $\mathcal{T}_0(X)$ with $d.i.(P) = (3,1)$ if and only if $n_1 + n_2 < n_3$ and $n_1 + n_3 < n_4$.*

*Proof.* (1) Put $M(X) = (\Lambda(v_1,v_2,v_3),0)$ with $|v_i| = n_i$. Then $Dv_1 = Dv_2 = 0$, $Dv_3 = v_1v_2t^{(n_3-n_1-n_2+1)/2} + t^{(n_3+1)/2}$ if and only if $d.i.([D]) = (2,1)$.
(2) Put $M(X) = (\Lambda(v_1,v_2,v_3,v_4),0)$. Then there is a differential $D$ with $Dv_3 = v_1v_2t^{(n_3-n_1-n_2+1)/2}$ and $Dv_4 = v_1v_3t^{(n_4-n_1-n_3+1)/2} + t^{(n_4+1)/2}$ if and only if there exists a bud $P$ of $d.i.(P) = (3,1)$. ■

**Example 3.5.** Let $X$ be the product of three odd-spheres. Then $r_0(X) = 3$. From Theorem 3.3, $\mathcal{T}_0(S^3 \times S^3 \times S^3)$ is given by the 4-points $\{P_0,P_1,P_2,P_3\}$ which is totally ordered as (1). But $\mathcal{T}_0(S^3 \times S^3 \times S^7)$ is given by the 5-points $\{P_0,P_1,P_2,P_3,P_4\}$ which is partially ordered as (2):

$$
(1)\quad
\begin{tikzcd}
P_3 \arrow[d, dash] \\
P_2 \arrow[d, dash] \\
P_1 \arrow[d, dash] \\
P_0
\end{tikzcd}
\qquad\qquad
(2)\quad
\begin{tikzcd}
P_3 \arrow[d, dash] & \\
P_2 \arrow[d, dash] & \\
P_1 \arrow[d, dash] & P_4 \\
P_0 \arrow[ur, dash] &
\end{tikzcd}
$$

where $P_1 = [(S^2 \times S^3 \times S^7)_{\mathbb{Q}}](= [(S^3 \times S^3 \times \mathbb{C}P^3)_{\mathbb{Q}}])$, $P_2 = [(S^2 \times S^2 \times S^7)_{\mathbb{Q}}]$ $(= [(S^2 \times S^3 \times \mathbb{C}P^3)_{\mathbb{Q}}])$ and $P_3 = [(S^2 \times S^2 \times \mathbb{C}P^3)_{\mathbb{Q}}]$. Here $P_4 = [Y_{\mathbb{Q}}]$ is given by $M(Y) = (\mathbb{Q}[t] \otimes \Lambda(x,y,z), D)$ with $Dx = Dy = 0$ and $Dz = xyt + t^4$ for $M(X) = (\Lambda(x,y,z), 0)$ of $|x| = |y| = 3$ and $|z| = 7$. Then $H^*(Y;\mathbb{Q}) \cong \Lambda(x,y) \otimes \mathbb{Q}[t]/(xyt + t^4)$, which is finite dimensional. Note $r_0(Y) = 0$ from Proposition 2.1. Indeed, suppose that there is a KS extension

$$(\mathbb{Q}[t_2], 0) \to (\mathbb{Q}[t_1, t_2] \otimes \Lambda(x,y,z), D') \to (\mathbb{Q}[t_1] \otimes \Lambda(x,y,z), D) = M(Y).$$

We have $D' \circ D' \neq 0$ for any non-trivial differential $D'x = f(t_1, t_2)$ and $D'y = g(t_1, t_2)$ in $\mathbb{Q}[t_1, t_2]$. Also if $D'x = D'y = 0$ and $D'z = xyt_1 + t_1^4 + axyt_2 + \sum a_{ij} t_1^i t_2^j$ $(a, a_{ij} \in \mathbb{Q})$, then $\dim H^*(\mathbb{Q}[t_1, t_2] \otimes \Lambda(x,y,z), D') = \infty$ for any $a$, $a_{ij}$. Thus $d.i.(P_4) = (2, 2-0-1) = (2,1)$.

