Classification of centers, their cyclicity and isochronicity for a class of polynomial differential systems of degree $d \geq 7$ odd

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Abstract

In this paper we classify the centers, the cyclicity of its Hopf bifurcation and the isochronicity of the polynomial differential systems in $\mathbb{R}^2$ of degree $d \geq 7$ odd that in complex notation $z = x + iy$ can be written as

$$\dot{z} = (\lambda + i)z + (z\overline{z})^{\frac{d-7}{2}}(Az^6\overline{z} + Bz^4\overline{z}^3 + Cz^2\overline{z}^5 + D\overline{z}^7),$$

where $\lambda \in \mathbb{R}$, and $A, B, C, D \in \mathbb{C}$.

1 Introduction and statement of the main results

Probably the two main problems in the qualitative theory of real planar polynomial differential systems are the determination of limit cycles and the center–focus problem; i.e. to distinguish when a singular point is either a focus or a center. The notion of *center* goes back to Poincaré in [16]. He defined it for a vector field on the real plane; i.e. a singular point surrounded by a neighborhood filled with periodic orbits with the unique exception of the singular point. This paper deals with the center–focus problem for a class of polynomial differential systems of degree $d \geq 7$ odd. Note that there are few results on families of centers of polynomial differential systems of arbitrary degree.

The classification of centers for the polynomial differential systems started with the quadratic ones with the works of Dulac [6], Kapteyn [10, 11], Bautin [2], Żołądek [18], ... see Schlomiuk [17] for an update on the quadratic centers.

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There are many partial results for the centers of polynomial differential systems of degree larger than 2 but, for instance, we are very far from obtaining a complete classification of the centers for the polynomial differential systems of degree 3.

In this paper we consider the polynomial differential systems in the real (x, y)–plane that have a singular point at the origin with eigenvalues $\lambda \pm i$ and that can be written as

$$\dot{z} = (\lambda + i)z + (z\overline{z})^{\frac{d-7}{2}}(Az^6\overline{z} + Bz^4\overline{z}^3 + Cz^2\overline{z}^5 + D\overline{z}^7), \tag{1}$$

where $z = x + iy$, $d \geq 7$ is odd, $\lambda \in \mathbb{R}$, and $A, B, C, D \in \mathbb{C}$. The vector field associated to this system has linear part $(\lambda + i)z$ and by a homogeneous polynomial of degree d formed by four monomials. For such systems we want to determine the conditions that ensure that the origin of (1) is a center. Of course these systems for $d = 7$ coincides with a class of seventh degree polynomial differential systems.

A similar study has been made for the systems of the form

$$\dot{z} = (\lambda + i)z + (z\overline{z})^{\frac{d-3}{2}}(Az^3 + Bz^2\overline{z} + Cz\overline{z}^2 + D\overline{z}^3),$$

with $d \geq 3$ an arbitrary odd integer, see [13]. The tools used in [13] and in this paper are essentially the same, but the computations for determining the centers, their cyclicity and their isochronicity are different and difficult. Moreover, clearly the families of centers obtained in both families are distinct and new.

The resolution of this problem implies the effective computation of the Liapunov constants. We write

$$A = a_1 + ia_2, \quad B = b_1 + ib_2, \quad C = c_1 + ic_2, \quad D = d_1 + id_2.$$

Indeed writing (1) in polar coordinates, i.e, doing the change of variables $r^2 = z\overline{z}$ and $\theta = \arctan(\mathrm{Im}z/\mathrm{Re}z)$, system (1) becomes

$$\frac{dr}{d\theta} = \frac{\lambda r + F(\theta)\, r^d}{1 + G(\theta)\, r^{d-1}}, \tag{2}$$

where

$$\begin{aligned} F(\theta) &= (a_1 + c_1)\cos 4\theta - (a_2 - c_2)\sin 4\theta + b_1 + d_1 \cos 8\theta + d_2 \sin 8\theta, \\ G(\theta) &= (a_2 + c_2)\cos 4\theta + (a_1 - c_1)\sin 4\theta + b_2 + d_2 \cos 8\theta - d_1 \sin 8\theta. \end{aligned} \tag{3}$$

Since the denominator of (2) is positive if r is sufficiently small, system (1) has a center at the origin if and only if system (2) has a center at the origin of the plane (r, θ).

The transformation $(r, \theta) \mapsto (\rho, \theta)$ defined by

$$\rho = \frac{r^{d-1}}{1 + G(\theta)\, r^{d-1}} \tag{4}$$

is a diffeomorphism from the region $\dot{\theta} > 0$ into its image. As far as we know the first in using this transformation was Cherkas in [4]. If we write equation (2) in the variable ρ, we obtain the following Abel differential equation

$$\begin{aligned}\frac{d\rho}{d\theta} = {} & (d-1)G(\theta)[\lambda G(\theta) - F(\theta)]\rho^3 + \\ & [(d-1)(F(\theta) - 2\lambda G(\theta)) - G'(\theta)]\rho^2 + (d-1)\lambda\rho \\ = {} & U(\theta)\rho^3 + V(\theta)\rho^2 + (d-1)\lambda\rho.\end{aligned} \tag{5}$$

These kind of differential equations appeared in the studies of Abel on the theory of elliptic functions. For more details on Abel differential equations, see [9], [3] or [8].

