Very ample linear series on real algebraic curves*

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Abstract

In this paper we transfer results on embeddings of complex algebraic curves into projective space (which are based on dimension theorems for special divisors) to the case of real algebraic curves with real points.

1 Introduction and Terminology

Let Y be a complex algebraic curve, i.e. a smooth and irreducible projective curve defined over $\mathbb{C}$, and let g be its genus. The distribution of complete and very ample linear series g^r_d of degree d and (projective) dimension r on Y is only known if they are not very special, i.e. if $h^1(Y, D) \leq 1$ holds for the index of speciality of a divisor $D \in g^r_d$: Y always has complete, very ample and non-special series of dimension r for all integers $r \geq 3$, and there is no complete, very ample and special series on Y if and only if Y is hyperelliptic. More precisely, Y has a complete, very ample, special but not very special linear series of dimension r (i.e. a complete g^r_{g-1+r}) for every integer r such that $3 \leq r < g$ provided that Y is not hyperelliptic, not bi-elliptic (i.e. not a double cover of an elliptic curve), not trigonal and not a smooth plane quintic; this statement is a consequence of Mumford's dimension theorem for the varieties of special divisors. (Cf. [ACGH], V, Ex. B; [CM1], 2.2 (iv); by the way, for $g \geq 6$ also the converse of this statement is true).

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It is the *aim of this paper* to prove the corresponding very ampleness-results for complete linear series on a real algebraic curve, i.e. on a smooth and geometrically irreducible projective curve defined over $\mathbb{R}$. We always assume that *the real curve has real points*. (On a real curve without real points linear series are restricted to even degree.)

Terminology: For details on real curves and the language of linear series on them we refer to [CM 2]. Linear series are always complete in this paper. X denotes a real (algebraic) curve of genus g, and $(X_\mathbb{C}, \sigma)$ denotes the complexification of X(σ is the anti-analytic involution on $X_\mathbb{C}$ induced by complex conjugation $^-$ on $\mathbb{C}$). If we assume that the set $X(\mathbb{R})$ of real points of X (or, what is the same, of $X_\mathbb{C}$) is non-empty it splits into $1 \leq s \leq g+1$ real components $C_1, \ldots, C_s$ of X, and we can identify the linear series g^r_d on the real curve X with the σ-invariant g^r_d on the complex curve $X_\mathbb{C}$ ([GH], section 2; [CM2], section 1). Such a "real" g^r_d on $X_\mathbb{C}$ defines, for $r \geq 1$, a rational map $\varphi : X_\mathbb{C} \to \mathbb{P}_\mathbb{C}{}^r$ which is σ-invariant (i.e. $\varphi(P^\sigma) = \overline{\varphi(P)}$ for any point $P \in X_\mathbb{C}$ where it is defined; this is due to the fact that $H^0(X_\mathbb{C}, D)$ for $D \in g^r_d$ has a basis of real functions because $(g^r_d)^\sigma = g^r_d$); so φ induces a map $\varphi : X \to \mathbb{P}^r_\mathbb{R} = \mathrm{Proj}(\mathbb{R}[x_0, \ldots, x_r])$.

A g^r_d on X is called *very ample* if the associated σ-invariant g^r_d on $X_\mathbb{C}$ is very ample, i.e. if the induced map φ is an isomorphism onto the image curve in $\mathbb{P}^r_\mathbb{C}$; then φ identifies the real curve X with a real curve in the real projective space $\mathbb{P}^r_\mathbb{R}$. The very ample g^r_d on X correspond to the elements of the subset $W^r_d(\mathbb{R}) \backslash (W^{r-1}_{d-2} + W_2)(\mathbb{R})$ of the Jacobian variety $\mathrm{Jac}(X_\mathbb{C})$ of $X_\mathbb{C}$; here W^r_d resp. $W^r_d(\mathbb{R})$ represents the set of g^r_d resp. of σ-invariant g^r_d on $X_\mathbb{C}$. A linear series g^r_d on X belonging to the - in general larger - set $W^r_d(\mathbb{R}) \backslash (W^{r-1}_{d-2}(\mathbb{R}) + W_2(\mathbb{R}))$ still provides a nice geometric description of X: it is easy to see that such a series is base point free and simple (i.e. φ is a birational morphism from $X_\mathbb{C}$ onto its image; so $r \geq 2$), and it has the defining two properties that the morphism φ separates conjugate points of $X_\mathbb{C}$ (i.e. $\varphi(P^\sigma) \neq \varphi(P)$ for every non-real point P of $X_\mathbb{C}$; in particular, the image curve $\varphi(X) \subset \mathbb{P}^r_\mathbb{R}$ has no isolated real points) and separates points and tangent vectors on $X(\mathbb{R})$ (so $X_\mathbb{C}$ has its singular points outside $X(\mathbb{R})$; in particular, the real contours $\varphi(C_1), \ldots, \varphi(C_s)$ of $\varphi(X_\mathbb{C})$ do not meet).