Notice that there is not a point of $(1,2)$ from [7, Lemma 2.12].

Thus the set $\mathcal{T}_0(X)$ is more sensitive than the number $r_0(X)$ about degrees of the rational homotopy group of $X$.

**Example 3.6.** Put $M(X) = (\Lambda(v_1, v_2, v_3, v_4, v_5), d)$ with $dv_1 = dv_2 = dv_4 = dv_5 = 0$, $dv_3 = v_1 v_2$. If
(1) $|v_1| = |v_2| = 3$, $|v_3| = 5$, $|v_4| = 9$, $|v_5| = 15$ or
(2) $|v_1| = |v_2| = 3$, $|v_3| = 5$, $|v_4| = 3$, $|v_5| = 7$,
then $\mathcal{H}(X)$ is given as

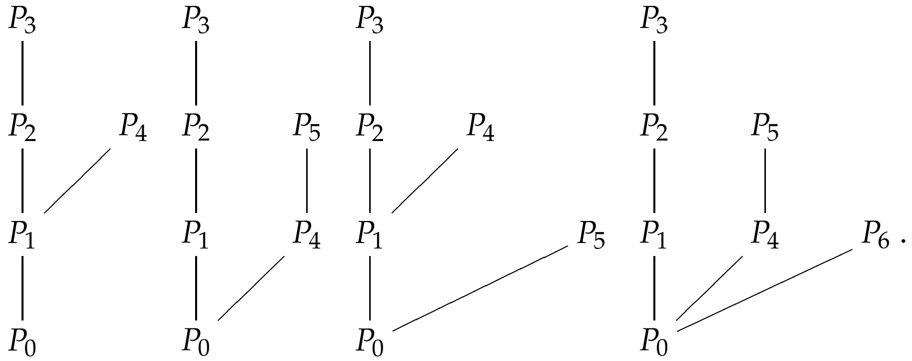
(1)
$$\begin{tikzcd}
P_3 \arrow[d, dash] & & \\
P_2 \arrow[d, dash] & P_5 \arrow[dl, dash] \arrow[d, dash] & \\
P_1 \arrow[d, dash] & P_4 \arrow[dl, dash] & P_6 \arrow[dll, dash] \\
P_0 & &
\end{tikzcd}$$

(2)
$$\begin{tikzcd}
P_3 \arrow[d, dash] & \\
P_2 \arrow[d, dash] & P_5 \arrow[dl, dash] \arrow[d, dash] \\
P_1 \arrow[d, dash] & P_4 \arrow[dl, dash] \\
P_0. &
\end{tikzcd}$$

In (1), $P_1 = [(\mathbb{Q}[t] \otimes \Lambda V, D)]$ with $Dv_1 = Dv_2 = Dv_5 = 0$, $Dv_4 = t^5$, $Dv_3 = v_1 v_2$.
$P_2 = [(\mathbb{Q}[t_1, t_2] \otimes \Lambda V, D)]$ with $Dv_1 = Dv_2 = Dv_5 = 0$, $Dv_3 = v_1 v_2 + t_2^3$, $Dv_4 = t_1^5$.
$P_3 = [(\mathbb{Q}[t_1, t_2, t_3] \otimes \Lambda V, D)]$ with $Dv_1 = Dv_2 = 0$, $Dv_3 = v_1 v_2 + t_2^3$, $Dv_4 = t_1^5$, $Dv_5 = t_3^8$.
$P_4 = [(\mathbb{Q}[t] \otimes \Lambda V, D)]$ with $Dv_1 = Dv_2 = Dv_5 = 0$, $Dv_3 = v_1 v_2$, $Dv_4 = v_1 v_3 t + t^5$.
$P_5 = [(\mathbb{Q}[t_1, t_2] \otimes \Lambda V, D)]$ with $Dv_1 = Dv_2 = 0$, $Dv_3 = v_1 v_2$, $Dv_4 = v_1 v_3 t_1 + t_1^5$, $Dv_5 = t_2^8$.
$P_6 = [(\mathbb{Q}[t] \otimes \Lambda V, D)]$ with $Dv_1 = Dv_2 = Dv_4 = 0$, $Dv_3 = v_1 v_2$, $Dv_5 = v_1 v_3 t^4 + v_2 v_4 t^2 + t^8$.

