Periodic solutions of a class of nonautonomous second order differential systems with (q, p)–Laplacian

Daniel Paşca

Abstract

Some existence theorems are obtained by the least action principle for periodic solutions of nonautonomous second-order differential systems with (q, p)–Laplacian.

1 Introduction

In the last years many authors starting with Mawhin and Willem (see [1]) proved the existence of solutions for problem

$$\begin{aligned} &\ddot{u}(t) = \nabla F(t, u(t)) \text{ a.e. } t \in [0, T], \\ &u(0) - u(T) = \dot{u}(0) - \dot{u}(T) = 0, \end{aligned} \tag{1}$$

under suitable conditions on the potential F (see [7]-[19]). Also in a series of papers (see [2]-[4]) we have generalized some of these results for the case when the potential F is just locally Lipschitz in the second variable x not continuously differentiable. Very recent (see [5] and [6]) we have considered the second order Hamiltonian inclusions systems with p–Laplacian.

The aim of this paper is to show how the results obtained in [14] can be generalized. More exactly our results represent the extensions to second-order differential systems with (q, p)–Laplacian.

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Consider the second order system

$$\begin{cases} \frac{d}{dt}\left(|\dot{u}_1(t)|^{q-2}\dot{u}_1(t)\right) = \nabla_{u_1}F(t,u_1(t),u_2(t)), \\ \frac{d}{dt}\left(|\dot{u}_2(t)|^{p-2}\dot{u}_2(t)\right) = \nabla_{u_2}F(t,u_1(t),u_2(t)) \text{ a.e. } t\in[0,T], \\ u_1(0)-u_1(T)=\dot{u}_1(0)-\dot{u}_1(T)=0, \\ u_2(0)-u_2(T)=\dot{u}_2(0)-\dot{u}_2(T)=0, \end{cases} \tag{2}$$

where $1<p,q<\infty$, $T>0$, and $F:[0,T]\times\mathbb{R}^N\times\mathbb{R}^N\to\mathbb{R}$ satisfy the following assumption (A):

- F is measurable in t for each $(x_1,x_2)\in\mathbb{R}^N\times\mathbb{R}^N$;
- F is continuously differentiable in (x_1,x_2) for a.e. $t\in[0,T]$;
- there exist $a_1,a_2\in C(\mathbb{R}_+,\mathbb{R}_+)$ and $b\in L^1(0,T;\mathbb{R}_+)$ such that

  $$|F(t,x_1,x_2)|,\ |\nabla_{x_1}F(t,x_1,x_2)|,\ |\nabla_{x_2}F(t,x_1,x_2)|\le\left[a_1(|x_1|)+a_2(|x_2|)\right]b(t)$$

  for all $(x_1,x_2)\in\mathbb{R}^N\times\mathbb{R}^N$ and a.e. $t\in[0,T]$.

Definition 1. *(see [14]) A function $G:\mathbb{R}^N\to\mathbb{R}$ is called to be (λ,μ)–subconvex if*

$$G(\lambda(x+y))\le\mu(G(x)+G(y))$$

for some $\lambda,\mu>0$ and all $x,y\in\mathbb{R}^N$.

Remark 1. *(see [14]) When $\lambda=\mu=\frac{1}{2}$, a function $(\frac{1}{2},\frac{1}{2})$–subconvex is called convex.*
When $\lambda=\mu=1$, a function $(1,1)$–subconvex is called subadditive.
When $\lambda=1$, $\mu>0$, a function $(1,\mu)$–subconvex is called μ–subadditive.

