Sharp inequalities and complete monotonicity for the Wallis ratio

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Abstract

The aim of this paper is to prove the complete monotonicity of a class of functions arising from Kazarinoff's inequality [Edinburgh Math. Notes 40 (1956) 19–21]. As applications, new sharp inequalities for the gamma and digamma functions are established.

1 Introduction and motivation

In this paper we study the complete monotonicity of the functions $f_a : (0, \infty) \to \mathbb{R}$,

$$f_a(x) = \ln \Gamma(x+1) - \ln \Gamma\left(x + \frac{1}{2}\right) - \frac{1}{2}\ln(x+a), \quad a \geq 0, \tag{1.1}$$

related to the Kazarinoff's inequality:

$$\frac{1}{\sqrt{\pi\left(n+\frac{1}{2}\right)}} < \frac{1 \cdot 3 \cdot 5 \cdot \ldots \cdot (2n-1)}{2 \cdot 4 \cdot 6 \cdot \ldots \cdot (2n)} < \frac{1}{\sqrt{\pi\left(n+\frac{1}{4}\right)}}, \quad n \geq 1. \tag{1.2}$$

For proof and other details, see [5, 13, 14, 16, 18].

As for the Euler's gamma function Γ (see [1, 8, 9]), we have

$$\Gamma(n+1) = n!, \quad \Gamma\left(n+\frac{1}{2}\right) = \frac{1 \cdot 3 \cdot \ldots \cdot (2n-1)}{2^n}\sqrt{\pi},$$

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for every positive integer n, the inequality (1.2) can be extended in the form

$$\sqrt{x+\frac{1}{4}}<\frac{\Gamma\left(x+1\right)}{\Gamma\left(x+\frac{1}{2}\right)}<\sqrt{x+\frac{1}{2}}, \quad x>0.$$

In the papers [3, 7, 13, 30, 32] the inequality (1.2) is proved mainly using the variation of the function $\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}$. Inequalities for the ratio $\frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)}$ (or more general, for ratio $\frac{\Gamma(x+1)}{\Gamma(x+s)}$, with $s>0$) have been studied extensively by many authors; for results and useful references, see, *e.g.*, [2, 4, 6, 11, 12, 15, 17, 19, 31, 33].

In the last section of this work, we prove the following sharp inequalities for $x\geq 1$,

$$\sqrt{x+\frac{1}{4}}<\frac{\Gamma\left(x+1\right)}{\Gamma\left(x+\frac{1}{2}\right)}\leq\omega\sqrt{x+\frac{1}{4}},$$

and

$$\mu\sqrt{x+\frac{1}{2}}\leq\frac{\Gamma\left(x+1\right)}{\Gamma\left(x+\frac{1}{2}\right)}<\sqrt{x+\frac{1}{2}},$$

where $\omega=\frac{4}{\sqrt{5\pi}}=1.009253...$ and $\mu=\frac{2\sqrt{2}}{\sqrt{3\pi}}=0.921317...$ are the best possible.

Then we establish some sharp inequalities for the digamma function ψ, that is the logarithmic derivative of the gamma function,

$$\psi\left(x\right)=\frac{d}{dx}\ln\Gamma\left(x\right)=\frac{\Gamma'\left(x\right)}{\Gamma\left(x\right)}.$$

More precisely, we prove that for every $x\geq 1$,

$$\frac{1}{2\left(x+\frac{1}{4}\right)}-\rho\leq\psi\left(x+1\right)-\psi\left(x+\frac{1}{2}\right)<\frac{1}{2\left(x+\frac{1}{4}\right)}$$

and

$$\frac{1}{2\left(x+\frac{1}{2}\right)}<\psi\left(x+1\right)-\psi\left(x+\frac{1}{2}\right)\leq\frac{1}{2\left(x+\frac{1}{2}\right)}+\sigma,$$

where the constants $\rho=\frac{7}{5}-2\ln 2=0.013706...$ and $\sigma=2\ln 2-\frac{4}{3}=0.052961...$ are the best possible.

2 A monotonicity result

The derivatives ψ', ψ'', ψ''', ... are known as polygamma functions. In what follows, we use the following integral representations, for every positive integer n,

$$\psi^{(n)}\left(x\right)=\left(-1\right)^{n-1}\int_0^\infty\frac{t^n e^{-xt}}{1-e^{-t}}dt \tag{2.1}$$

and for every $r > 0$,

$$\frac{1}{x^r} = \frac{1}{\Gamma(r)} \int_0^\infty t^{r-1} e^{-xt} dt. \tag{2.2}$$

See, *e.g.*, [1, 18].