In (2), $P_4 = [(\mathbb{Q}[t] \otimes \Lambda V, D)]$ with $Dv_1 = Dv_2 = Dv_4 = 0$, $Dv_3 = v_1 v_2$, $Dv_5 = v_1 v_4 t + t^4$.
$P_5 = [(\mathbb{Q}[t_1, t_2] \otimes \Lambda V, D)]$ with $Dv_1 = Dv_2 = Dv_4 = 0$, $Dv_3 = v_1 v_2 + t_2^3$, $Dv_5 = v_1 v_4 t_1 + t_1^4$.

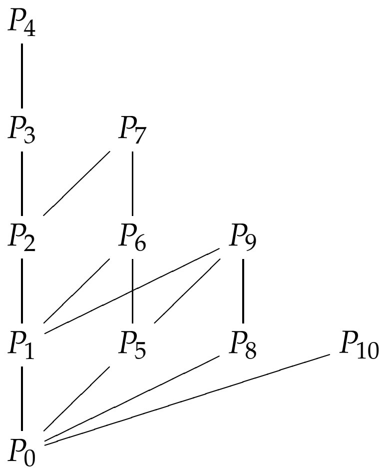
**Remark 3.7.** *If $r_0(X) = 3$, $\mathcal{H}(X)$ is given as one of Example 3.5, Example 3.6 or the diagrams:*



*Thus we see that* $4 \le \sharp\{\mathcal{H}(X) | r_0(X) = 3\} \le 8$.

Finally we give two examples with the same Hasse diagrams.

**Example 3.8.** Put $M(X_1) = (\Lambda(v_1, v_2, v_3, v_4, v_5, v_6), d)$ with $dv_1 = dv_2 = dv_4 = dv_5 = dv_6 = 0$, $dv_3 = v_1 v_2$ and $|v_1| = |v_2| = 3$, $|v_3| = 5$, $|v_4| = 9$, $|v_5| = 13$, $|v_6| = 17$. Then $\mathcal{H}(X_1)$ is given as



where $P_4 = [(\mathbb{Q}[t] \otimes \wedge V, D)]$ with $Dv_1 = Dv_2 = 0$, $Dv_3 = v_1 v_2 + t^3$, $Dv_4 = t^5$, $Dv_5 = t^7$, $Dv_6 = t^9$.
$P_5 = [(\mathbb{Q}[t] \otimes \wedge V, D)]$ with $Dv_1 = Dv_2 = Dv_5 = Dv_6 = 0$, $Dv_3 = v_1 v_2$, $Dv_4 = v_1 v_3 t + t^5$.
$P_8 = [(\mathbb{Q}[t] \otimes \wedge V, D)]$ with $Dv_1 = Dv_2 = Dv_4 = Dv_6 = 0$, $Dv_3 = v_1 v_2$, $Dv_5 = v_2 v_4 t + v_1 v_3 t^4 + t^7$.
$P_9 = [(\mathbb{Q}[t_1, t_2] \otimes \wedge V, D)] = [(\mathbb{Q}[t_1, t_2] \otimes \wedge V, D')] = [(\mathbb{Q}[t_1, t_2] \otimes \wedge V, D'')]$ with $Dv_1 = Dv_2 = 0$, $Dv_3 = v_1 v_2$ (same for $D'$ and $D''$) and

$$Dv_4 = v_1 v_3 t_2 + t_1^5,\ Dv_5 = 0,\ Dv_6 = v_2 v_5 t_2 + t_2^9,$$

$$D'v_4 = v_1 v_3 t_1 + t_1^5,\ D'v_5 = 0,\ D'v_6 = v_2 v_5 t_2 + t_2^9,$$

$$D''v_4 = v_1 v_3 t_1 + t_2^5,\ D''v_5 = 0,\ D''v_6 = v_2 v_5 t_1 + t_1^9.$$

Note that $P_1 < P_9$ is given by $D$, $P_5 < P_9$ by $D'$ and $P_8 < P_9$ by $D''$.
$P_{10} = [(\mathbb{Q}[t] \otimes \wedge V, D)]$ with $Dv_1 = Dv_2 = Dv_4 = Dv_5 = 0$, $Dv_3 = v_1v_2$ and $Dv_6 = v_1v_5t + v_2v_4t^3 + v_1v_3t^5 + t^9$.