2 Main results

Theorem 1. *Assume that $F=F_1+F_2$, where F_1, F_2 satisfy assumption (A) and the following conditions:*

(i) $F_1(t,\cdot,\cdot)$ is (λ,μ)–subconvex with $\lambda>1/2$ and $0<\mu<2^{r-1}\lambda^r$ for a.e. $t\in[0,T]$ where $r=\min(q,p)$;

(ii) there exist $f_i,g_i\in L^1(0,T;\mathbb{R}_+)$, $i=1,2$ and $\alpha_1\in[0,q-1)$, $\alpha_2\in[0,p-1)$ such that

$$|\nabla_{x_1}F_2(t,x_1,x_2)|\le f_1(t)|x_1|^{\alpha_1}+g_1(t)$$
$$|\nabla_{x_2}F_2(t,x_1,x_2)|\le f_2(t)|x_2|^{\alpha_2}+g_2(t)$$

for all $(x_1,x_2)\in\mathbb{R}^N\times\mathbb{R}^N$ and a.e. $t\in[0,T]$;

(iii)

$$\frac{1}{|x_1|^{q'\alpha_1}+|x_2|^{p'\alpha_2}}\left[\frac{1}{\mu}\int_0^T F_1(t,\lambda x_1,\lambda x_2)dt+\int_0^T F_2(t,x_1,x_2)dt\right]\longrightarrow+\infty$$

as $|x|=\sqrt{|x_1|^2+|x_2|^2}\to\infty$, where $\frac{1}{q}+\frac{1}{q'}=1$ and $\frac{1}{p}+\frac{1}{p'}=1$.

Then the problem (2) has at least one solution which minimizes the function $\varphi : W = W_T^{1,q} \times W_T^{1,p} \to \mathbb{R}$ given by

$$\varphi(u_1,u_2) = \frac{1}{q}\int_0^T |\dot{u}_1(t)|^q dt + \frac{1}{p}\int_0^T |\dot{u}_2(t)|^p dt + \int_0^T F(t,u_1(t),u_2(t))dt.$$

Theorem 2. *Assume that $F = F_1 + F_2$, where F_1, F_2 satisfy assumption (A) and the following conditions:*

(iv) *$F_1(t,\cdot,\cdot)$ is (λ,μ)–subconvex with $\lambda > 1/2$ and $0 < \mu < 2^{r-1}\lambda^r$ for a.e. $t \in [0,T]$ where $r = \min(q,p)$, and there exists $\gamma \in L^1(0,T;\mathbb{R})$, $h_1,h_2 \in L^1(0,T;\mathbb{R}^N)$ with $\int_0^T h_i(t)dt = 0, i = 1,2$ such that*

$$F_1(t,x_1,x_2) \geq ((h_1(t),h_2(t)),(x_1,x_2)) + \gamma(t)$$

for all $(x_1,x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $t \in [0,T]$;

(v) *there exist $g_1, g_2 \in L^1(0,T;\mathbb{R}_+)$, $c_0 \in \mathbb{R}$ such that*

$$|\nabla_{x_1}F_2(t,x_1,x_2)| \leq g_1(t)$$
$$|\nabla_{x_2}F_2(t,x_1,x_2)| \leq g_2(t)$$

for all $(x_1,x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $t \in [0,T]$, and

$$\int_0^T F_2(t,x_1,x_2)dt \geq c_0$$

for all $x \in \mathbb{R}^N$;

(vi)

$$\frac{1}{\mu}\int_0^T F_1(t,\lambda x_1,\lambda x_2)dt + \int_0^T F_2(t,x_1,x_2)dt \longrightarrow +\infty$$

as $|x| = \sqrt{|x_1|^2 + |x_2|^2} \to \infty$.

Then the problem (2) has at least one solution which minimizes φ on W.

Theorem 3. *Assume that $F = F_1 + F_2$, where F_1, F_2 satisfy assumption (A) and the following conditions:*

(vii) *$F_1(t,\cdot,\cdot)$ is (λ,μ)–subconvex with $\lambda > 1/2$ and $0 < \mu < 2^{r-1}\lambda^r$ for a.e. $t \in [0,T]$ where $r = \min(q,p)$, and there exists $\gamma \in L^1(0,T;\mathbb{R})$, $h_1,h_2 \in L^1(0,T;\mathbb{R}^N)$ with $\int_0^T h_i(t)dt = 0, i = 1,2$ such that*

$$F_1(t,x_1,x_2) \geq ((h_1(t),h_2(t)),(x_1,x_2)) + \gamma(t)$$

for all $(x_1,x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $t \in [0,T]$;