Note that these notions are also meaningful for a real curve without real points; in this paper, however, we assume that $X(\mathbb{R}) \neq \phi$ (i.e. $s \geq 1$).

Example 1: There are real curves of genus 4 with a given number $s, 1 \leq s \leq 5$, of real components and without a g^1_3 ([GH], p. 178). Let X be such a curve and let $|K_{X_\mathbb{C}}|$ denote the canonical series of $X_\mathbb{C}$. Since $|K_{X_\mathbb{C}}|$ is σ-invariant so is $|K_{X_\mathbb{C}} - P|$ for any $P \in X_\mathbb{C}(\mathbb{R}) = X(\mathbb{R})$; thus the ∞^1 nets g^2_5 on X belong to $(W^1_3 + W_2)(\mathbb{R})$ but clearly not to $W^1_3(\mathbb{R}) + W_2(\mathbb{R}) = \phi$. ∎

A *pseudo-line* for a g^r_d $(r \geq 1)$ on X is a real component of X on which some (hence every) divisor in this g^r_d has odd degree. The number δ of pseudo-lines for a g^r_d on X obviously satisfies $0 \leq \delta \leq$ Min (s, d) and $\delta \equiv d \bmod 2$, and if the g^r_d is special (i.e. $r > d - g$) then $\delta \leq 2g - 2 - d$ ([H]). In particular, the canonical series $|K_{X_{\mathbb{C}}}|$ of $X_{\mathbb{C}}$ has no pseudo-lines.

2 Non-special very ample series

We identify the real points of X with the σ-invariant points of $X_{\mathbb{C}}$. Recall that X has $s \geq 1$ real components.

Lemma 1: *Let d, δ be non-negative integers such that $\delta \leq$ Min(s, d) and $\delta \equiv d$ mod 2. Choose δ general points $P_1, \ldots, P_\delta$ on real components $C_1, \ldots, C_\delta$ of X, respectively, and $\frac{d-\delta}{2}$ general points $Q_1, \ldots, Q_{\frac{d-\delta}{2}}$ in $X_{\mathbb{C}}$. Let D be the real ($=\sigma$-invariant) and effective divisor $P_1 + \cdots + P_\delta + (Q_1 + Q_1^\sigma) + \cdots + (Q_{\frac{d-\delta}{2}} + Q^\sigma_{\frac{d-\delta}{2}})$ of degree d on $X_{\mathbb{C}}$. Then* $\dim |D| =$Max$(0, d-g)$.

Proof: Clearly, by Riemann-Roch, $r := \dim |D| \geq$ Max $(0, d-g)$. Our real divisors D of degree d on $X_{\mathbb{C}}$ form a family of real dimension at least d. Taking the complete linear series they define we see that $\dim_{\mathbb{R}}(W^r_d(\mathbb{R})) \geq d - r$.
Let $d < g$. Then Max$(0, d-g) = 0$, and we have $d - r \leq \dim_{\mathbb{R}}(W^r_d(\mathbb{R})) \leq \dim(W^r_d) \leq d - 2r$, by Martens' dimension theorem ([ACGH], IV, 5.1). Hence $r = 0$, and we are done.