**Example 3.9.** Put $M(X_2) = (\Lambda(v_1, v_2, v_3, v_4, v_5, v_6, w_1, w_2, w_3, w_4, w_5, w_6, u, y, z, v), d)$ with $dv_1 = \cdots = dv_6 = dw_1 = \cdots = dw_6 = du = dy = dz = 0$,

$$dv = v_1v_2v_3v_4 + v_1v_2v_5v_6 + w_1w_2w_3w_4 + w_1w_2w_5w_6 + u^2$$

and $|v_1| = \cdots = |v_6| = |w_1| = \cdots = |w_6| = 3$, $|u| = 6$, $|y| = |z| = 7$, $|v| = 11$. We see $\dim H^*(X_2; \mathbb{Q}) < \infty$ since $M(X_2)$ is the total space of a KS extension

$$(\Lambda(v_1, \cdots, v_6, w_1, \cdots, w_6, y, z), 0) \to (\Lambda V, d) \to (\Lambda(u, v), \overline{d}) = M(S^6),$$

where $\overline{d}u = 0$ and $\overline{d}v = u^2$. Remark the space $X_1$ in Example 3.8 is not a formal space but $X_2$ is formal [3]. Note that $r_0(X') = 0$ for a space $X'$ with $X_{2\mathbb{Q}} \simeq (X' \times S^7 \times S^7)_{\mathbb{Q}}$, where $M(X') = (\Lambda(v_1, v_2, v_3, v_4, v_5, v_6, w_1, w_2, w_3, w_4, w_5, w_6, u, v), d)$.

Then $\mathcal{H}(X_2)$ is given as

$$\begin{tikzcd}
P_4 \arrow[d, dash] & & & \\
P_3 \arrow[d, dash] & P_7 \arrow[dl, dash] \arrow[d, dash] & & \\
P_2 \arrow[d, dash] & P_6 \arrow[dl, dash] \arrow[d, dash] & P_9 \arrow[dll, dash] \arrow[dl, dash] \arrow[d, dash] & \\
P_1 \arrow[d, dash] & P_5 \arrow[dl, dash] & P_8 \arrow[dll, dash] & P_{10} \arrow[dlll, dash] \\
P_0 & & &
\end{tikzcd}$$

where $P_1 = [(\mathbb{Q}[t] \otimes \Lambda V, D)]$ such that $Dv_i = 0$ for $i \neq 4$, $Dv_4 = t^2$, $Dw_i = Du = Dz = 0$, $Dy = v_2v_3t$, $Dv = dv - v_1yt$.
$P_4 = [(\mathbb{Q}[t_1, t_2, t_3, t_4] \otimes \Lambda V, D)]$ with