The solution $\rho(\theta,\gamma)$ of (5) satisfying that $\rho(0,\gamma) = \gamma$ can be expanded in a convergent power series of $\gamma \geq 0$ sufficiently small. Thus

$$\rho(\theta,\gamma) = \rho_1(\theta)\gamma + \rho_2(\theta)\gamma^2 + \rho_3(\theta)\gamma^3 + \ldots \tag{6}$$

with $\rho_1(\theta) = 1$ and $\rho_k(0) = 0$ for $k \geq 2$. Let $P : [0,\gamma_0] \to \mathbb{R}$ be the Poincaré map defined by $P(\gamma) = \rho(2\pi,\gamma)$ and for a convenient $\gamma_0 > 0$. Then, the values of $\rho_k(2\pi)$ for $k \geq 2$ controle the behavior of the Poincaré map in a neighborhood of $\rho = 0$. Therefore system (1) has a center at the origin if and only if $\rho_1(2\pi) = 1$ and $\rho_k(2\pi) = 0$ for every $k \geq 2$. Assuming that $\rho_2(2\pi) = \cdots = \rho_{m-1}(2\pi) = 0$ we say that $v_m = \rho_m(2\pi)$ is the m-th *Liapunov* or *Liapunov–Abel* constant of system (1).

The problem of computing the Liapunov constants for determining a center goes back from the very beginning of the qualitative theory of differential equations, see for instance [16] and [12]. In the case of polynomial differential systems each of the Liapunov constants is a polynomial in the coefficients of the system. The set of coefficients for which all the Liapunov constants vanish is called the *center variety* of the family of polynomial differential systems. By the Hilbert Basis Theorem the center variety is an algebraic set. Then a natural question arises: How to characterize the center variety of a given family of polynomial differential systems? That is, find necessary and sufficient conditions such that a given system of the family has a center at the origin.

In general it is very difficult to distinguish between the centers and the foci, because it requires a good knowledge, not only of the common zeros of the Liapunov constants, but also of the finitely generated ideal that they generate in the ring of polynomials taking as variables the coefficients of the polynomial differential system. Furthermore, in general the calculation of the Liapunov constants is not easy, and the computational complexity of finding their common zeros grows very quickly. A number of algorithms have been developed to compute them automatically up to a certain order (see for instance [5, 14, 15] and the references therein). We also want to mention that even if we are able to obtain the Liapunov constants it is in general extremely difficult to decompose the resulting variety into irreducible components.

In this paper we also want to study the maximum number of limit cycles bifurcating from the origin of the class of polynomial differential systems (1). This has been studied for many classes of polynomial differential systems and this information allows to obtain estimates on the number of limit cycles of the system. More concretely if we denote by E_d the class of all polynomial differential systems of degree $d \geq 7$ odd of the form (1) we say that the origin of any system

$\dot{z} = w(z, \bar{z})$ with $w \in E_d$ has cyclicity k with respect to E_d if any perturbation in E_d of this system has k or fewer limit cycles in a small neighborhood of the origin and k is the maximal number with this property.

Now we wan to characterize which of the centers of system (1) with $d \geq 7$ odd are isochronous. In that case, let $z = 0$ be a center (that is, we assume that we are under the hypotheses that guarantee that $z = 0$ is a center) and let V be a neighborhood of $z = 0$ covered with periodic orbits surrounding $z = 0$. We can define a function, *the period function* of $z = 0$ by associating to every point z of V the minimal period of the periodic orbits passing through z. The center $z = 0$ of system (1) is isochronous if the period of all integral curves in $V \setminus \{z = 0\}$ is constant.

The study of isochronous centers started with Huygens where he studied the cycloidal pendulum. This pendulum has isochronous oscillations (see for instance [7]).

If we take the equation of θ' and we apply the change of variables given in (4) we obtain

$$T = \int_0^{2\pi} \frac{1}{1 + G(\theta) r^{d-1}}\, d\theta = \int_0^{2\pi} (1 - G(\theta)\rho)\, d\theta = 2\pi - \int_0^{2\pi} G(\theta)\rho\, d\theta,$$

where $\rho(\theta) = \sum_{j \geq 1} \rho_j(\theta)\gamma^j$ is given in (6) and $\rho_j(\theta)$ are the functions such that $\rho_j(2\pi)$ are the Lyapunov–Abel constants. Then system (1) has an isochronous center at the origin if it is a center and satisfies

$$\int_0^{2\pi} G(\theta)\varrho(\theta)\, d\theta = \sum_{j \geq 1} \left(\int_0^{2\pi} G(\theta)\rho_j(\theta)\, d\theta \right) \gamma^j = 0,$$

that is, if

$$T = 2\pi - \sum_{j \geq 1} T_j \gamma^j = 2\pi,$$

with

$$T_j(\gamma) = \int_0^{2\pi} G(\theta)\rho_j(\theta)\, d\theta = 0, \quad \text{for} \quad j \geq 1. \tag{7}$$

The constants T_j will be called the *period Abel constants* or simply the *period contants*.