(viii) there exist $f_i, g_i \in L^1(0,T;\mathbb{R}_+)$, $i = 1,2$ *and* $\alpha_1 \in [0, q-1)$, $\alpha_2 \in [0, p-1)$ *such that*

$$|\nabla_{x_1} F_2(t, x_1, x_2)| \leq f_1(t)|x_1|^{\alpha_1} + g_1(t)$$
$$|\nabla_{x_2} F_2(t, x_1, x_2)| \leq f_2(t)|x_2|^{\alpha_2} + g_2(t)$$

for all $(x_1, x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ *and a.e.* $t \in [0,T]$;

(ix)

$$\frac{1}{|x_1|^{q'\alpha_1} + |x_2|^{p'\alpha_2}} \int_0^T F_2(t, x_1, x_2)dt \longrightarrow +\infty$$

as $|x| = \sqrt{|x_1|^2 + |x_2|^2} \to \infty$.

Then the problem (2) has at least one solution which minimizes φ *on* W.

Remark 2. *Theorems 1, 2 and 3 generalizes the corresponding Theorems 1, 2 and 3 of [14]. In fact, it follows from these theorems letting* $p = q = 2$ *and* $F(t, x_1, x_2) = F_1(t, x_1)$.

3 The proofs of the theorems

We introduce some functional spaces. Let $T > 0$ be a positive number and $1 < q, p < \infty$. We use $|\cdot|$ to denote the Euclidean norm in $\mathbb{R}^N$. We denote by $W_T^{1,p}$ the Sobolev space of functions $u \in L^p(0,T;\mathbb{R}^N)$ having a weak derivative $\dot{u} \in L^p(0,T;\mathbb{R}^N)$. The norm in $W_T^{1,p}$ is defined by

$$\|u\|_{W_T^{1,p}} = \Big(\int_0^T \big(|u(t)|^p + |\dot{u}(t)|^p\big)dt \Big)^{\frac{1}{p}}.$$

Moreover, we use the space W defined by

$$W = W_T^{1,q} \times W_T^{1,p}$$

with the norm $\|(u_1, u_2)\|_W = \|u_1\|_{W_T^{1,q}} + \|u_2\|_{W_T^{1,p}}$. It is clear that W is a reflexive Banach space.
We recall that

$$\|u\|_p = \Big(\int_0^T |u(t)|^p dt \Big)^{\frac{1}{p}} \text{ and } \|u\|_\infty = \max_{t \in [0,T]} |u(t)|.$$

For our aims it is necessary to recall some very well know results (for proof and details see [1]).

Proposition 4. *Each* $u \in W_T^{1,p}$ *can be written as* $u(t) = \bar{u} + \tilde{u}(t)$ *with*

$$\bar{u} = \frac{1}{T} \int_0^T u(t)dt, \qquad \int_0^T \tilde{u}(t)dt = 0.$$

We have the Sobolev's inequality

$$\|\tilde{u}\|_\infty \le C_1\|\dot{u}\|_p \qquad \text{for each } u \in W_T^{1,p},$$

and Wirtinger's inequality

$$\|\tilde{u}\|_p \le C_2\|\dot{\tilde{u}}\|_p \qquad \text{for each } u \in W_T^{1,p}.$$

In [11] the authors have proved the following result (see Lemma 3.1) which generalize a very well known result proved by Jean Mawhin and Michel Willem (see Theorem 1.4 in [1]):

Lemma 5. *Let $L : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$, $(t,x_1,x_2,y_1,y_2) \to L(t,x_1,x_2,y_1,y_2)$ be measurable in t for each (x_1,x_2,y_1,y_2) for a.e. $t \in [0,T]$. If there exist $a_i \in C(\mathbb{R}_+,\mathbb{R}_+)$, $b \in L^1(0,T;\mathbb{R}_+)$, and $c_1 \in L^{q'}(0,T;\mathbb{R}_+), c_2 \in L^{p'}(0,T;\mathbb{R}_+)$, $1 < q,p < \infty, \frac{1}{q}+\frac{1}{q'} = 1, \frac{1}{p}+\frac{1}{p'} = 1$, such that for a.e. $t \in [0,T]$ and every $(x_1,x_2,y_1,y_2) \in \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N$, one has*