$$Dv_1 = Dv_2 = Dv_3 = Dv_5 = Dw_1 = Dw_2 = Dw_3 = Dw_5 = Du = 0, \cdots (*)$$

$$Dv_4 = t_1^2,\ Dv_6 = t_2^2,\ Dw_4 = t_3^2,\ Dw_6 = t_4^2,$$

$$Dy = v_2v_3t_1 + v_1v_5t_2,\ Dz = w_2w_3t_3 + w_1w_5t_4,$$

$$Dv = dv - v_1yt_1 + v_2yt_2 - w_1zt_3 + w_2zt_4.$$

Then $D \circ D = 0$ and $\dim H^*(\mathbb{Q}[t_1, t_2, t_3, t_4] \otimes \Lambda V, D) < \infty$. Thus $r_0(X_2) \geq 4$ and we deduce $r_0(X_2) < 5$ by the direct (but complicated) calculations that $\dim H^*(\mathbb{Q}[t_1, t_2, t_3, t_4, t_5] \otimes \Lambda V, D) = \infty$ for any $D$. Note the part $(*)$ of $P_4$ is applied for all differentials below.
$P_5 = [(\mathbb{Q}[t] \otimes \Lambda V, D)]$ with $Dv = dv$, $Dy = t^4$, $Dv_i = Dw_i = Du = Dz = 0$.
$P_6 = [(\mathbb{Q}[t_1, t_2] \otimes \Lambda V, D)]$ such that $Dv_i = 0$ for $i \neq 4$, $Dv_4 = t_1^2$, $Dw_i = Du = 0$,

$Dy = v_2v_3t_1$, $Dv = dv - v_1yt_1$, $Dz = t_2^4$.
$P_8 = [(\mathbb{Q}[t] \otimes \Lambda V, D)]$ with $Dv = dv - v_1yt + v_2yt - w_1zt$, $Dy = v_2v_3t + v_1v_5t$, $Dz = w_2w_3t$, $Dv_4 = Dv_6 = Dw_4 = t^2$, $Dw_6 = 0$.
$P_9 = [(\mathbb{Q}[t_1,t_2] \otimes \Lambda V, D)] = [(\mathbb{Q}[t_1,t_2] \otimes \Lambda V, D')] = [(\mathbb{Q}[t_1,t_2] \otimes \Lambda V, D'')]$ with

$$Dv_4 = t_1^2,\ Dv_6 = Dw_4 = Dw_6 = t_2^2,\ Dy = v_2v_3t_1 + v_1v_5t_2,$$

$$Dz = w_2w_3t_2 + w_1w_5t_2,\ Dv = dv - v_1yt_1 + v_2yt_2 - w_1zt_2 + w_2zt_2,$$

$$D'v_i = D'w_i = D'u = 0,\ D'y = t_1^4,\ D'z = t_2^4,\ D'v = dv,$$

$$D''v_4 = t_2^2,\ D''v_6 = D''w_4 = D''w_6 = t_1^2,\ D''y = v_2v_3t_2 + v_1v_5t_1,$$

$$D''z = w_2w_3t_1 + w_1w_5t_1,\ D''v = dv - v_1yt_2 + v_2yt_1 - w_1zt_1 + w_2zt_1.$$

Note that $P_1 < P_9$ is given by $D$, $P_5 < P_9$ by $D'$ and $P_8 < P_9$ by $D''$.
$P_{10} = [(\mathbb{Q}[t] \otimes \Lambda V, D)]$ with $Dv_4 = Dv_6 = Dw_4 = Dw_6 = t^2$, $Dy = v_2v_3t + v_1v_5t$, $Dz = w_2w_3t + w_1w_5t$, $Dv = dv - v_1yt + v_2yt - w_1zt + w_2zt$.

## 4 Proof of Theorem 1.5

Let $G$ be a connected, non-directed, finite, simple(i.e., without multiple edges, loops), based graph with the vertex set $V(G) = \{v_0, v_1, .., v_N\}$ of the base point $v_0$. For the set of distances $D_0 = \{d(v_0, v_i) | v_i \in V(G)\}_i$ between the points of $G$ and $v_0$, put $n = maxD_0$. Suppose that a path of length $n$

$$l_0 :\quad v_0 \to v_{i_1} \to \cdots \to v_{i_{n-1}} \to v_{i_n}$$

with $d(v_0, v_{i_n}) = n$ is unique. $\cdots (0)$
Then put $\psi(v_0) := (0,0)$ and

$$\psi(v_{i_u}) := (0, u) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$$

for $u = 1, .., n$.