In general it is very difficult to study the isochronous centers, because to do it requires first the knowledge of the conditions to be a center, and second a good knowledge, not only of the common zeros of the period Abel constants, but also of the finite generated ideal that they generate in the ring of polynomials taking as variables the coefficients of the polynomial differential system. Furthermore, in general the calculation of the period Abel constants is not easy, and the computational complexity of finding their common zeros grows very quickly.

The main results in this paper are Theorem 1 where we classify the centers of the polynomial differential systems (1) determining the conditions on the parameters λ, A, B, C and D in order that the origin of the polynomial differential system (1) of degree $d \geq 7$ odd be a center; Theorem 5 where we provide the cyclicity of its Hopf bifurcation, and Theorem 6 where we classify the isochronous centers.

The first main result in this paper is the following.

Theorem 1. *System* (1) *with* $d \geq 7$ *odd has a center at the origin if and only if one of the following conditions*

(c.1) $\lambda = b_1 = 3A + \overline{C} = 0$ *(Hamiltonian case),*

(c.2) $\lambda = b_1 = \mathrm{Im}(AC) = \mathrm{Re}(A^2 D) = \mathrm{Re}(\overline{C}^2 D) = 0$ *(reversible case),*

holds

For proving Theorem 1 we shall prove the next results.

Proposition 2. *If one of the conditions* $(c.1)$ *or* $(c.2)$ *holds, then system* (1) *has a center at the origin.*

Set

$$K_d(C, D) = 512(d-1)^2|C|^2 + (d^4 - 24d^3 + 66d^2 + 1728d - 9963)|D|^2, \quad (8)$$

Proposition 3. *The Liapunov constants of system* (1)*, with* $d \geq 7$ *odd, are*

$$\begin{array}{ll} V_1 = & e^{2\pi(d-1)\lambda}, \\ V_2 = & b_1, \\ V_3 = & -\mathrm{Im}(AC), \\ V_4 = & -\mathrm{Re}\Big((3A+\bar{C})D((d-9)A + (d+7)\bar{C})\Big), \\ V_5 = & -\mathrm{Im}\Big((3A+\bar{C})DB(A-\bar{C})\Big), \\ V_6 = & \mathrm{Re}\Big((3A+\overline{C})D(Au_1 + \bar{C}u_2)\Big), \\ V_7 & = 0, \\ V_8 & = \mathrm{Re}\Big((3A+\overline{C})DA|D|^4\Big), \end{array}$$

with

$$\begin{cases} u_1 = 4|A|^2 - 3|D|^2,\ u_2 = 0, & \text{if } d = 9, \\ u_1 = -K_d(C,D),\ u_2 = -u_1, & \text{if } d \geq 7 \text{ odd and } d \neq 9. \end{cases}$$

We remark that $V_k \equiv \rho_k(2\pi)$ $(\mathrm{mod.}\ \{\lambda, V_2, \ldots, V_{k-1}\})$ *for* $k = 1, \ldots, 8$ *and also modulo a positive constant.*

Proposition 4. *For* $d \geq 15$ *odd,* $V_1 = 1$ *and* $V_2 = V_3 = V_4 = V_5 = V_6 = 0$ *imply that either* $(c.1)$ *or* $(c.2)$ *holds. Moreover, for* $d \in \{7, 9, 11, 13\}$*,* $V_1 = 1$*,* $V_2 = V_3 = V_4 = V_5 = V_6 = V_7 = V_8 = 0$ *imply that either* $(c.1)$ *or* $(c.2)$ *holds.*

The eigenvalues at the singular point located at the origin of system (1) are $\lambda \pm i$. Therefore the origin is either a weak focus or a center if $\lambda = 0$, see for instance [1, 15], otherwise it is a strong focus.

From Proposition 3 it follows that system (1), with $d \geq 7$ odd, has by definition

(a) a *strong focus* at the origin if $\lambda \neq 0$, stable if $V_1 < 0$, otherwise unstable;

(b) a *weak focus of first order* at the origin if $\lambda = 0$ and $V_2 \neq 0$, stable if $V_2 < 0$, otherwise unstable;

(c) a *weak focus of second order* at the origin if $\lambda = V_2 = 0$ and $V_3 \neq 0$, stable if $V_3 < 0$, otherwise unstable;

(d) a *weak focus of third order* at the origin if $\lambda = V_2 = V_3 = 0$ and $V_4 \neq 0$, stable if $V_4 < 0$, otherwise unstable;

