Some results on the qualitative theory of semiflows

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Abstract

We point out some results on the qualitative theory of three-parameter semiflows by analyzing the stability of the associated linear skew-product semiflow. Extensions of the well-known theorems due to Datko, Pazy, Rolewicz are obtained.

1 Introduction and Preliminaries

It is known that the qualitative theory of (semi)flows on (locally) compact spaces or ($\sigma-$)finite measure spaces uses notions like stability or exponential dichotomy of the associated linear skew-product (semi)flow. In the finite-dimensional context, the Sacker-Sell spectrum provides an important and useful characterization of these properties (see [25, 26 and 27]). Also, recent extensions to norm-continuous linear skew-product (semi)flows on infinite-dimensional Banach spaces have been obtained by Latushkin and Stepin in [13]. However, all truly infinite-dimensional situations, e.g. flows originating from partial differential equations and functional differential equations, only yield linear skew-product (semi)flows. In recent decades, significant progress has been made in the study of asymptotic behavior of linear skew-product (semi)flows and nonautonomous Cauchy problems (see [5, 15, 25]), giving a unified answer to an impressive list of classical problems. Also, in last few years important contributions were done

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in infinite-dimensional case (see [3, 4, 9, 12, 27]) and in the applications. For instance, it is known by now that well-known equations like Navier-Stokes, Bubnov-Galerkin, Taylor-Couette can be modeled asymptotically by associating a linear skew-product (semi)flow (for details we refer the reader to [22]).

Let us consider (Θ, d) a metric space, X a Banach space, $\mathcal{B}(X)$ the space of all bounded operators acting on X and $\Delta = \{(t, t_0) \in \mathbb{R}^2 : t \geq t_0 \geq 0\}$. We denote the norm of vectors on X and operators on $\mathcal{B}(X)$ by $||\cdot||$. Also $\mathbb{R}^n$ will denote the classical $n-$dimensional Euclidean space and $\mathbb{R}^{n\times n}$ will denote the set of all real $n \times n$ matrices.

Definition 1.1. *A two-parameter (nonlinear) semiflow $\sigma : \Theta \times \mathbb{R}_+ \to \Theta$ is defined by the properties:*

i) $\sigma(\theta, 0) = \theta$, for all $\theta \in \Theta$;

ii) $\sigma(\theta, t+s) = \sigma(\sigma(\theta, s), t)$; for all $\theta \in \Theta$ and $t, s \in \mathbb{R}_+$

If in addition $(\theta, t) \to \sigma(\theta, t)$ is continuous, then σ is called a *continuous* two-parameter (nonlinear) semiflow on Θ .

Remark 1.2. If the above properties hold for any $t, s \in \mathbb{R}$ then σ is said to be a (nonlinear) two-parameter *flow* on Θ.

Definition 1.3. *A family $\{T(t)\}_{t\geq 0}$ of linear and bounded operators acting on X, is said to be a C_0-semigroup on X if the following conditions hold:*

i) $T(0) = I$;

ii) $T(t+s) = T(t)T(s)$, for all $t, s \geq 0$;

iii) there exists $\lim\limits_{t\to 0_+} T(t)x = x$, for all $x \in X$

If the second property holds for any $t, s \in \mathbb{R}$ then $\{T(t)\}_{t\in\mathbb{R}}$ is called a C_0-group.

For a general presentation of the theory of C_0-semigroups we refer the reader to [8], [17] or [19].

Remark 1.4. It is well-known the connection between (nonlinear) (semi)flows, first order differential operators, and (linear) (semi)groups. For instance, consider a continuously differentiable vector field $F : \mathbb{R}^n \longrightarrow \mathbb{R}^n$ with $\sup_{\theta\in\mathbb{R}^n} ||DF(\theta)|| < \infty$, for the derivative $DF(\theta)$ of F and $\theta \in \mathbb{R}^n$. Take the first order differential operator on

$$X := C_0(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R}^n : \text{ f is continuous vanishing at infinity }\}$$

corresponding to the vector field F,

$$Af(\theta) = \langle grad f(\theta), F(\theta)\rangle = \sum_{i=1}^{n} F_i(\theta)\frac{\partial f}{\partial \theta_i}(\theta),$$

for $f \in C_c^1(\mathbb{R}^n) = \{f : \mathbb{R}^n \to \mathbb{R}^n :$ f continuously differentiable, with compact support $\}$, and $\theta \in \mathbb{R}^n$. It can be easily shown that A is dissipative. Since F is globally Lipschitz it follows from standard arguments that there exists a continuous two-parameter flow $\sigma : \mathbb{R}^n \times \mathbb{R} \to \mathbb{R}^n$ which solves the differential equation

$$\frac{\partial}{\partial t}\sigma(\theta, t) = F(\sigma(\theta, t)), \text{ for all } t \in \mathbb{R} \text{ and } \theta \in \mathbb{R}^n \text{ (see [1, Thm. 10.3]).}$$

To such a flow we associate a one-parameter group of linear operators on $C_0(\mathbb{R}^n)$ given by

$$(T(t)f)(\theta) := f(\sigma(\theta, t)), \text{ for } \theta \in \mathbb{R}^n,\ t \in \mathbb{R},$$

the so-called group induced by the flow σ. It can be proved that the generator of the above group is the closure of the first order differential operator A. The domain of the generator will be $D(A) = C_c^1(\mathbb{R}^n)$. For details we refer the reader to [8].