$$\begin{aligned}
|L(t,x_1,x_2,y_1,y_2)| &\le \left[a_1(|x_1|)+a_2(|x_2|)\right]\left[b(t)+|y_1|^q+|y_2|^p\right],\\
|D_{x_1}L(t,x_1,x_2,y_1,y_2)| &\le \left[a_1(|x_1|)+a_2(|x_2|)\right]\left[b(t)+|y_2|^p\right],\\
|D_{x_2}L(t,x_1,x_2,y_1,y_2)| &\le \left[a_1(|x_1|)+a_2(|x_2|)\right]\left[b(t)+|y_1|^q\right],\\
|D_{y_1}L(t,x_1,x_2,y_1,y_2)| &\le \left[a_1(|x_1|)+a_2(|x_2|)\right]\left[c_1(t)+|y_1|^{q-1}\right],\\
|D_{y_2}L(t,x_1,x_2,y_1,y_2)| &\le \left[a_1(|x_1|)+a_2(|x_2|)\right]\left[c_2(t)+|y_2|^{p-1}\right],
\end{aligned}$$

then the function $\varphi : W_T^{1,q} \times W_T^{1,p} \to \mathbb{R}$ defined by

$$\varphi(u_1,u_2) = \int_0^T L(t,u_1(t),u_2(t),\dot{u}_1(t),\dot{u}_2(t))dt$$

is continuously differentiable on $W_T^{1,q} \times W_T^{1,p}$ and

$$\begin{aligned}
\langle \varphi'(u_1,u_2),(v_1,v_2)\rangle = \int_0^T &\left[(D_{x_1}L(t,u_1(t),u_2(t),\dot{u}_1(t),\dot{u}_2(t)),v_1(t))\right.\\
&+(D_{y_1}L(t,u_1(t),u_2(t),\dot{u}_1(t),\dot{u}_2(t)),\dot{v}_1(t))\\
&+(D_{x_2}L(t,u_1(t),u_2(t),\dot{u}_1(t),\dot{u}_2(t)),v_2(t))\\
&\left.+(D_{y_2}L(t,u_1(t),u_2(t),\dot{u}_1(t),\dot{u}_2(t)),\dot{v}_2(t))\right]dt.
\end{aligned}$$

Corollary 6. *Let $L : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ be defined by*

$$L(t,x_1,x_2,y_1,y_2) = \frac{1}{q}|y_1|^q + \frac{1}{p}|y_2|^p + F(t,x_1,x_2)$$

where $F : [0,T] \times \mathbb{R}^N \times \mathbb{R}^N \to \mathbb{R}$ satisfy condition (A). If $(u_1,u_2) \in W_T^{1,q} \times W_T^{1,p}$ is a solution of the corresponding Euler equation $\varphi'(u_1,u_2) = 0$, then (u_1,u_2) is a solution of (2).

Remark 3. *The function φ is weakly lower semi-continuous (w.l.s.c.) on W as the sum of two convex continuous functions and of a weakly continuous one.*

Proof of Theorem 1. Like in [14] we obtain

$$F_1(t,x_1,x_2) \le \Big(2^{\frac{\beta}{2}+1}\mu\big(|x_1|^\beta + |x_2|^\beta\big) + 1\Big)(a_{10}+a_{20})b(t)$$

for all $(x_1,x_2) \in \mathbb{R}^N \times \mathbb{R}^N$ and a.e. $t \in [0,T]$, where $\beta < r$, and $a_{i0} = \max_{0\le s\le 1} a_i(s), i = 1,2$.