(e) a *weak focus of fourth order* at the origin if $\lambda = V_2 = V_3 = V_4 = 0$ and $V_5 \neq 0$, stable if $V_5 < 0$, otherwise unstable;

(f) a *weak focus of fifth order* at the origin if $\lambda = V_2 = V_3 = V_4 = V_5 = 0$ and $V_6 \neq 0$, stable if $V_6 < 0$, otherwise unstable;

(g) a center at the origin if and only if $\lambda = V_2 = V_3 = V_4 = V_5 = V_6 = 0$ and $d \geq 15$ odd;

(g) a *weak focus of sixth order* at the origin if $\lambda = V_2 = V_3 = V_4 = V_5 = V_6 = 0$ and $V_8 \neq 0$ for $d \in \{7, 9, 11, 13\}$, stable if $V_8 < 0$, otherwise unstable; and

(h) a center at the origin if and only if $\lambda = V_2 = V_3 = V_4 = V_5 = V_6 = V_8 = 0$ and $d \in \{7, 9, 11, 13\}$.

We classify the cyclicity of the Hopf bifurcation of the centers obtained in Theorem 1.

Theorem 5. *The cyclicity of the equilibrium point $z = 0$ of system* (1) *with respect to E_d is less or equal five for $d \geq 41$ odd, and less or equal six for $d \in \{7, 9, \ldots, 39\}$. More precisely the cyclicity is*

(a) 0 *for* $\lambda \neq 0$;

(b) 1 *for* $\lambda = 0$ *and* $b_1 \neq 0$;

(c) 2 *for* $\lambda = b_1 = 0 \neq 0$ *and* $\mathrm{Im}(AC) \neq 0$;

(d) 3 *for* $\lambda = b_1 = \mathrm{Im}(AC) = 0$ *and* $(d-9)A + (d+7)\overline{C} \neq 0$;

(e) 4 *for* $\lambda = b_1 = \mathrm{Im}(AC) = (d-9)A + (d+7)\overline{C} = 0$ *and* $b_2 \neq 0$;

(f) 5 *for any odd with* $d \geq 15$, *when* $\lambda = B = \mathrm{Im}(AC) = (d-9)A + (d+7)\overline{C} = 0$;

(g) 5 *for* $d = 9$, $\lambda = B = \mathrm{Im}(AC) = (d-9)A + (d+7)\overline{C} = 0$ *and* $4|A|^2 \neq 3|d|^2$;

(h) 5 *for* $d \in \{7, 11, 13\}$, $\lambda = B = \mathrm{Im}(AC) = (d-9)A + (d+7)\overline{C} = 0$ *and* $K_d(C, D) \neq 0$ *(see* (8)*)*;

(i) 6 *for* $d = 9$, $\lambda = B = \mathrm{Im}(AC) = (d-9)A + (d+7)\overline{C} = 4|A|^2 - 3|D|^2 = 0$ *and* $D \neq 0$;

(j) 6 *for* $d \in \{7, 11, 13\}$, $\lambda = B = \mathrm{Im}(AC) = (d-9)A + (d+7)\overline{C} = K_d(C, D) = 0$, *and* $D \neq 0$.

The problem now is to determine which of the centers described in Theorem 1 are isochronous. This is the last main statement in the paper.

Theorem 6. *System* (1) *with $d \geq 7$ odd has an isochronous center at the origin if and only if one of the following two conditions holds.*

(d.1) $\lambda = B = D = 0$ *and* $C = \overline{A}$.

(d.2) $\lambda = B = D = 0$ *and* $C = (5-d)\overline{A}/(3+d)$.

For proving Theorem 6 we will show the next propositions.

Proposition 7. *If either conditions* $(d.1)$ *or* $(d.2)$ *holds, then system* (1) *has an isochronous center at the origin.*

Proposition 8. *If system* (1) *has an isochronous center at the origin, then either conditions* $(d.1)$ *or* $(d.2)$ *holds.*

The paper has been organized as follows. Propositions 2, 3 and 4, and Theorem 5 are proved in Sections 2, 3, 4 and 5 respectively. Finally the proof of Propositions 7 and 8 are given in Sections 6 and 7, respectively.

2 Proof of Proposition 2

We separate the proof of the proposition into different lemmas.