The general relation between (nonlinear) two-parameter semiflows and linear semigroups is given in the remark below.

Remark 1.5. Let Θ be a compact metric space and take $X = \mathcal{C}(\Theta)$, where

$$\mathcal{C}(\Theta) = \{f : \Theta \to \mathbb{C} : f \text{ continuous on } \Theta\}$$

i) The two-parameter (nonlinear) semiflow σ is continuous if and only if it induces a strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on X by the formula:

$$(T(t)f)(\theta) := f(\sigma(\theta, t)), \text{ for } \theta \in \Theta,\ t \geq 0,\ f \in X \qquad (1.5.1)$$

ii) The generator $(A, D(A))$ of $\{T(t)\}_{t\geq 0}$ is a derivation.

iii) Every strongly continuous semigroup $\{T(t)\}_{t\geq 0}$ on X that consists of algebra homomorphisms originates, via (1.5.1), from a continuous two-parameter (nonlinear) semiflow on Θ. (see[17 ,B-II, Thm. 3.4])

For details we refer the reader to [8, page 95].

Taking into account the relation between two-parameter (nonlinear) semiflows and one-parameter semigroups (see the example above) we consider in the next a three-parameter (nonlinear) semiflow to approach and extend some well-known theorems given in the case of two-parameter evolution families. We will define the three-parameter (nonlinear) (semi)flow as in [23].

Definition 1.6. *A three-parameter(nonlinear) semiflow $\sigma : \Theta \times \Delta \to \Theta$ is defined by the properties:*

i) $\sigma(\theta, t, t) = \theta$, for all $t \in \mathbb{R}_+$, and all $\theta \in \Theta$;

ii) $\sigma(\sigma(\theta, s, t_0), t, s) = \varphi(\theta, t, t_0)$ for all $t \geq s \geq t_0 \geq 0$, and all $\theta \in \Theta$.

If in addition $(\theta, t, t_0) \mapsto \sigma(\theta, t, t_0)$ is continuous then σ is called a *continuous* three-parameter (nonlinear) semiflow on Θ.

If instead of $\Delta = \{(t, t_0) \in \mathbb{R}^2 : t \geq t_0 \geq 0\}$ we consider $\mathbb{I} \times \mathbb{I}$ (where $\mathbb{I}$ can be any of the intervals $(-\infty, o], [0, \infty)$ or $\mathbb{R}$) then σ will be called a three-parameter flow on Θ.

Remark 1.7. It can be seen that the above concept is a natural extension of the classical two-parameter semiflow and it appears when we pass to nonautonomous differential equations. The set Θ is the so-called *phase space* and $\mathbb{I} \times \Theta$ is called the *extended phase space*. It is known that, in the finite-dimensional setting, the general solution of a nonautonomous differential equation $\dot{x} = f(t, x)$ is a three-parameter flow (of course, if $f : \mathbb{R} \times \mathbb{R}^n \to \mathbb{R}^n$ satisfy conditions guaranteeing global existence and uniqueness of solutions). In case $\Theta = \mathbb{R}^n$, a three-parameter flow is called linear if

$$\sigma(a\theta_1 + b\theta_2, t, t_0) = a\sigma(\theta_1, t, t_0) + b\sigma(\theta_2, t, t_0),$$

for all $\theta_1, \theta_2 \in \mathbb{R}^n, a, b \in \mathbb{R}$, and $t, t_0 \in \mathbb{I}$.

For instance a linear three-parameter flow is generated by a linear nonautonomous differential equation $\dot{x} = A(t)x$, where $A : \mathbb{R} \to \mathbb{R}^{n \times n}$ is continuous. Given a linear three-parameter flow there exists a corresponding matrix valued function $\Phi : \mathbb{I} \times \mathbb{I} \to \mathbb{R}^{n \times n}$ with $\Phi(t, t_0)\theta = \sigma(\theta, t, t_0)$ for all $t, t_0 \in \mathbb{R}$ and $\theta \in \mathbb{R}^n$.