Let $d \geq g$, i.e. Max$(0, d-g) = d - g$. Assume that $r > d - g$. Then $r > 0$. Let $d' := 2g - 2 - d$, $r' := g - 1 - d + r \geq 0$. By Martens' dimension theorem, $d - r \leq \dim_{\mathbb{R}}(W^r_d(\mathbb{R})) \leq \dim(W^r_d) = \dim(\kappa - W^{r'}_{d'}) = \dim(W^{r'}_{d'}) \leq d' - 2r' = d - 2r$, again; here $\{\kappa\} = W^{g-1}_{2g-2}$ is the canonical point on Jac$(X_{\mathbb{C}})$ corresponding to the canonical series $|K_{X_{\mathbb{C}}}|$ of $X_{\mathbb{C}}$. It follows that $r \leq 0$, a contradiction. ∎

Proposition 1: *Let $r \geq 3$ and $\delta \geq 0$ be integers such that $\delta \leq s$ and $\delta \equiv g + r$ mod 2. Then for δ assigned real components $C_1, \ldots, C_\delta$ of X there is a complete and very ample g^r_{g+r} on X having precisely the pseudo-lines $C_1, \ldots, C_\delta$.*

Proof: The proof is just a straightforward modification of the proof of [CM2], Proposition 1. Choose D as in the lemma, with $d := g + r$; then $\dim|D| = d - g = r$. For every real divisor $D' \in |D|$ and every real component C of X we have $\deg(D'|_C) \equiv \deg(D|_C)$ mod 2; so, by construction of D, precisely $C_1, \ldots, C_\delta$ are the pseudo-lines for $|D|$.

Let $r \geq 3$ and assume that no such $|D|$ is very ample. Then these series constitute a $d - r = g$-dimensional subset of Jac$(X_{\mathbb{C}})(\mathbb{R})$ contained in $(W^{r-1}_{g+r-2} + W_2)(\mathbb{R})$. Hence $\dim_{\mathbb{R}}((W^{r-1}_{g+r-2} + W_2)(\mathbb{R})) \geq g$. But $W^{r-1}_{g+r-2} = \kappa - W_{g-r}$ has dimension $g - r$ (if $r \leq g$; otherwise it is empty), and so we have $\dim_{\mathbb{R}}((W^{r-1}_{g+r-2} + W_2)(\mathbb{R})) \leq \dim(W^{r-1}_{g+r-2} + W_2) \leq (g - r) + 2 < g$, a contradiction. ∎

For $r = 2$ we can, of course, only expect a weaker result:

Proposition 2: *Let $\delta \geq 0$ be an integer such that $\delta \leq s$ and $\delta \equiv g \bmod 2$. Then for δ assigned real components $C_1, \ldots, C_\delta$ of X there is a complete, base point free and simple net g^2_{g+2} on X having precisely the pseudo-lines $C_1, \ldots, C_\delta$. And if X is not hyperelliptic (i.e. has no g^1_2) the real plane curve obtained by this net has only points of multiplicity at most 2.*

Proof: Let $W^1_g \times W_2 \to \mathrm{Jac}(X_{\mathbb{C}})$ be the summation map and $W(g,1)$ be the closed and σ-invariant sublocus of $\mathrm{Jac}(X_{\mathbb{C}})$ consisting of points whose fibre under this map is not finite. Then $\mathrm{Jac}(X_{\mathbb{C}})(\mathbb{R}) \backslash W(g,1)(\mathbb{R})$ collects the complete, base point free and simple nets g^2_{g+2} on X. Since $\dim_{\mathbb{R}}(W(g,1)(\mathbb{R})) \leq \dim(W(g,1)) < \dim(W^1_g \times W_2) = \dim(W^1_g) + 2 = \dim(\kappa - W_{g-2}) + 2 = g$ we see that the g-dimensional subset of $\mathrm{Jac}(X_{\mathbb{C}})(\mathbb{R})$ made up by the series $|D|$ constructed as in the proof of Proposition 1 cannot be contained in $W(g,1)(\mathbb{R})$.

Likewise, if the image curve of the morphism defined by $|D|$ has a singular point of multiplicity $m \geq 3$ than $|D|$ represents a point in $(W^1_{g+2-m} + W_m)(\mathbb{R})$. But, by Martens' dimension theorem ([ACGH], IV, 5.1), $\dim(W^1_{g+2-m}) \leq g - 1 - m$ if $X_{\mathbb{C}}$ is not hyperelliptic, and then $\dim(W^1_{g+2-m} + W_m) \leq g - 1$ only. ∎