Lemma 9. *If condition* $(c.1)$ *holds, then system* (1) *has a center at the origin.*

Proof. Under conditions (c.1) if we rescale system (1) by $|z|^{d-7}$ it becomes

$$\dot{z} = iz|z|^{7-d} + Az^6\bar{z} + i\mathrm{Im}Bz^4\bar{z}^3 - 3\bar{A}z^2\bar{z}^5 + D\bar{z}^7 = i\frac{\partial H}{\partial \bar{z}},$$

where for $d \geq 7$ odd we have

$$H = \frac{2}{9-d}|z|^{9-d} - i\frac{A}{2}z^6\bar{z}^2 + i\frac{\bar{A}}{2}z^2\bar{z}^6 + \frac{\mathrm{Im}B}{4}z^4\bar{z}^4 - \frac{i}{8}(D\bar{z}^8 - \bar{D}z^8) \quad \text{for } d \neq 9$$

and

$$H = \log|z|^2 - i\frac{A}{2}z^6\bar{z}^2 + i\frac{\bar{A}}{2}z^2\bar{z}^6 + \frac{\mathrm{Im}B}{4}z^4\bar{z}^4 - \frac{i}{8}(D\bar{z}^8 - \bar{D}z^8) \quad \text{for } d = 9.$$

Note that the integral $\exp(H)$ for $d = 9$ and H for $d \geq 7$ odd with $d \neq 9$, are real and well defined at the origin. Therefore the origin is a center. ■

We recall that systems (1) are *reversible* with respect to a straight line through the origin if they are invariant under the change of variables $\overline{w} = e^{i\gamma}z$, $\tau = -t$ for some γ real. For systems (1) we have the following result.

Lemma 10. *System* (1) *is reversible if and only if* $A = -\overline{A}e^{4i\gamma}$, $C = -\overline{C}e^{-4i\gamma}$, $D = -\overline{D}e^{-8i\gamma}$ *and* $B = -\overline{B}$ *for some* $\gamma \in \mathbb{R}$. *Furthermore, in this situation the origin of system* (1) *is a center.*

Proof. The proof follows directly from its definition. For more details see [5]. ■

Lemma 11. *If condition* (*c*.2) *holds, then system* (1) *has a center at the origin.*

Proof. We will see that if condition (c.2) is satisfied then (1) is a reversible system and thus the proof of this case will follow from Proposition 10. We consider that condition (c.2) in Theorem 3 holds and rewrite it as

$$B = -\bar{B}, \quad \frac{\bar{A}}{A} = \frac{C}{\bar{C}}, \quad \left(\frac{\bar{A}}{A}\right)^2 = \frac{D}{\bar{D}}, \quad \left(\frac{\bar{C}}{C}\right)^2 = \frac{\bar{D}}{D}. \tag{9}$$

Now let θ_1, θ_2 and θ_3 such that $e^{i\theta_1} = -\bar{A}/A$, $e^{i\theta_2} = -\bar{C}/C$ and $e^{i\theta_3} = -\bar{D}/D$. Then by (9) we obtain

$$\theta_1 = -\theta_2(\text{mod.}2\pi) \quad \text{and} \quad 2\theta_2 = \theta_3(\text{mod.}2\pi). \tag{10}$$

Now, take $\gamma = -\theta_1/4$. Using (10) we have

$$e^{4i\gamma} = e^{-i\theta_1} = -\frac{A}{\bar{A}}, \quad e^{-4i\gamma} = e^{i\theta_1} = e^{-i\theta_2} = -\frac{C}{\bar{C}},$$

and

$$e^{-8i\gamma} = e^{2i\theta_1} = e^{-2i\theta_2} = e^{-i\theta_3} = -\frac{D}{\bar{D}},$$

which clearly implies that system (1) under condition (c.2) is reversible and thus has a center at the origin. ■

3 Proof of Proposition 3

Solving $\rho_1(\theta)' = (d-1)\lambda\rho_1(\theta)$ and evaluating at $\theta = 2\pi$ we obtain $v_1 = \rho_1(2\pi) = e^{2\pi(d-1)\lambda}$. Then $V_1 = e^{2\pi(d-1)\lambda}$. In what follows we take $\lambda = 0$.

Substituting (6) into (5) we get that the functions $\rho_k(\theta)$ must satisfy

$$\begin{aligned}
&\rho_2' = V\rho_1^2, \\
&\rho_3' = U\rho_1^3 + 2V\rho_1\rho_2, \\
&\rho_4' = 3U\rho_1^2\rho_2 + V(\rho_2^2 + 2\rho_1\rho_3), \\
&\rho_5' = 3U(\rho_1\rho_2^2 + \rho_1^2\rho_3) + 2V(\rho_2\rho_3 + \rho_1\rho_4), \\
&\rho_6' = U(\rho_2^3 + 6\rho_1\rho_2\rho_3 + 3\rho_1^2\rho_4) + V(\rho_3^2 + 2\rho_2\rho_4 + 2\rho_1\rho_5), \\
&\rho_7' = 3U(\rho_2^2\rho_3 + \rho_1\rho_3^2 + 2\rho_1\rho_2\rho_4 + \rho_1^2\rho_5) + 2V(\rho_3\rho_4 + \rho_2\rho_5 + \rho_1\rho_6), \\
&\rho_8' = 3U(\rho_2\rho_3^2 + \rho_2^2\rho_4 + 2\rho_1\rho_3\rho_4 + 2\rho_1\rho_2\rho_5 + \rho_1^2\rho_6) \\
&\qquad + V(\rho_4^2 + 2\rho_3\rho_5 + 2\rho_2\rho_6 + 2\rho_1\rho_7),
\end{aligned}$$

where we have omitted that all the functions depend on θ. Note that all these differential equations can be solved recursively doing a integral between 0 and θ, and recalling that $\rho_k(0) = 0$ for $k \geq 2$. We have done all the computations of this paper with the help of the algebraic manipulator mathematica. These computations are not difficult but are long and tedious.