Definition 1.8. *A family of linear and bounded operators $\{U(t, t_0)\}_{t \geq t_0 \geq 0}$ is said to be a two-parameter evolution family if it satisfies to the following conditions:*

i) $U(t, t) = I$, for all $t \geq 0$;

ii) $U(t, s)U(s, t_0) = U(t, t_0)$, for all $t \geq s \geq t_0 \geq 0$;

iii) $U(\cdot, t_0)x$ is continuous on $[t_0, \infty)$, for all $t_0 \geq 0,\ x \in X$;

$U(t, \cdot)x$ is continuous on $[0, t]$, for all $t \geq 0,\ x \in X$;

If in addition $\{U(t, t_0)\}_{t \geq t_0 \geq 0}$ satisfies the condition below

iv) there exist $M, \omega > 0$ such that

$$||U(t, t_0)|| \leq Me^{\omega(t - t_0)}, \text{ for all } t \geq t_0 \geq 0.$$

then we say that $\{U(t, t_0)\}_{t \geq t_0 \geq 0}$ is an evolution family with exponential growth.

For a general presentation of the theory of two-parameter evolution families we refer the reader to [2] and [5].

Example 1.9. *Take $X = \mathcal{C}(\Theta)$. If σ is a continuous three-parameter (nonlinear)semiflow on Θ, then*

$$(U(t, t_0))(f)(\theta) = f(\sigma(\theta, t, t_0)).$$

defines a two-parameter evolution family on X.

Example 1.10. *If $\Theta = \mathbb{R}_+$ (or $\mathbb{R}$) then $\sigma : \Theta \times \Delta \to \Theta$, $\sigma(\theta, t, t_0) = \theta + t - t_0$ is a linear three-parameter semiflow on Θ.*

As we pointed out in the introduction section, the qualitative theory of (nonlinear) (semi)flows relies on notions like stability or dichotomy for the associated linear skew-product (semi)flow. Taking into account this aspect, we will associate in the classical fashion a linear skew-product three-parameter semiflow to the above three-parameter (nonlinear) (semi)flow. It is interesting to note that the linear skew-product three-parameter semiflow that we consider, arises as solution operators for the variational equation:

$$\begin{cases} \dot{x}(t) = A(\sigma(\theta, t, t_0))x(t), & t \geq t_0 \geq 0 \\ x(t_0) = x_0 \end{cases}$$

For details we refer the reader to Example 1.12.

Definition 1.11. *The pair $\pi = (\Phi, \sigma)$ is said to be a linear skew-product three-parameter semiflow on X if $\Phi : \Theta \times \Delta \to \mathcal{B}(X)$ satisfies the following properties:*

i) $\Phi(\theta, t, t) = I$, for all $t \in \mathbb{R}_+$ and all $\theta \in \Theta$, where I represents the identity operator on X;

ii) $\Phi(\sigma(\theta, s, t_0), t, s)\Phi(\theta, s, t_0) = \Phi(\theta, t, t_0)$, for all $t \geq s \geq t_0 \geq 0$ and all $\theta \in \Theta$.

iii) $t \mapsto \Phi(\theta, t, t_0)x : [t_0, \infty) \to X$ is continuous;
$\tau \mapsto \Phi(\theta, t, \tau)x : [0, t] \to X$ is continuous;

iv) there exists $M, \omega \in \mathbb{R}$, $M \geq 1$ such that

$$||\Phi(\theta, t, t_0)|| \leq Me^{\omega(t-t_0)}, \text{ for all } t \geq t_0 \geq 0, \text{ and } \theta \in \Theta.$$

Example 1.12. *Let σ be a continuous three-parameter (nonlinear) semiflow on Θ, $A : \Theta \to \mathcal{B}(X)$ a continuous map and f a locally integrable function on X.*

It is easy to see that the solution of the homogeneous Cauchy problem:

$$\begin{cases} \dot{x}(t) = A(\sigma(\theta, t, t_0))x(t), & t \geq t_0 \geq 0 \\ x(t_0) = x_0 \end{cases}$$

verifies the integral equation

$$x(t) = x_0 + \int_{t_0}^{t} A(\sigma(\theta, \tau, t_0))x(\tau)d\tau, \quad (1.12.1)$$

and that of the inhomogeneous Cauchy problem:

$$\begin{cases} \dot{x}(t) = A(\sigma(\theta, t, t_0))x(t) + f(t), & t \geq t_0 \geq 0 \\ x(t_0) = x_0 \end{cases}$$

verifies the integral equation

$$x(t) = \Phi(\theta, t, t_0)x_0 + \int_{t_0}^{t} A(\sigma(\theta, \tau, t_0), t, \tau)x(\tau)d\tau + \int_{t_0}^{t} f(\tau)d\tau \quad (1.12.2)$$

due to the fact that in both cases, the solution of the variational Cauchy problem is an absolutely continuous function.