Next put $D_1 = \{d(v_0, v_j) | v_j \in V_1 = V(G) - V(l_0)\}_j$ and the set of paths with length $n_1 = maxD_1$ as

$$L_1 = \{l_{1,j}\}_j = \{v_0 \to v_{j_1} \to \cdots \to v_{j_{n_1-1}} \to v_{j_{n_1}} |\, d(v_0, v_{j_{n_1}}) = n_1, v_{j_{n_1}} \in V_1\}_j.$$

Here $V(l_0) = \{v_0, v_{i_1}, .., v_{i_n}\}$. Suppose that (for some $c$)

$$j_m \neq i_m \text{ for } m > c \text{ if } j_c \neq i_c. \quad \cdots (1)$$

For a path $l_{1,j}$ of $L_1$, if $v_{j_c} = v_{i_c}$ for $c = 0, .., m-1$ and $v_{j_m} \neq v_{i_m}$, put

$$\psi(v_{j_u}) := (n - n_1, u) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$$

for $u = m, m+1, .., n_1$.

Next put $D_2 = \{d(v_0, v_k) | v_k \in V_2 = V(G) - (V(l_0) \cup V(L_1))\}_k$ and the set of paths of with length $n_2 = maxD_2$

$$L_2 = \{l_{2,k}\}_k = \{v_0 \to v_{k_1} \to \cdots \to v_{k_{n_2-1}} \to v_{k_{n_2}} |\, d(v_0, v_{k_{n_2}}) = n_2, v_{k_{n_2}} \in V_2\}_k.$$

Suppose that (for some $c$)

$$k_m \neq i_m \text{ for } m > c \text{ if } k_c \neq i_c \quad \text{and}$$

$$k_m \neq i_m, j_m \text{ for } m > c \text{ if } k_c \neq j_c. \quad \cdots (2)$$

For a path $l_{2,k}$ of $L_2$, if $v_{k_c} = v_{i_c}$ or $v_{k_c} = v_{j_c}$ for $c = 0, .., m-1$ but $v_{k_m} \neq v_{i_m}$ and $v_{k_m} \neq v_{j_m}$, put

$$\psi(v_{k_u}) := (n - n_2, u) \in \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$$

for $u = m, m+1, .., n_2$.

Iterating this argument, we have an injection $\psi : V(G) \to \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0}$ and it is naturally extended to the map from the set of edges, $\tilde{\psi} : E(G) \to \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$ as $\tilde{\psi}(v_a v_b) = \psi(v_a) - \psi(v_b)$, the line segment with extremal points $\psi(v_a)$ and $\psi(v_b)$, for any edge $v_a v_b$ of $G$. Thus there is the embedding of $G$ into $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$

$$\begin{tikzcd} V(G) \arrow[r, "\psi"] \arrow[d, "\cap"'] & \mathbb{Z}_{\geq 0} \times \mathbb{Z}_{\geq 0} \arrow[d, "\cap"] \\ G \arrow[r, "\tilde{\psi}"] & \mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}. \end{tikzcd}$$

Notice that two graphs $G$ and $G'$ satisfying (0),(1),(2),.. are isomorphic as based graphs if and only if $\tilde{\psi}G = \tilde{\psi}G'$.

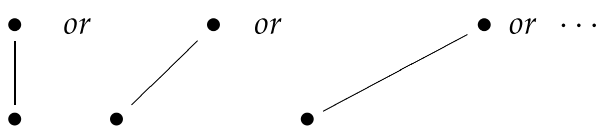
*Proof of Theorem 1.5.* If $G = G\mathcal{H}(X)$, the above conditions (0), (1), (2),.. are satisfied from Lemma 1.3. Thus the above map $\tilde{\psi}$ is defined and we see $\tilde{\psi}G\mathcal{H}(X) = \tilde{\phi}G\mathcal{H}(X)$ in $\mathbb{R}_{\geq 0} \times \mathbb{R}_{\geq 0}$. Suppose $\mathcal{H}(X) \neq \mathcal{H}(Y)$. Then $\tilde{\psi}G\mathcal{H}(X) = \tilde{\phi}G\mathcal{H}(X) \neq \tilde{\phi}G\mathcal{H}(Y) = \tilde{\psi}G\mathcal{H}(Y)$. Thus $G\mathcal{H}(X)$ and $G\mathcal{H}(Y)$ are not isomorphic as based graphs. ∎

## 5 Appendix

Recall an edge of $G\mathcal{H}(X)$ is represented by a rationalized Borel fibration

$$Y_{\mathbb{Q}} \to (ES^1 \times_{S^1} Y)_{\mathbb{Q}} \to BS^1_{\mathbb{Q}}$$

where $Y_{\mathbb{Q}} \in \mathcal{X}_n$ and $(ES^1 \times_{S^1} Y)_{\mathbb{Q}} \in \mathcal{X}_{n+1}$ for some $n$. It is given as




in $\tilde{\phi}G\mathcal{H}(X)$.