It follows from (i), (ii) and Sobolev's inequality that

$$\begin{aligned}
&\Big|\int_0^T \big[F_2(t,u_1(t),u_2(t)) - F_2(t,\bar{u}_1,\bar{u}_2)\big]dt\Big| \le \\
&\le \Big|\int_0^T \big[F_2(t,u_1(t),u_2(t)) - F_2(t,u_1(t),\bar{u}_2)\big]dt\Big| + \\
&+\Big|\int_0^T \big[F_2(t,u_1(t),\bar{u}_2) - F_2(t,\bar{u}_1,\bar{u}_2)\big]dt\Big| = \\
&= \Big|\int_0^T\int_0^1 (\nabla_{x_2}F_2(t,u_1(t),\bar{u}_2 + s\tilde{u}_2(t)),\tilde{u}_2(t))dsdt\Big| + \\
&+\Big|\int_0^T\int_0^1 (\nabla_{x_1}F_2(t,\bar{u}_1 + s\tilde{u}_1(t),\bar{u}_2),\tilde{u}_1(t))dsdt\Big| \le \\
&\le \int_0^T\int_0^1 f_2(t)|\bar{u}_2 + s\tilde{u}_2(t)|^{\alpha_2}|\tilde{u}_2(t)|dsdt + \int_0^T\int_0^1 g_2(t)|\tilde{u}_2(t)|dsdt + \\
&+\int_0^T\int_0^1 f_1(t)|\bar{u}_1 + s\tilde{u}_1(t)|^{\alpha_1}|\tilde{u}_1(t)|dsdt + \int_0^T\int_0^1 g_1(t)|\tilde{u}_1(t)|dsdt \le \\
&\le 2\big(|\bar{u}_2|^{\alpha_2} + \|\tilde{u}_2\|_\infty^{\alpha_2}\big)\|\tilde{u}_2\|_\infty\int_0^T f_2(t)dt + \|\tilde{u}_2\|_\infty\int_0^T g_2(t)dt + \\
&+2\big(|\bar{u}_1|^{\alpha_1} + \|\tilde{u}_1\|_\infty^{\alpha_1}\big)\|\tilde{u}_1\|_\infty\int_0^T f_1(t)dt + \|\tilde{u}_1\|_\infty\int_0^T g_1(t)dt = \\
&= c_{11}\|\tilde{u}_1\|_\infty^{\alpha_1+1} + 2c_{12}|\bar{u}_1|^{\alpha_1}\|\tilde{u}_1\|_\infty + c_{13}\|\tilde{u}_1\|_\infty + \\
&+c_{21}\|\tilde{u}_2\|_\infty^{\alpha_2+1} + 2c_{22}|\bar{u}_2|^{\alpha_2}\|\tilde{u}_2\|_\infty + c_{23}\|\tilde{u}_2\|_\infty \le \\
&\le \tilde{c}_{11}\|\dot{u}_1\|_q^{\alpha_1+1} + 2\tilde{c}_{12}|\bar{u}_1|^{\alpha_1}\|\dot{u}_1\|_q + \tilde{c}_{13}\|\dot{u}_1\|_q + \\
&+\tilde{c}_{21}\|\dot{u}_2\|_p^{\alpha_2+1} + 2\tilde{c}_{22}|\bar{u}_2|^{\alpha_2}\|\dot{u}_2\|_p + \tilde{c}_{23}\|\dot{u}_2\|_p \le \\
&\le \tilde{c}_{11}\|\dot{u}_1\|_q^{\alpha_1+1} + \frac{1}{2q}\|\dot{u}_1\|_q^q + \tilde{c}_{13}\|\dot{u}_1\|_q + \tilde{c}_{12}|\bar{u}_1|^{q'\alpha_1} + \\
&+\tilde{c}_{21}\|\dot{u}_2\|_p^{\alpha_2+1} + \frac{1}{2p}\|\dot{u}_2\|_p^p + \tilde{c}_{23}\|\dot{u}_2\|_p + \tilde{c}_{22}|\bar{u}_2|^{p'\alpha_2}
\end{aligned}$$

for all $(u_1,u_2) \in W$ and some positive constants $\tilde{c}_{11},\dots,\tilde{c}_{22}$. Hence we have