With a similar argument as in the proof of the existence and uniqueness theorems for the non-autonomous systems (see for instance J.L. Daleckij, M.G. Krein [5] and J.L. Massera, J.J. Schäffer [15]), one can show that the solution of the variational homogeneous equation (1.12.1) is $\Phi(\theta, t, t_0)x_0 = x(t)$, where $\pi = (\Phi, \sigma)$ is a linear skew-product three-parameter semiflow, and that (1.12.2) has the solution

$$x(t) = \Phi(\theta, t, t_0)x_0 + \int_{t_0}^{t} \Phi(\sigma(\theta, \tau, t_0), t, \tau) f(\tau) d\tau.$$

Example 1.13. *Let $\Theta = \mathbb{R}$ and σ be the three-parameter linear semiflow given in Example 1.10. Considering the continuous and bounded mapping $a : \mathbb{R} \to \mathbb{R}$ and the variational equation*

$$\begin{cases} \dot{x}(t) = a(\theta + t - t_0)x(t) \\ x(t_0) = 1 \end{cases}$$

Then

$$x(t) = e^{\int_{t_0}^{t} a(\theta + s - t_0) ds}.$$

By taking

$$\Phi(\theta, t, t_0) = e^{\int_{t_0}^{t} a(\theta + s - t_0) ds}$$

we have that $\pi = (\Phi, \sigma)$ is a linear skew-product three-parameter semiflow on Θ.

Example 1.14. *If $\{U(t, t_0)\}_{t \geq t_0 \geq 0}$ is a two-parameter evolution family (on X) with exponential growth and σ is a three-parameter semiflow on Θ, then $\pi = (\Phi, \sigma)$ is a linear skew-product three-parameter semiflow, where*

$$\Phi(\theta, t, t_0) = U(t, t_0), \text{ for all } t \geq t_0 \geq 0.$$

Conversely, considering $\Theta = \mathbb{R}_+$, $\sigma : \mathbb{R}_+ \times \Delta \to \mathbb{R}_+$, $\sigma(\theta, t, t_0) = \theta$, and $\pi = (\Phi, \sigma)$ a linear skew-product three-parameter semiflow on X, we have that $\{U(t, t_0)\}_{t \geq t_0 \geq 0}$, $U(t, t_0) = \Phi(0, t, t_0)$ is an evolution family on X.

Thus, we can consider that evolution families with exponential growth are particular cases of linear skew-product three-parameter semiflows.

Example 1.15. *Let $\Theta = \mathbb{R}_+$ and σ be the linear three-parameter semiflow given in Example 1.10. If $\{U(t, t_0)\}_{t \geq t_0 \geq 0}$ is a two-parameter evolution family (on X) with exponential growth, then $\pi = (\Phi, \sigma)$ is a linear skew-product three-parameter semiflow, where*

$$\Phi(\theta, t, t_0) = U(\theta + t - t_0, \theta), \text{ for all } t \geq t_0 \geq 0.$$

Example 1.16. *Let $\mathcal{A}$ be the set of all continuous functions $u : \mathbb{R} \to \mathbb{R}$ endowed with the topology of uniform convergence on compact sets. $\mathcal{A}$ is metrizable with respect to the following distance:*

$$\delta(u, v) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{\delta_n(u, v)}{1 + \delta_n(u, v)}, \text{ where } \delta_n = \sup_{t \in [-n, n]} |u(t) - v(t)|.$$

Let $u_n : \mathbb{R}_+ \to \left(\frac{1}{2n+1}, \frac{1}{2n}\right)$, $n \in \mathbb{N}^*$ be a decreasing function with

$$\lim_{t\to\infty} u_n(t) = \frac{1}{2n+1},$$

we denote $u_n^{t_0} = u_n(t+t_0),\ t, t_0 \geq 0.$

Let Θ *be the closure (in* $\mathcal{A}$*) of the set* $\{u_n^{t_0},\ n \in \mathbb{R}_+\}$*. The mapping*

$$\sigma : \Theta \times \Delta \to \Theta,\ \sigma(u, t, t_0) = u_{t-t_0},\ \textit{where } u_{t-t_0}(s) = u(t - t_0 + s),\ s \geq 0$$

is a three-parameter semiflow on Θ*. Let* $X = \mathbb{R}^n$*,* $n \geq 1$ *endowed with the norm* $||(x_1, ..., x_n)|| = |x_1| + ... + |x_n|$*.*

The pair $\pi = (\Phi, \sigma)$ *is a linear three-parameter skew-product semiflow for*

$$\Phi : \Theta \times \Delta \to \mathcal{B}(X),\ \Phi(u, t, t_0) = \left(e^{a_1 \int_{t_0}^{t} u(s-t_0)ds} x_1, ..., e^{a_n \int_{t_0}^{t} u(s-t_0)ds} x_n\right)$$