Example 2: Let $\delta = g$ (maximal) in Proposition 2. By [H], 2.7, then, the g^2_{g+2} on X separates conjugate points, and its induced morphism φ restricts to an isomorphism on every real components of X. Let $\Gamma_1, \ldots, \Gamma_g$ be the images (under φ) of the g pseudo-lines $C_1, \ldots, C_g$ of X. Since any two of the Γ_i intersect in $\mathbb{P}^2_{\mathbb{C}}(\mathbb{R})$ we have at least $\binom{g}{2}$ such intersection points, and since the singular plane curve $\varphi(X_{\mathbb{C}})$ cannot have more than $p_a(\varphi(X_{\mathbb{C}})) - p_g(\varphi(X_{\mathbb{C}})) = \binom{g+1}{2} - g = \binom{g}{2}$ singular points we see that $\varphi(X) \subset \mathbb{P}^2_{\mathbb{R}}$ has precisely the $\binom{g}{2}$ points of intersection of the real contours Γ_i as its singularities.

Let $g \geq 2$. Since $\Gamma_i \cap \Gamma_i \neq \phi$ for $i \neq j$ there are points $P \in C_i, Q \in C_j$ such that $\dim |g^2_{g+2} - P - Q| = 1$; so $g^2_{g\,+\,2} \in W^1_g(\mathbb{R}) + C_i + C_j \subseteq W^1_g(\mathbb{R}) + W_2(\mathbb{R})$ but $g^2_{g+2} \notin W^1_g(\mathbb{R}) + C_i + C_i$ for every i. ∎

Remarks: (i) For $r \geq 3$ let $W^{r-2}_{g-4+r} \times W_4 \to \mathrm{Jac}(X_{\mathbb{C}})$ be the summation map and $W(g-4+r, r-2)$ be the closed and σ-invariant sublocus of $\mathrm{Jac}(X_{\mathbb{C}})$ consisting of points whose fibre under this map is not finite. Assume that X is not hyperelliptic. If, then, the very ample series g^r_{g+r} found in Proposition 1 would admit infinitely many m-secant line divisors for some $m \geq 4$ (i.e. if the curve of degree $g + r$ in $\mathbb{P}^r$ embedded by our g^r_{g+r} would have ∞^1 quadrisecant lines) we would have $\dim_{\mathbb{R}}(W(g-4+r, r-2)(\mathbb{R})) \geq g$ which implies $g \leq \dim(W(g-4+r, r-2) < \dim(W^{r-2}_{g-4+r} \times W_4) = \dim(W^{r-2}_{g-4+r}) + 4 \leq ((g-4+r) - 2(r-2) - 1) + 4 = g - r + 3$, by Martens' dimension theorem; so $r \leq 2$. This contradiction shows that our g^r_{g+r} has only a finite number of m-secant line divisors ($m \geq 4$).

(ii) The result in (i) resp. the last statement in Proposition 2 (on singularities) are false for hyperelliptic X since $g^r_{g+r} = |g^1_2 + g^{r-2}_{g+r-2}|$ then. ■

3 Special very ample series

The following lemma is well-known and due to the fact that a reduced and irreducible complex curve in $\mathbb{P}^r (r \geq 3)$ cannot have ∞^1 singular points resp. ∞^2 trisecant lines ([ACGH], III, ex. L).

Lemma 2: *Let $L = g^r_d$ $(r \geq 3)$ be a base point free and simple linear series on the complex curve $X_{\mathbb{C}}$. Then there are only finitely many points $P \in X_{\mathbb{C}}$ such that $g^{r-1}_{d-1} := |L - P|$ is not both base point free and simple.* ■

The lemma implies

Proposition 3: *Let X be non-hyperelliptic and $2 \leq r < g$ and $\delta \geq 0$ be integers such that $\delta \leq \mathrm{Min}(s, g-1-r)$ and $\delta \not\equiv g - r \bmod 2$. Then for assigned δ real components $C_1, \ldots, C_\delta$ of X there is a complete, base point free and simple g^r_{g-1+r} on X having precisely the pseudo-lines $C_1, \ldots, C_\delta$.*

Proof: Note that the conditions on δ are necessary for δ pseudo-lines for a (complete and special) g^r_{g-1+r} on X.