Recall that a function g is completely monotonic in an interval I if g has derivatives of all orders in I such that

$$(-1)^n g^{(n)}(x) \geq 0, \tag{2.3}$$

for all $x \in I$ and $n = 0, 1, 2, 3\ldots$. Dubourdieu [10] proved that if a non constant function g is completely monotonic, then strict inequalities hold in (2.3). Completely monotonic functions involving $\ln \Gamma(x)$ are important because they produce sharp bounds for the polygamma functions, see, *e.g.*, [2, 4, 17, 20-29]. The famous Hausdorff-Bernstein-Widder theorem [34, p. 161] states that g is completely monotonic on $[0, \infty)$ if and only if

$$g(x) = \int_0^\infty e^{-xt} d\mu(t),$$

where μ is a non-negative measure on $[0, \infty)$ such that the integral converges for all $x > 0$.

Lemma 2.1. *Let $(w_k)_{k\geq 2}$ be the sequence defined by*

$$w_k = a^k - \left(a + \frac{1}{2}\right)^k + \frac{1}{2}, \quad k \geq 2.$$

(i) *If $a \in \left[0, \frac{1}{4}\right]$, then $w_k \geq 0$, for every $k \geq 2$.*

(ii) *If $a \in \left[\frac{1}{2}, \infty\right)$, then $w_k \leq 0$, for every $k \geq 2$.*

Proof. Regarded as a function of a, $w_k = w_k(a)$ is strictly decreasing, since

$$\frac{d}{da}\left(a^k - \left(a + \frac{1}{2}\right)^k + \frac{1}{2}\right) = k\left(a^{k-1} - \left(a + \frac{1}{2}\right)^{k-1}\right) < 0.$$

For $a \leq \frac{1}{4}$ and $k \geq 2$, we have

$$w_k = w_k(a) \geq w_k\left(\frac{1}{4}\right) = \frac{1}{4^k} - \left(\frac{3}{4}\right)^k + \frac{1}{2} \geq 0.$$

For $a \geq \frac{1}{2}$ and $k \geq 2$, we have

$$w_k = w_k(a) \leq w_k\left(\frac{1}{2}\right) = \frac{1}{2^k} - \frac{1}{2} < 0. \quad \blacksquare$$

Now we are in position to give the following

Theorem 2.1. (i) *The function f_a given by (1.1) is completely monotonic, for every $a \in \left[0, \frac{1}{4}\right]$.*

(ii) *The function* $-f_b$ *is completely monotonic, for every* $b \in \left[\frac{1}{2}, \infty\right)$.

Proof. We have

$$f_a'(x) = \psi(x+1) - \psi\left(x + \frac{1}{2}\right) - \frac{1}{2(x+a)}$$

and

$$f_a''(x) = \psi'(x+1) - \psi'\left(x + \frac{1}{2}\right) + \frac{1}{2(x+a)^2}.$$

Using (2.1)-(2.2), we get

$$f_a''(x) = \int_0^\infty \frac{te^{-(x+1)t}}{1-e^{-t}} dt - \int_0^\infty \frac{te^{-\left(x+\frac{1}{2}\right)t}}{1-e^{-t}} dt + \frac{1}{2}\int_0^\infty te^{-(x+a)t} dt,$$

or

$$f_a''(x) = \int_0^\infty \frac{te^{-(x+1+a)t}}{1-e^{-t}} \varphi_a(t)\, dt,$$

where

$$\varphi_a(t) = e^{at} - e^{\left(a+\frac{1}{2}\right)t} + \frac{1}{2}\left(e^t - 1\right) = \sum_{k=2}^{\infty} w_k t^k.$$

(i) If $a \in \left[0, \frac{1}{4}\right]$, then $w_k \geq 0$ and then $\varphi_a > 0$. In consequence, f_a'' is completely monotonic, that is

$$(-1)^n f_a^{(n)}(x) > 0, \tag{2.4}$$

for every $x \in (0, \infty)$ and $n \geq 2$. Further, $f_a'' > 0$, so f_a' is strictly increasing. As $\lim_{x\to\infty} f_a'(x) = 0$, we have $f_a'(x) < 0$, for every $x > 0$, so f_a is strictly decreasing. As $\lim_{x\to\infty} f_a(x) = 0$, it results that $f_a > 0$. Now (2.4) holds also for $n = 1$ and $n = 0$, meaning that f_a is completely monotonic.