Definition 1.17. *Let* $\pi = (\Phi, \sigma)$ *be a linear skew-product three-parameter semiflow.* $\pi = (\Phi, \sigma)$ *is said to be uniformly exponentially stable if there exists* $N \geq 1$ *and* $\nu > 0$ *such that:*

$$||\Phi(\theta, t, t_0)x|| \leq Ne^{-\nu(t-t_0)}||x||, \textit{ for all } x \in X,\ \theta \in \Theta, \textit{ and } t \geq t_0 \geq 0.$$

Definition 1.18. *Let* $\pi = (\Phi, \sigma)$ *be a linear skew-product three-parameter semiflow.* $\pi = (\Phi, \sigma)$ *is said to be uniformly exponentially unstable if there exists* $N \geq 1$ *and* $\nu > 0$ *such that:*

$$||\Phi(\theta, t, t_0)x|| \geq Ne^{\nu(t-t_0)}||x||, \textit{ for all } x \in X,\ \theta \in \Theta, \textit{ and } t \geq t_0 \geq 0.$$

Remark 1.19. A trivial consequence of Definition 1.18. is that $\Phi(\theta, t, t_0)$ is one-to-one, for each $\theta \in \Theta$ and $t \geq t_0 \geq 0$.

2 Results

In this section we will extend the well-known Datko's theorem (regarding the uniform exponential stability of two-parameters evolution families) to the case of linear skew-product three-parameter semiflows. Variants for the exponential instability are also given.

For convenience, we will briefly introduce the reader to Datko's theorem and related results. The history of this subject goes back to 1970, when Datko shows in [6] that all the trajectories $T(\cdot)x$ (of a C_0-semigroup $\{T(t)\}_{t\geq 0}$) have an exponential decay as $t \to \infty$, if and only if, the function $t \to ||T(t)x||$ lies in $L^2(\mathbb{R}_+, \mathbb{R})$, for each vector $x \in X$. Later, A.Pazy shows in [18] that the result remains valid if we replace $L^2(\mathbb{R}_+, \mathbb{R})$ with $L^p(\mathbb{R}_+, \mathbb{R})$, where $p \in [1, \infty)$. In 1972, R.Datko generalizes the above result for two-parameter evolution families. He points out in [7] that a two-parameter evolution family $\{U(t, t_0)\}_{t\geq t_0 \geq 0}$ with exponential growth is uniformly exponentially stable (i.e. there exist $N, \nu > 0$ such that $||U(t, t_0)|| \leq Ne^{-\nu(t-t_0)}$, for all $t \geq t_0 \geq 0$) if and only if there exists $p \in [1, \infty)$ such that $\sup_{t_0 \geq 0} \int_{t_0}^{\infty} ||U(t, t_0)x||^p dt < \infty$, for each $x \in X$. An improvement of the above Datko's result was obtained by Rolewicz in 1986 (see [24]) when he proved that if $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, nondecreasing function with $\varphi(0) = 0$ and $\varphi(u) > 0$ for each strictly positive u, and if $\{U(t,t_0)\}_{t \geq t_0 \geq 0}$ is an evolution family (on X) with exponential growth such that $\sup\limits_{t_0 \geq 0} \int\limits_{t_0}^{\infty} \varphi(||U(t,t_0)x||)dt < \infty$ for each $x \in X$ then $\{U(t,t_0)\}_{t \geq t_0 \geq 0}$ is uniformly exponentially stable. A shorter proof of the Rolewicz theorem was given by Q. Zheng [30] who removed the continuity assumption about φ. Also we note that an analogous result was obtained independently by Littman [14] in 1989, in the case of C_0-semigroups, and again without the assumption of continuity for φ. A discrete-time version of Rolewicz theorem was obtained in 1974 by Zabczyk [29] for the particular case of C_0-semigroups.

We establish below a technical characterization of the uniform exponential stability for a linear skew-product three-parameter semiflow.

Theorem 2.1. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product three-parameter semiflow. Then $\pi = (\Phi, \sigma)$ is uniformly exponentially stable if and only if there exists $c \in (0,1)$ and $h > 0$ such that for each $t_0 \geq 0$, $\theta \in \Theta$ and $x \in X$ we can find $u \in (0,h]$ (u depends on t_0, θ and x) with the property that*

$$||\Phi(\theta, t_0 + u, t_0)x|| \leq c||x||. \qquad (2.1.1)$$

Proof. The *necessity* is immediate if we consider in Definition 1.17, $t = t_0 + u$ and $c = e^{-\nu h}$.