Since X is not hyperelliptic so is its complexification $X_{\mathbb{C}}$. Consequently, $X_{\mathbb{C}}$ has a canonical divisor $K_{X_{\mathbb{C}}}$ which is real and very ample, and from section 1 we know that $|K_{X_{\mathbb{C}}}|$ has no pseudo-lines. This proves the Proposition for $r = g - 1$. Assume that the Proposition is true for some $3 \leq r < g$; we want to prove it for $r - 1$ by using Lemma 2, i.e. we want to show the existence of a base point free and simple g^{r-1}_{g-2+r} on X with δ assigned pseudo-lines $C_1, \ldots, C_\delta$ provided that $0 \leq \delta \leq \mathrm{Min}(s, g-r)$ and $\delta \equiv g - r \bmod 2$.

If $\delta > 0$ take a base point free and simple g^r_{g-1+r} on X having exactly the $\delta - 1$ pseudo-lines $C_1, \ldots, C_{\delta-1}$; it represents a σ-invariant series on $X_{\mathbb{C}}$. By Lemma 2 we can choose a point $P_\delta \in C_\delta$ such that $|g^r_{g-1+r} - P_\delta|$ is a σ-invariant, base point free and simple g^{r-1}_{g-2+r} on $X_{\mathbb{C}}$. Since $|K_{X_{\mathbb{C}}} - g^r_{g-1+r}|$ has the pseudo-lines $C_1, \ldots, C_{\delta-1}$ the series $|K_{X_{\mathbb{C}}} - g^r_{g-1+r} + P_\delta|$ and hence also its dual $|g^r_{g-1+r} - P_\delta|$ have the pseudo-lines $C_1, \ldots, C_\delta$.

If $\delta = 0$ (note that this implies $r < g - 1$) we take a base point free and simple g^r_{g-1+r} on X with exactly one pseudo-line C, and by Lemma 2 we can find a point $P \in C$ such that (on $X_{\mathbb{C}}$) $|g^r_{g-1+r} - P|$ is base point free and simple, again. Since this series has even degree on C we see that it has no pseudo-line. ■

Corollary: *Let $g \geq 5$ and $s \geq 3$ ($s \geq 2$ suffices for odd g). Then $W^1_{g-1}(\mathbb{R}) \neq \phi$.*

Proof: If X is hyperelliptic we have $W^1_2(\mathbb{R}) \neq \phi$, and so $W^1_2(\mathbb{R}) + W_1(\mathbb{R}) + \cdots + W_1(\mathbb{R})$ (with $g-3$ varieties $W_1(\mathbb{R})$) is contained in $W^1_{g-1}(\mathbb{R})$, for $g \geq 3$. If X is not hyperelliptic we apply Proposition 3 with $r = 2$ and $\delta = 2$ for odd g resp. $\delta = 3$ for even g. Then the image curve in $\mathbb{P}^2_{\mathbb{C}}$ under the birational morphism induced by the chosen net g^2_{g+1} on $X_{\mathbb{C}}$ has real singular points (the points of intersection of the images of the $\delta \geq 2$ pseudo-lines), and the projection off such a singularity onto $\mathbb{P}^1_{\mathbb{C}}$ given us a σ-invariant pencil of degree at most $g-1$ on $X_{\mathbb{C}}$. ∎

Finally, for $r \geq 3$ we have the

Theorem: *Assume that X has no g^1_3, is not a smooth plane quintic and not a double cover of a real elliptic curve. Let $3 \leq r < g$ and $\delta \geq 0$ be integers such that $\delta \leq \mathrm{Min}(s, g-1-r)$ and $\delta \not\equiv g-r \bmod 2$. Then for δ assigned real components $C_1, \ldots, C_\delta$ of X there is a complete and very ample g^r_{g-1+r} on X having precisely $C_1, \ldots, C_\delta$ as its pseudo-lines.*

Proof: The inequality $3 \leq r < g$ implies $g \geq 4$, and for $g = 4$ we have $r = 3$, i.e. g^r_{g-1+r} is the canonical series on X, and so $\delta = 0$. Hence we may assume that $g \geq 5$. Choose D as in Lemma 1, with $d = g-1-r \geq 0$ there. Then $\dim|D| = 0$, and the series $|D|$ form a subset of real dimension $g-1-r$ in $W_{g-1-r}(\mathbb{R})$. Since $X_{\mathbb{C}}$ has a real canonical divisor $K_{X_{\mathbb{C}}}$ the series $|K_{X_{\mathbb{C}}} - D|$ is a complete and σ-invariant g^r_{g-1+r} on $X_{\mathbb{C}}$ with $\delta(K_{X_{\mathbb{C}}} - D) = \delta(D) = \delta$, and since $\deg((K_{X_{\mathbb{C}}} - D)|_C) \equiv \deg(D|_C) \bmod 2$ for every real component C of X we see that precisely the δ assigned real components $C_1, \ldots, C_\delta$ of X are the pseudo-lines for these g^r_{g-1+r}.