$$\begin{aligned}
\varphi(u_1,u_2) &= \frac{1}{q}\int_0^T |\dot{u}_1(t)|^q dt + \frac{1}{p}\int_0^T |\dot{u}_2(t)|^p dt + \int_0^T F_1(t,u_1(t),u_2(t))dt + \\
&+\int_0^T \big[F_2(t,u_1(t),u_2(t)) - F_2(t,\bar{u}_1,\bar{u}_2)\big]dt + \int_0^T F_2(t,\bar{u}_1,\bar{u}_2)dt \ge
\end{aligned}$$

$$\geq \frac{1}{2q}\|\dot{u}_1\|_q^q - \tilde{c}_{11}\|\dot{u}_1\|_q^{\alpha_1+1} - \tilde{c}_{13}\|\dot{u}_1\|_q - \tilde{c}_{12}|\bar{u}_1|^{q'\alpha_1} +$$

$$+\frac{1}{2p}\|\dot{u}_2\|_p^p - \tilde{c}_{21}\|\dot{u}_2\|_p^{\alpha_2+1} - \tilde{c}_{23}\|\dot{u}_2\|_p - \tilde{c}_{22}|\bar{u}_2|^{p'\alpha_2} + \int_0^T F_2(t,\bar{u}_1,\bar{u}_2)dt +$$

$$+\frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u}_1,\lambda\bar{u}_2)dt - \int_0^T F_1(t,-\tilde{u}_1(t),-\tilde{u}_2(t))dt \geq$$

$$\geq \frac{1}{2q}\|\dot{u}_1\|_q^q - \tilde{c}_{11}\|\dot{u}_1\|_q^{\alpha_1+1} - \tilde{c}_{13}\|\dot{u}_1\|_q - \tilde{c}_{12}|\bar{u}_1|^{q'\alpha_1} +$$

$$+\frac{1}{2p}\|\dot{u}_2\|_p^p - \tilde{c}_{21}\|\dot{u}_2\|_p^{\alpha_2+1} - \tilde{c}_{23}\|\dot{u}_2\|_p - \tilde{c}_{22}|\bar{u}_2|^{p'\alpha_2} + \int_0^T F_2(t,\bar{u}_1,\bar{u}_2)dt +$$

$$+\frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u}_1,\lambda\bar{u}_2)dt - \left[2^{\frac{\beta}{2}+1}\mu\left(\|\tilde{u}_1\|_\infty^\beta + \|\tilde{u}_2\|_\infty^\beta\right) + 1\right](a_{10}+a_{20})\int_0^T b(t)dt \geq$$

$$\geq \frac{1}{2q}\|\dot{u}_1\|_q^q - \tilde{c}_{11}\|\dot{u}_1\|_q^{\alpha_1+1} - \tilde{c}_{13}\|\dot{u}_1\|_q - \tilde{c}_{31}\|\dot{u}_1\|_q^\beta +$$

$$+\frac{1}{2p}\|\dot{u}_2\|_p^p - \tilde{c}_{21}\|\dot{u}_2\|_p^{\alpha_2+1} - \tilde{c}_{23}\|\dot{u}_2\|_p - \tilde{c}_{31}\|\dot{u}_2\|_p^\beta - \tilde{c}_{32} + \int_0^T F_2(t,\bar{u}_1,\bar{u}_2)dt +$$

$$+\frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u}_1,\lambda\bar{u}_2)dt - \max(\tilde{c}_{12},\tilde{c}_{22})\left(|\bar{u}_1|^{q'\alpha_1} + \tilde{c}_{22}|\bar{u}_2|^{p'\alpha_2}\right) =$$

$$= \frac{1}{2q}\|\dot{u}_1\|_q^q - \tilde{c}_{11}\|\dot{u}_1\|_q^{\alpha_1+1} - \tilde{c}_{13}\|\dot{u}_1\|_q - \tilde{c}_{31}\|\dot{u}_1\|_q^\beta +$$

$$+\frac{1}{2p}\|\dot{u}_2\|_p^p - \tilde{c}_{21}\|\dot{u}_2\|_p^{\alpha_2+1} - \tilde{c}_{23}\|\dot{u}_2\|_p - \tilde{c}_{31}\|\dot{u}_2\|_p^\beta - \tilde{c}_{32} +$$