Solving the equation $\rho_2' = V\rho_1^2$ we get that $\rho_2(2\pi) = 2\pi(d-1)b_1$. Then $V_2 = b_1$. From now on we take $b_1 = 0$.

Now we compute the solution $\rho_3(\theta)$ of $\rho_3' = U\rho_1^3 + 2V\rho_1\rho_2$, and we get that $\rho_3(2\pi) = 2\pi(1-d)\,\mathrm{Im}(AC)$. Then $V_3 = -\mathrm{Im}(AC)$.

Computing the solution $\rho_4(\theta)$ from the differential equation for $\rho_4(\theta)$, we get $\rho_4(\theta)$ and in particular we obtain the expression of $v_4 = \rho_4(2\pi)$ given in the statement of Proposition 3 modulo $\rho_2(2\pi) = \rho_3(2\pi) = 0$ and a positive constant. More precisely we can check that if we multiply v_4 by $16/(\pi(d-1))$ then

$$v_4 = V_4 - 4V_3(d+3)d_2.$$

Solving the differential equation for $\rho_k(\theta)$ we get $\rho_k(\theta)$ and in particular we obtain from the expression of $v_k = \rho_k(2\pi)$ modulo $\rho_2(2\pi) = \rho_3(2\pi) = \ldots = \rho_{k-1}(2\pi) = 0$ and modulo a positive constant for $k = 5, 6, 7, 8$. In these two cases proceeding in the same way as we did for V_4, we get V_5, V_6, V_7 and V_8 as stated in the proposition. This completes the proof.

4 Proof of Proposition 4

From the fact that $V_1 = 1$ we get that $\lambda = 0$. Furthermore to make $V_3 = 0$ we will consider two different cases: $C = 0$ and $C \neq 0$. In this last case we have that $A = \mu\bar{C}$, with $\mu \in \mathbb{R}$.

Case 1: $C = 0$. In that case

$$V_4 = -3(d-9)\mathrm{Re}(A^2D).$$

In view of the factors of V_4 and since $d \geq 7$ odd, we need to consider two different cases.

Case 1.1: $\mathrm{Re}(A^2D) = 0$. In this case we are under the hypotheses of condition $(c.2)$.

Case 1.2: $\mathrm{Re}(A^2D) \neq 0$ and $d = 9$. In this case, since $b_1 = 0$, we have

$$V_5 = -3\mathrm{Im}(A^2BD) = -3b_2\mathrm{Re}(A^2D).$$

To have $V_5 = 0$ we must impose $b_2 = 0$, that is, $B = 0$. Then

$$V_6 = 3(4|A|^2 - 3|D|^2)\mathrm{Re}(A^2D).$$

In order to have $V_5 = 0$ we must impose $4|A|^2 = 3|D|^2$. Then $V_7 = 0$ and

$$V_8 = 3|D|^4\mathrm{Re}(A^2D).$$

Since $\mathrm{Re}(A^2D) \neq 0$ (and thus in particular $D \neq 0$) we have that $V_8 \neq 0$ and this case we do not have any center.

Case 2: $A = \mu\bar{B}$, $\mu \in \mathbb{R}$. In this case

$$V_4 = -(3\mu+1)((d-9)\mu + (d+7))\mathrm{Re}(\bar{C}^2D).$$

In view of the factors in V_3 we need to consider three different cases.

Case 2.1: $\mu = -1/3$. In this case we are under the hypotheses of condition (c.1).

Case 2.2: $\mathrm{Re}(\overline{C}^2 D) = 0$. In this case we are under the hypotheses of condition (c.2).

Case 2.3: $\mu = -(d+7)/(d-9)$, $\mathrm{Re}(\overline{C}^2 D) \neq 0$ and $d \neq 9$. In this case, since $b_1 = 0$, we have

$$V_5 = -\frac{4(d-1)(d+15)}{(d-9)^2} b_2 \mathrm{Re}(\bar{C}^2 D).$$

Then, since $d \geq 11$ odd, $V_5 = 0$ if and only if $b_2 = 0$, that is $B = 0$. Computing V_6 we obtain

$$V_6 = -2\frac{d-1}{d-9} K_d(C, D) \mathrm{Re}(\bar{C}^2 D),$$

with $K_d(C, D)$ introduced in (8). Then $V_6 = 0$ if and only if $d^4 - 24d^3 + 66d^2 + 1728d - 9963 < 0$ (see (8)). We note that if $d \geq 15$ odd, this is not possible and consequently it has not a center.