For the *sufficiency* we consider $x \in X$, $\theta \in \Theta$ and $t_0 \geq 0$.Then there exists $u \in (0,h]$ such that:

$$||\Phi(\theta, t_0 + u, t_0)x|| \leq c||x||.$$

Denoting $y = \Phi(\theta, t_0 + u, t_0)x$, $\theta' = \sigma(\theta, t_0 + u, t_0)$ and $t' = t_0 + u$ and applying (2.1.1) we obtain that there exists $u' \in (0,h]$ such that:

$$||\Phi(\sigma(\theta, t', t_0), t_0 + u', t')\Phi(\theta, t', t_0)x|| \leq c^2||x||, \text{ for all } x \in X.$$

Consider now $s_0 = t_0$, $s_1 = s_0 + u$, $s_2 = s_1 + u'$, and keep going like this we can find a sequence $s_{n+1} > s_n$, $s_{n+1} - s_n \in (0,h]$ such that

$$||\Phi(\theta, s_n, t_0)x|| \leq c^n||x||, \text{ for all } n \in \mathbb{N} \text{ and } x \in X.$$

Let $t > t_0 \geq 0$ and $\theta \in \Theta$. Then if $s_n \to \infty$ we obtain that there exists $n \in \mathbb{N}$ such that $s_n \leq t < s_{n+1}$, which implies that

$$||\Phi(\theta, t, t_0)x|| = ||\Phi(\sigma(\theta, s_n, t_0)t, s_n)\Phi(\theta, s_n, t_0)x|| \leq Me^{\omega(s_n - t)}c^n||x||$$

$$\leq Me^{\omega h}c^n||x|| = Me^{\omega h}e^{-\nu n h}e^{\nu h}e^{-\nu h}||x||$$

$$= \frac{Me^{\omega h}}{c}e^{-\nu(n+1)h}||x|| \leq \frac{Me^{\omega h}}{c}e^{-\nu(t - t_0)}||x||,$$

where $\nu = -\frac{1}{h}\ln c > 0$ and $N = \frac{Me^{\omega h}}{c}$.

On the other hand, if $s_n \to s \in \mathbb{R}_+$ then $s \geq t_0$. Since

$$||\Phi(\theta, s_n, t_0)x|| \leq c^n ||x||, \text{ for all } n \in \mathbb{N},$$

for $n \to \infty$ it follows that

$$||\Phi(\theta, s, t_0)|| = 0,$$

which implies that

$$\Phi(\theta, s, t_0)x = 0.$$

Consider now $t \geq s \geq t_0$. Then

$$\Phi(\theta, t, t_0)x = \Phi(\sigma(\theta, s, t_0)t, s)\Phi(\theta, s, t_0)x = 0.$$

Furthermore, if $s > t \geq t_0$ and $s_n \to s$ we have that there exists $n \in \mathbb{N}$ such that $s_n \leq t < s_{n+1}$. This leads us to

$$||\Phi(\theta, t, t_0)x|| = ||\Phi(\sigma(\theta, s_n, t_0)t, s_n)\Phi(\theta, s_n, t_0)x|| \leq$$

$$Me^{\omega h} c^n ||x|| \leq \frac{Me^{\omega h}}{c} e^{-\nu(t-t_0)} ||x||.$$

Therefore

$$||\Phi(\theta, t, t_0)x|| \leq Ne^{-\nu(t-t_0)} ||x||, \text{ for all } t \geq t_0 \geq 0,\ x \in X \text{ and } \theta \in \Theta,$$

where $\nu = -\frac{1}{h} \ln c > 0$ and $N = \frac{Me^{\omega h}}{c}$. ∎

Theorem 2.2. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product three-parameter semiflow. Then $\pi = (\Phi, \sigma)$ is exponentially stable if and only if there exist $k, p > 0$ such that*

$$\Big(\int_{t_0}^{\infty} ||\Phi(\theta, t, t_0)x||^p dt\Big)^{\frac{1}{p}} \leq k||x||, \textit{ for all } x \in X,\ \theta \in \Theta,\ t_0 \geq 0.$$

Proof. The *necessity* is immediate, taking $k = \frac{1}{\nu p}$. For the *sufficiency* we assume that Φ is not exponentially stable. Applying Theorem 2.1 we obtain that for all $c \in (0, 1)$ and all $h > 0$ there exist $t_0 \geq 0$, $\theta \in \Theta$ and $x \in X$ with $||x|| = 1$ such that

$$||\Phi(\theta, t_0 + u, t_0)x|| > c, \text{ for all } u \in (0, h].$$

This implies that

$$\Big(\int_0^h ||\Phi(\theta, t_0 + u, t_0)x||^p du\Big)^{\frac{1}{p}} > ch^{\frac{1}{p}}$$

and therefore

$$\Big(\int_{t_0}^{t_0+h} ||\Phi(\theta, s, t_0)x||^p ds\Big)^{\frac{1}{p}} > ch^{\frac{1}{p}}.$$