**Definition 5.1.** *Suppose $Y_{\mathbb{Q}} \in \mathcal{X}_n$. For two elements $Y_1 = (ES^1 \times^{\mu_1}_{S^1} Z_1)_{\mathbb{Q}}$ and $Y_2 = (ES^1 \times^{\mu_2}_{S^1} Z_2)_{\mathbb{Q}}$ of $\mathcal{X}_{n+1}$, we denote*

$$Y_1 \underset{Y_3}{\sim} Y_2$$

*if there exists a homotopy commutative diagram of fiber inclusions of rationalized Borel fibrations over $BS^1_{\mathbb{Q}}$*

$$\begin{tikzcd}
Y_{\mathbb{Q}} \arrow[r] \arrow[d] & (ES^1_2 \times_{S^1_2} Z_2)_{\mathbb{Q}} = Y_2 \arrow[d] \\
Y_1 = (ES^1_1 \times_{S^1_1} Z_1)_{\mathbb{Q}} \arrow[r] & (E(S^1_1 \times S^1_2) \times_{S^1_1 \times S^1_2} Z_3)_{\mathbb{Q}} =: Y_3
\end{tikzcd}$$

*where $Z_{1\mathbb{Q}} \simeq Z_{2\mathbb{Q}} \simeq Z_{3\mathbb{Q}} \simeq Y_{\mathbb{Q}}$ and $\dim H^*(Y_3;\mathbb{Q}) < \infty$.*

Note $Y_3 \in \mathcal{X}_{n+2}$ and in general $r_0(Y_1) \neq r_0(Y_2)$. The Sullivan model is given as the DGA-homotopy commutative diagram of natural projections

$$\begin{tikzcd}
M(Y) \cong (\Lambda W, d_W) & (\mathbb{Q}[t_2] \otimes \Lambda W, D_2) \arrow[l] \\
(\mathbb{Q}[t_1] \otimes \Lambda W, D_1) \arrow[u] & (\mathbb{Q}[t_1,t_2] \otimes \Lambda W, D) \arrow[l] \arrow[u]
\end{tikzcd}$$

with $\dim H^*(\mathbb{Q}[t_1,t_2] \otimes \Lambda W, D) < \infty$. Here $W = \mathbb{Q}(t_3,..,t_{n+2}) \oplus V$ for $M(X) = (\Lambda V, d)$ and $d_W|V = d$. Remark that Definition 5.1 is not an equivalence relation.

**Definition 5.2.** *For edges (1-cells) $P_aP_b$, $P_aP_d$, $P_bP_c$ and $P_dP_c$ in $G\mathcal{H}(X)$, which is given as (a horizontal deformation of)*

$$\begin{tikzcd}[column sep=small]
 & P_c \arrow[dl, dash] \arrow[d, dash] \\
P_b \arrow[d, dash] & P_d \arrow[dl, dash] \\
P_a &
\end{tikzcd}$$

*in $\tilde{\phi}G\mathcal{H}(X)$, we say that a 2-cell attaches on the 1-cycle $\Box P_aP_bP_cP_d$ (or simply that $\Box P_aP_bP_cP_d$ makes a leaf) and denote as $\partial e^2 = \Box P_aP_bP_cP_d$ if $Y_1 \sim_{Y_3} Y_2$ for $[Y_{\mathbb{Q}}] = P_a$, $[Y_1] = P_b$, $[Y_2] = P_d$ and $[Y_3] = P_c$.*