If $d \in \{7, 11, 13\}$ odd, then we impose $K_d(C, D) = 0$ (see (8)). We have

$$V_8 = -\frac{2(d-1)}{d-9} |D|^4 \mathrm{Re}(\overline{C}^2 D).$$

Then $V_8 = 0$ if and only if $D = 0$, a contradiction with the fact that $\mathrm{Re}(\bar{C}^2 D) \neq 0$, and thus $V_8 \neq 0$. So we do not have a center. This completes the proof of Proposition 4.

5 Proof of Theorem 5

Due to the relation between the Liapunov constants and the coefficients of the Poincaré map near the origin of system (1) (see the introduction and the references quoted there) in order to prove Theorem 5 it is well known that if we can choose $d \geq 15$ odd with the six focal values satisfying $|V_1| << |V_2| << |V_3| << |V_4| << |V_5| << |V_6|$ and $V_k V_{k-1} < 0$, for $k = 2, \ldots, 6$, then the ciclicity is five. Moreover if for $d \in \{7, 9, 11, 13\}$ we can choose the seven focal values satisfying $|V_1| << |V_2| << |V_3| << |V_4| << |V_5| << |V_6| << |V_8|$ and $V_k V_{k-1} < 0$, for $k = 2, \ldots, 6$ and $V_8 V_6 < 0$, then the ciclicity is six. From the expressions of the Liapunov constants given in Proposition 3 and also using Proposition 4, it follows easily that the previous inequalities hold and consequently Theorem 5 is proved.

6 Proof of Proposition 7

From Section 1 in order to prove Proposition 7 it is enough to show that

$$\int_0^{2\pi} \frac{d\theta}{\theta'} = 2\pi \tag{11}$$

where θ' is (1) in polar coordinates under conditions (d.1), or (d.2). Furthermore, since in assumptions (d.1), or (d.2) we can assume that $A \neq 0$ (otherwise we will obtain the linear center), we can make the change of variables

$$w = \xi z \quad \text{where} \quad \xi = \frac{A^{(d+3)/(8(d-1))}}{\overline{A}^{(d-5)/(8(d-1))}}, \tag{12}$$

and system (1) with hypotheses $(d.1)$ and after the change of variables (12) becomes

$$w' = iw + (w\bar{w})^{(d-7)/2}(w^6\bar{w} + w^2\overline{w}^5), \tag{13}$$

while system (1) with hypotheses $(d.2)$ and after the change of variables (12) becomes

$$w' = iw + (w\bar{w})^{(d-7)/2}\left(w^6\bar{w} + \frac{d+3}{5-d}w^2\overline{w}^5\right). \tag{14}$$

From the introduction it follows that in order to prove Proposition 7 it is enough to show that

$$\int_0^{2\pi} \frac{d\theta}{\theta'} = 2\pi \tag{15}$$

where $\theta' = 1 + G(\theta)r^{d-1}$ (see (3)) under conditions (d.1), or (d.2) and after the change of variables (12).

The proof of Proposition 7 will come straightforward from the following two auxiliary lemmas.

Lemma 12. *System* (13) *has an isochronous center at the origin.*

Proof. We rewrite (13) in polar coordinates we obtain

$$r' = 2r^d\cos(2\theta) \quad \text{and} \quad \theta' = 1.$$

Then

$$\int_0^{2\pi} \frac{dt}{\theta'} = \int_0^{2\pi} dt = 2\pi.$$ ∎

Lemma 13. *System* (14) *has a isochronous center at the origin.*

Proof. We rewrite (14) in polar coordinates and we obtain

$$r' = \frac{8}{d+3}r^d\cos(4\theta) \quad \text{and} \quad \theta' = 1 + \frac{2(d-1)}{d+3}r^{d-1}\sin(4\theta). \tag{16}$$

Therefore

$$\frac{dr}{d\theta} = \frac{8r^d\cos(4\theta)}{d+3+2(d-1)r^{d-1}\sin(4\theta)} \quad \text{with} \quad r(0) = r_0.$$

Then integrating it and since $r(\theta) \geq 0$ for any θ we get that

$$r(\theta) = \left(\frac{-2(d-1)\sin(4\theta) + \sqrt{(d+3)^2 r_0^{2-2d} + 4(d-1)^2 \sin^2(4\theta)}}{d+3} \right)^{1/(1-d)}. \tag{17}$$

Note that

$$\sqrt{(d+3)^2 r_0^{2-2d} + 4(d-1)^2 \sin^2(4\theta)} \geq |2(d-1)\sin(4\theta)|.$$

Thus $r(\theta)$, given in (17), is positive. Therefore introducing (17) into (16) we have that

$$\int_0^{2\pi} \frac{d\theta}{\theta'} = \int_0^{2\pi} \left(1 - \frac{2(d-1)\sin(4\theta)}{\sqrt{(d+3)^2 r_0^{2-2d} + 4(d-1)^2 \sin^2(4\theta)}} \right) d\theta = 2\pi, \tag{18}$$

because the function $2(d-1)\sin(4\theta)/\sqrt{4(d-1)^2\sin^2(4\theta) + (d+3)^2 r_0^{2-2d}}$ is odd in θ. ∎

7 Proof of Proposition 8

We note that since $u_1(\theta) = 1$, then from (7) and (3) we have

$$T_1 = 2\pi b_2.$$

Therefore in order to have $T_1 = 0$ we must impose $b_2 = 0$. Moreover, since either (c.1) or (c.2) holds, we also have $b_1 = 0$. From now on we will impose $B = 0$.