Thus

$$k \geq \Big(\int_{t_0}^{\infty} ||\Phi(\theta, s, t_0)x||^p ds\Big)^{\frac{1}{p}} \geq \Big(\int_{t_0}^{t_0+h} ||\Phi(\theta, s, t_0)x||^p ds\Big)^{\frac{1}{p}} > ch^{\frac{1}{p}},$$

for all $c \in (0, 1)$ and $h > 0$. For $h \to \infty$ we obtain that

$$\frac{k}{c} \geq \infty,$$

which is a contradiction. ∎

Theorem 2.3. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product three-parameter semiflow, such that $\Phi(\theta, t, t_0)$ is one-to-one, for each $\theta \in \Theta$ and $t \geq t_0 \geq 0$. If there exists $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a nondecreasing function, $\varphi(t) > 0$ for all $t > 0$, then $\pi = (\Phi, \sigma)$ is exponentially stable if and only if there exists $k > 0$ such that*

$$\int_{t_0}^{\infty} \varphi(||\Phi(\theta, t, t_0)x||)dt \leq k\varphi(||x||), \text{ for all } x \in X,\ \theta \in \Theta, \text{ and } t_0 \geq 0.$$

Proof. It is similar to that of Theorem 2.2., taking into consideration the properties of φ. ■

Corollary 2.4. *(R. Datko, 1972)*

Let $\{U(t, t_0)\}_{t \geq t_0 \geq 0}$ be a two-parameter evolution family with exponential growth. Then $\{U(t, t_0)\}_{t \geq t_0 \geq 0}$ is uniformly exponentially stable (i.e. there exist $N, \nu > 0$ such that $||U(t, t_0)|| \leq Ne^{-\nu(t-t_0)}$, for all $t \geq t_0 \geq 0$) if and only if there exists $p \in [1, \infty)$ such that

$$\sup_{t_0 \geq 0} \int_{t_0}^{\infty} ||U(t, t_0)x||^p dt < \infty, \text{for each } x \in X.$$

Proof. It follows immediately from Theorem 2.2 and Example 1.14. ■

Corollary 2.5. *(S. Rolewicz, 1986)*

If $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ is a continuous, nondecreasing function with $\varphi(0) = 0$ and $\varphi(u) > 0$ for each strictly positive u, and if $\{U(t, t_o)\}_{t \geq t_0 \geq 0}$ is a two-parameter evolution family (on X) with exponential growth,c such that

$$\sup_{t_0 \geq 0} \int_{t_0}^{\infty} \varphi(||U(t, t_0)x||)dt < \infty \text{ for each } x \in X$$

then $\{U(t, t_0)\}_{t \geq t_0 \geq 0}$ is uniformly exponentially stable.

Proof. It follows trivially from Theorem 2.3. and Example 1.14. ■

We obtain below variants of Theorem 2.1, Theorem 2.2, Theorem 2.3 for the exponential instability of a linear skew-product three-parameter semiflow.

Theorem 2.6. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product three-parameter semiflow. Then $\pi = (\Phi, \sigma)$ is exponentially unstable if and only if there exist $c, h > 1$ such that for all $t_0 \geq 0$, $\theta \in \Theta$ and $x \in X$ we can find $u \in (0, h]$ (u depends on θ, t_0 and x) with the property:*

$$||\Phi(\theta, t_0 + u, t_0)x|| \geq c||x||. \qquad (2.6.1.)$$

Proof. The *necessity* is immediate and for *sufficiency* let us consider $x \in X$ and $\theta \in \Theta$. From the hypothesis we have that there exist $c, h > 1$ and $u \in (0, h]$ such that

$$||\Phi(\theta, t_0 + u, t_0)x|| \geq c||x||.$$

Taking $y = \Phi(\theta, t_0 + u, t_0)x$, $\theta' = \sigma(\theta, t_0 + u, t_0)$ and $t' = t_0 + u$, applying (2.6.1.) we have that there exist $u' \in (0, h]$ such that

$$||\Phi(\sigma(\theta, t', t_0), t_0 + u', t')\Phi(\theta, t', t_0)x|| \geq c^2||x||.$$

Consider now $s_0 = t_0$, $s_1 = s_0 + u$, $s_2 = s_1 + u'$ and keep going like this we can find a sequence $(s_n)_{n\in\mathbb{N}}$, $s_{n+1} > s_n$ such that

$$||\Phi(\theta, s_n, t_0)x|| \geq c^n ||x||, \text{ for all } n \in \mathbb{N}.$$

Let $t > t_0 \geq 0$ and $\theta \in \Theta$. If $s_n \to s \in \mathbb{R}_+$ and $s \geq t_0$ and taking into account that

$$||\Phi(\theta, s_n, t_0)x|| \geq c^n ||x||$$

we have that

$$||\Phi(\theta, s, t_0)x|| \geq \infty, \text{ for } n \to \infty,$$

which is a contradiction, and therefore $s_n \to \infty$.