The series g^r_{g-1+r} thus constructed constitute a $(g-1-r)$-dimensional subset Z of $W^r_{g-1+r}(\mathbb{R}) = (\kappa - W_{g-1-r})(\mathbb{R}) = \kappa - W_{g-1-r}(\mathbb{R})$. Assume that none of these g^r_{g-1+r} is very ample, i.e. $Z \subseteq (W^{r-1}_{g-3+r} + W_2)(\mathbb{R}) = ((\kappa - W^1_{g-r+1}) + W_2)(\mathbb{R}) = \kappa - (W^1_{g-r+1} - W_2)(\mathbb{R})$. Then $Z' := \kappa - Z$ is a $(g-1-r)$-dimensional subset of $W_{g-1-r}(\mathbb{R}) \cap (W^1_{g-r+1} - W_2)(\mathbb{R}) \subseteq (W_{g-1-r} \cap (W^1_{g-r+1} - W_2))(\mathbb{R})$.

Claim 1: $\dim(W^1_{g-r+1}) \leq g-3-r$ for $r \geq 3$.

In fact, the claim is clear provided that the complex curve $X_{\mathbb{C}}$ is not hyperelliptic, not trigonal, not bi-elliptic and not a smooth plane quintic, according to Mumford's dimension theorem ([ACGH], IV, 5.2). Assume that $X_{\mathbb{C}}$ is hyperelliptic or trigonal. Then $X_{\mathbb{C}}$ has a unique g^1_2 resp. a unique g^1_3 (recall $g \geq 5$) which - being unique - must be σ-invariant; so X has a g^1_3 which contradicts our hypotheses. If $X_{\mathbb{C}}$ is bi-elliptic and $g \geq 6$ the covered elliptic curve $E_{\mathbb{C}}$ is unique ([ACGH], VIII, Ex. C - 1). Then σ moves $E_{\mathbb{C}}$ into itself whence there is a real elliptic curve E doubly covered by X whose complexification is $E_{\mathbb{C}}$. This again contradicts our hypotheses. If $X_{\mathbb{C}}$ is bi-elliptic and $g = 5$ we have $r = 3$ or $r = 4$ which implies $W^1_{g-r+1} = \phi$ ([ACGH], VIII, Ex. C -1). Finally, if $X_{\mathbb{C}}$ is a smooth plane quintic ($g = 6$) it has a unique net g^2_5 which (being unique) must be σ-invariant. Then X is a smooth real plane quintic, a case we have excluded. This proves the claim.

Claim 2: $W_{g-1-r} \not\subseteq W^1_{g-r+1} - W_2$.

In fact, claim 1 yields $\dim(W^1_{g-r+1} - W_2) \leq g - 1 - r$. Then $W_{g-1-r} \subseteq W^1_{g-r+1} - W_2$ would imply that there is an irreducible component Y of W^1_{g-r+1} of dimension $g - 3 - r$ such that $W_{g-1-r} = Y - W_2$. But since $Y \not\subseteq W^2_{g-r+1}$ we have $Y - W_2 \not\subseteq W_{g-1-r}$.

Claim 2 implies that $W_{g-1-r} \cap (W^1_{g-r+1} - W_2)$ is properly contained in the irreducible variety W_{g-1-r} of dimension $g - 1 - r$, i.e. we have $g - 1 - r > \dim(W_{g-1-r} \cap (W^1_{g-r+1} - W_2)) \geq \dim_{\mathbb{R}}((W_{g-1-r} \cap (W^1_{g-r+1} - W_2))(\mathbb{R})) \geq \dim_{\mathbb{R}}(Z') = g - 1 - r$. This contradiction shows that our assumption $Z \subseteq (W^{r-1}_{g-3+r} + W_2)(\mathbb{R})$ is false. ∎

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