Now we compute T_2, using $u_2(\theta)$ computed in Proposition 3, (7) and (3), and we get

$$T_2 = \frac{\pi}{8}\big(2(d-5)|A|^2 - 2(d+3)|C|^2 - (d+7)|D|^2 + 16\mathrm{Re}(AC)\big).$$

We distinguish two different cases.

Case 1: $A = 0$. In this case T_2 becomes

$$T_2 = -\frac{\pi}{8}\big(2(d+3)|C|^2 + (d+7)|D|^2\big).$$

In order that $T_2 = 0$ we must impose $C = D = 0$. Then $A = B = C = D = 0$, which is not possible, otherwise we get the linear center.

Case 2: $A \neq 0$. In this case since from $V_3 = 0$ we have that $\mathrm{Im}(AC) = 0$, we get that $C = \mu\overline{A}$ with $\mu \in \mathbb{R}$. We will consider two different subcases.

Subcase 2.1: $\mu = -3$. In this case $C = -3\overline{A}$ and we are under the hypotheses (c.1). Then T_2 becomes

$$T_2 = -\frac{\pi(d+7)}{8}\big(|D|^2 + 16|A|^2\big).$$

In order that $T_2 = 0$ we must impose $A = D = 0$. Then $A = B = C = D = 0$, which is not possible as above.

Subcase 2.2: $\mu \in \mathbb{R} \setminus \{-3\}$. In this case $C = \mu\overline{A}$ and we are in hypotheses (c.2). Doing the change of variables given in (12) we can rewrite system (1) as

$$w' = iw + (w\overline{w})^{(d-7)/2}[w^6\overline{w} + \mu w^2\overline{w}^5 + \tilde{D}\overline{w}^7], \quad \tilde{D} = \frac{DA^{1/2}}{\overline{A}^{3/2}}. \tag{19}$$

Since we are in hypotheses (c.2) we have $\mathrm{Re}(A^2D) = 0$. In view of (19) we have

$$\tilde{d}_1 = \mathrm{Re}(\tilde{D}) = \mathrm{Re}\left(\frac{DA^{1/2}}{\overline{A}^{3/2}}\right) = \frac{1}{A^{3/2}\overline{A}^{3/2}}\mathrm{Re}(DA^2) = 0.$$

Computing the period constants of (19) and taking into account that $\tilde{D} = \tilde{d}_2 i$ we get

$$\begin{aligned} T_2 &= 2(d-5) + 16\mu - 2(d+3)(d+7)\mu^2\tilde{d}_2^2, \\ T_3 &= \tilde{d}_2\big(124 + 49\tilde{d}_2^2 - 190\mu + 70\mu^2 + d^2(4\mu^2 + 2\mu - 2 + \tilde{d}_2^2) \\ &\quad + 2d(19\mu^2 - 18\mu - 5 + 7\tilde{d}_2^2)\big). \end{aligned}$$

Here the constants T_2 and T_3 have been computed modulo a nonzero real constant.

Computing the common zeros of T_2 and T_3 we obtain the following subcases.

Subcase 2.2.1: $\mu = 1$ and $\tilde{d}_2 = 0$. In that case $C = \overline{A}$ and $D = 0$. Therefore we obtain the conditions (d.1).

Subcase 2.2.2: $\mu = (5-d)/(3+d)$ and $\tilde{d}_2 = 0$. In that case $C = (5-d)\overline{A}/(3+d)$ and $D = 0$. Therefore we obtain the conditions (d.2).

Subcase 2.2.3: $\mu = (9-d)/(7+d)$ and $\tilde{d}_2 = \pm 4\sqrt{2}(d-1)/(d+7)^{3/2}$. We compute T_4 using $u_4(\theta)$ given in Proposition 3, (7) and (3). We obtain

$$T_4 = -\frac{(d-1)^5(7d-15)\pi}{6(d+7)^5} \neq 0.$$

Therefore this case does not provide any isochronous center.

Subcase 2.2.4: $\mu = 3/(2+d)$ and $\tilde{d}_2 = \pm\sqrt{2}(d-1)/(d+2)(d+7)^{1/2}$. We compute T_4 using $u_4(\theta)$ given in Proposition 3, (7) and (3). We obtain

$$T_4 = -\frac{(d-1)^4(d+1)^2(d-5)(3d-11)\pi}{512(d+2)^4(d+7)} \neq 0.$$

Therefore this case does not provide any isochronous center and the proof of the proposition is completed.

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