Consider now $t \geq 0$. Then there exist $n \in \mathbb{N}$ such that $s_n \leq t < s_{n+1}$, which implies that

$$||\Phi(\theta, s_{n+1}, t_0)x|| = ||\Phi(\sigma(\theta, t, t_0), s_{n+1}, t)\Phi(\theta, t, t_0)x||,$$

and therefore

$$c^{n+1}||x|| \leq Me^{\omega(t-s_{n+1})}||\Phi(\theta, t, t_0)x||.$$

Since $s_{n+1} - s_n \in (0, h]$ we have that $t - s_{n+1} \in (0, h]$ and

$$c^{n+1}||x|| \leq Me^{\omega h}||\Phi(\theta, t, t_0)x||.$$

Denoting $c = e^{\nu h}$, $\nu > 0$ we obtain that

$$e^{\nu(n+1)h}||x|| \leq Me^{\omega h}||\Phi(\theta, t, t_0)x||.$$

From $t < s_{n+1} \leq t_0 + (n+1)h$ it results that

$$e^{\nu(t-t_0)}||x|| \leq Me^{\omega h}||\Phi(\theta, t, t_0)x||$$

and therefore

$$||\Phi(\theta, t, t_0)x|| \geq \frac{1}{Me^{\omega h}} e^{\nu(t-t_0)}||x||, \text{ for all } t \geq t_0 \geq 0,\ \theta \in \Theta, \text{ and } x \in X,$$

which implies that Φ is exponentially unstable. ∎

Theorem 2.7. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product three-parameter semiflow, such that $\Phi(\theta, t, t_0)$ is one-to-one, for each $\theta \in \Theta$ and $t \geq t_0 \geq 0$. Then $\pi = (\Phi, \sigma)$ is exponentially unstable if and only if there exist $k, p > 0$ such that*

$$\left(\int_{t_0}^{\infty} \frac{dt}{||\Phi(\theta, t, t_0)x||^p}\right)^{\frac{1}{p}} \leq \frac{k}{||x||}, \text{ for all } x \in X \setminus \{0\}, \text{ for all } \theta \in \Theta,\ t_0 \geq 0.$$

Proof. The *necessity* is trivial and for the *sufficiency* we assume that Φ is not exponentially unstable. From Theorem 2.6 we have that for all $c > 1$ and $h > 0$ there exist $t_0 \geq 0$, $\theta \in \Theta$ and $x \in X$ with $||x|| = 1$ such that

$$||\Phi(\theta, t_0 + u, t_0)x|| < c, \text{ for all } u \in (0, h].$$

Thus

$$\frac{1}{||\Phi(\theta, t_0 + u, t_0 x)||} > \frac{1}{c},$$

and therefore

$$\Big(\int_0^h \frac{du}{||\Phi(\theta, t_0 + u, t_0)x||^p}\Big)^{\frac{1}{p}} > \frac{h^{\frac{1}{p}}}{c}.$$

This implies that

$$\Big(\int_{t_0}^{t_0+h} \frac{ds}{||\Phi(\theta, s, t_0)x||^p}\Big)^{\frac{1}{p}} > \frac{h^{\frac{1}{p}}}{c},$$

which leads to

$$k > \Big(\int_{t_0}^{\infty} \frac{ds}{||\Phi(\theta, s, t_0)x||^p}\Big)^{\frac{1}{p}} \geq \Big(\int_{t_0}^{t_0+h} \frac{ds}{||\Phi(\theta, s, t_0)x||^p}\Big)^{\frac{1}{p}} > \frac{h^{\frac{1}{p}}}{c},$$

and for $h \to \infty$ we get that $kc \geq \infty$, which is a contradiction. ■

Corollary 2.8. *Let $\pi = (\Phi, \sigma)$ be a linear skew-product three-parameter semiflow, such that $\Phi(\theta, t, t_0)$ is one-to-one, for each $\theta \in \Theta$ and $t \geq t_0 \geq 0$. If there exists $\varphi : \mathbb{R}_+ \to \mathbb{R}_+$ a nondecreasing function, $\varphi(t) > 0$ for all $t > 0$, then $\pi = (\Phi, \sigma)$ is exponentially unstable if and only if there exists $k > 0$ such that*

$$\int_{t_0}^{\infty} \varphi\Big(\frac{1}{||\Phi(\theta, t, t_0)x||}\Big)dt \leq k\varphi\Big(\frac{1}{||x||}\Big), \textit{ for all } x \in X \setminus \{0\},\ \theta \in \Theta,\ t_0 \geq 0$$

Proof. It is similar to that of Theorem 2.7, taking into consideration the properties of φ. ■

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