(ii) If $b \in \left[\frac{1}{2}, \infty\right)$, then $w_k \leq 0$ and then $\varphi_b < 0$. In consequence, $-f_b''$ is completely monotonic, that is

$$(-1)^n f_b^{(n)}(x) < 0, \tag{2.5}$$

for every $x \in (0, \infty)$ and $n \geq 2$. Further, $f_b'' < 0$, so f_b' is strictly decreasing. As $\lim_{x\to\infty} f_b'(x) = 0$, we have $f_b'(x) > 0$, for every $x > 0$, so f_b is strictly increasing. As $\lim_{x\to\infty} f_b(x) = 0$, it results that $f_b < 0$. Now (2.5) holds also for $n = 1$ and $n = 0$, meaning that $-f_b$ is completely monotonic. ■

3 Applications

In view of their importance, the gamma and polygamma functions have incited the work of many researches, so that numerous remarkable estimates were discovered. We refer here to [4, 15, 17].

We establish in this section some new sharp inequalities for the gamma and digamma functions, using the monotonicity results stated in Theorem 2.1.

More precisely, for $a = \frac{1}{4}$, the function

$$f_{1/4}(x) = \ln\Gamma(x+1) - \ln\Gamma\left(x+\frac{1}{2}\right) - \frac{1}{2}\ln\left(x+\frac{1}{4}\right)$$

is completely monotonic, in particular it is strictly decreasing. In consequence, we have, for every $x \geq 1$,

$$0 = \lim_{x\to\infty} f_{1/4}(x) < f_{1/4}(x) \leq f_{1/4}(1).$$

By exponentiating, we obtain the sharp inequalities for $x \geq 1$,

$$\sqrt{x+\frac{1}{4}} < \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} \leq \omega\sqrt{x+\frac{1}{4}},$$

where the constant $\omega = \exp f_{1/4}(1) = \frac{4}{\sqrt{5\pi}} = 1.009253...$ is the best possible.

The function

$$f'_{1/4}(x) = \psi(x+1) - \psi\left(x+\frac{1}{2}\right) - \frac{1}{2\left(x+\frac{1}{4}\right)}$$

is strictly increasing. In consequence, for every $x \geq 1$, we have

$$f'_{1/4}(1) \leq f'_{1/4}(x) < \lim_{x\to\infty} f'_{1/4}(x) = 0,$$

thus

$$\frac{1}{2\left(x+\frac{1}{4}\right)} - \rho \leq \psi(x+1) - \psi\left(x+\frac{1}{2}\right) < \frac{1}{2\left(x+\frac{1}{4}\right)},$$

where the constant $\rho = -f'_{1/4}(1) = \frac{7}{5} - 2\ln 2 = 0.013706...$ is the best possible.

For $b = \frac{1}{2}$, the function $-f_{1/2}$ is completely monotonic, in particular, the function

$$g(x) = \ln\Gamma(x+1) - \ln\Gamma\left(x+\frac{1}{2}\right) - \frac{1}{2}\ln\left(x+\frac{1}{2}\right)$$

is strictly increasing. In consequence, for every $x \geq 1$, we have

$$g(1) \leq g(x) \leq \lim_{x\to\infty} g(x) = 0.$$

By exponentiating, we obtain the sharp inequalities for $x \geq 1$,

$$\mu\sqrt{x+\frac{1}{2}} \leq \frac{\Gamma(x+1)}{\Gamma\left(x+\frac{1}{2}\right)} < \sqrt{x+\frac{1}{2}},$$

where the constant $\mu = \exp g(1) = \frac{2\sqrt{2}}{\sqrt{3\pi}} = 0.921\,32...$ is the best possible.

The function

$$f'_{1/2}(x) = \psi(x+1) - \psi\left(x+\frac{1}{2}\right) - \frac{1}{2\left(x+\frac{1}{2}\right)}$$

is strictly decreasing. In consequence, for every $x \geq 1$, we have

$$0 = \lim_{x \to \infty} f'_{1/2}(x) < f'_{1/2}(x) \leq f'_{1/2}(1),$$

thus

$$\frac{1}{2\left(x+\frac{1}{2}\right)} < \psi(x+1) - \psi\left(x+\frac{1}{2}\right) \leq \frac{1}{2\left(x+\frac{1}{2}\right)} + \sigma,$$

where the constant $\sigma = f'_{1/2}(1) = 2\ln 2 - \frac{4}{3} = 0.052961...$ is the best possible.

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