The existence of a leaf may depend on the degree of certain freedom of $\{D\}$ that represent the upper right point $P_c$ of a cycle $\Box P_aP_bP_cP_d$. In Example 3.8, we easily find that $\Box P_0P_5P_9P_8$ makes a leaf by $Dv_1 = Dv_2 = Dv_4 = 0$, $Dv_3 = v_1v_2$,

$$Dv_5 = v_1v_4t_1 + t_1^7 \quad \text{and}$$
$$Dv_6 = v_2v_4t_2^3 + v_1v_3t_2^5 + t_2^9,$$

where $[D] = P_9$, $[D_1] = P_5$ and $[D_2] = P_8$. Indeed, then the above DGA-diagram is commutative. But, in Example 3.9, the author can not find a differential $D$ that makes the above homotopy commutative diagram for the 1-cycle $\Box P_0P_5P_9P_8$.

In general, if $G\mathcal{H}(X)$ contains (a horizontal deformation of)

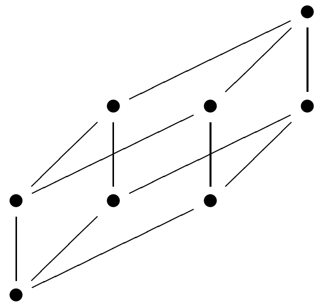
$$\begin{tikzcd}[column sep=small]
 & & R \arrow[dll, dash] \arrow[dl, dash] \arrow[d, dash] \\
Q_1 \arrow[d, dash] & Q_2 \arrow[dl, dash] & Q_3 \arrow[dll, dash] \\
P & &
\end{tikzcd}$$

as a sub-graph with $\partial e_1^2 = \Box PQ_1RQ_2$, $\partial e_2^2 = \Box PQ_2RQ_3$ and $\partial e_3^2 = \Box PQ_1RQ_3$, then $\mathcal{K}(X)$ contains

$$(\, e_1^2 \cup e_2^2 \,) \cup_{\Box PQ_1RQ_3} e_3^2 \;\cong\; S^2.$$

Thus three pieces of leaf can make a 2-sphere.

**Remark 5.3.** *To append certain further informations of $\mathcal{X}$ on $\mathcal{T}_0(X)$, it may be suitable to regard ($\mathcal{T}_0(X)$ as the 0-skeleton and) the based graph $G\mathcal{H}(X)$ as the 1-skeleton of a finite CW complex $\mathcal{T}(X)$, which is obtained by generalizing Definition 5.2. When $\tilde{\phi}G\mathcal{H}(X)$ contains (a horizontal deformation of)*



*as a sub-graph, then a 3-cell of $\mathcal{T}(X)$ is given by the existence of the homotopy commutative diagram of natural projections*

$$\begin{tikzcd}
& (\mathbb{Q}[t_1,t_2,t_3]\otimes\Lambda W, D) \arrow[dl] \arrow[d] \arrow[dr] & \\
(\mathbb{Q}[t_1,t_2]\otimes\Lambda W, D_{12}) \arrow[d] \arrow[dr] & (\mathbb{Q}[t_1,t_3]\otimes\Lambda W, D_{13}) \arrow[dl] \arrow[dr] & (\mathbb{Q}[t_2,t_3]\otimes\Lambda W, D_{23}) \arrow[dl] \arrow[d] \\
(\mathbb{Q}[t_1]\otimes\Lambda W, D_1) \arrow[dr] & (\mathbb{Q}[t_2]\otimes\Lambda W, D_2) \arrow[d] & (\mathbb{Q}[t_3]\otimes\Lambda W, D_3) \arrow[dl] \\
& (\Lambda W, d_W) &
\end{tikzcd}$$

*which represents the above sub-graph. Similarly we can construct higher dimensional CW-structure, which makes a complex $\mathcal{T}(X)$. It must be a topological homotopy invariant of spaces. (The complex $\mathcal{T}(X)$ is at most 2-dimensional if $r_0(X) \le 5$.) If $\mathcal{T}(X)$ is compared to a plant, then the base point $X_{\mathbb{Q}}$ corresponds to the seed (that grows up to be the plant), and $BS^1_{\mathbb{Q}}$, the water (that is necessary for its growth).*

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