$$+\left(|\bar{u}_1|^{q'\alpha_1} + \tilde{c}_{22}|\bar{u}_2|^{p'\alpha_2}\right)\Big\{\frac{1}{|\bar{u}_1|^{q'\alpha_1} + \tilde{c}_{22}|\bar{u}_2|^{p'\alpha_2}}\Big[\frac{1}{\mu}\int_0^T F_1(t,\lambda\bar{u}_1,\lambda\bar{u}_2)dt +$$

$$+\int_0^T F_2(t,\bar{u}_1,\bar{u}_2)dt\Big] - \max(\tilde{c}_{12},\tilde{c}_{22})\Big\}$$

for all $(u_1,u_2) \in W$, which imply that $\varphi(u_1,u_2) \to +\infty$ as $\|(u_1,u_2)\|_W \to \infty$ due to (iii). By Theorem 1.1 in [1] and Corollary 6 we complete our proof.

Proof of Theorem 2. Let (u_{1k},u_{2k}) be a minimizing sequence of φ. It follows from (iv), (v) and Sobolev's inequality that

$$\varphi(u_{1k},u_{2k}) = \frac{1}{q}\int_0^T |\dot{u}_{1k}(t)|^q dt + \frac{1}{p}\int_0^T |\dot{u}_{2k}(t)|^p dt + \int_0^T F_1(t,u_{1k}(t),u_{2k}(t))dt +$$

$$+\int_0^T F_2(t,u_{1k}(t),u_{2k}(t))dt \geq \frac{1}{q}\|\dot{u}_{1k}\|_q^q + \frac{1}{p}\|\dot{u}_{2k}\|_p^p +$$

$$+\int_0^T \big((h_1(t),h_2(t)),(u_{1k}(t),u_{2k}(t))\big)dt + \int_0^T \gamma(t)dt + \int_0^T F_2(t,\bar{u}_{1k},\bar{u}_{2k})dt +$$

$$+\int_0^T\int_0^1 (\nabla_{x_2}F_2(t,u_{1k}(t),\bar{u}_{2k}+s\tilde{u}_{2k}(t)),\tilde{u}_{2k}(t))dsdt +$$

$$-\|\tilde{u}_{1k}\|_\infty \int_0^T |h_1(t)|dt - \|\tilde{u}_{2k}\|_\infty \int_0^T |h_2(t)|dt + \int_0^T \gamma(t)dt + c_0 -$$
$$-\int_0^T \int_0^1 (\nabla_{x_1} F_2(t, \bar{u}_{1k} + s\tilde{u}_{1k}(t), \bar{u}_{2k}), \tilde{u}_{1k}(t))dsdt \geq$$
$$\geq \frac{1}{q}\|\dot{u}_{1k}\|_q^q + \frac{1}{p}\|\dot{u}_{2k}\|_p^p - \|\tilde{u}_{1k}\|_\infty \int_0^T g_1(t)dt - \|\tilde{u}_{2k}\|_\infty \int_0^T g_2(t)dt \geq$$
$$\geq \frac{1}{q}\|\dot{u}_{1k}\|_q^q + \frac{1}{p}\|\dot{u}_{2k}\|_p^p - \tilde{c}_1\|\dot{u}_{1k}\|_q - \tilde{c}_2\|\dot{u}_{2k}\|_p + \tilde{c}_3$$

for all k and some constants $\tilde{c}_1, \tilde{c}_2, \tilde{c}_3$, which implies that $(\tilde{u}_{1k}, \tilde{u}_{2k})$ is bounded. On the other hand, in a way similar to the proof of Theorem 1, one has

$$\Big|\int_0^T \big[F_2(t, u_1(t), u_2(t)) - F_2(t, \bar{u}_1, \bar{u}_2)\big]dt\Big| \leq \tilde{c}_{13}\|\dot{u}_1\|_q + \tilde{c}_{23}\|\dot{u}_2\|_p$$

for all $(u_1, u_2) \in W$ and some positive constants $\tilde{c}_{13}, \tilde{c}_{23}$, which implies that

$$\varphi(u_{1k}, u_{2k}) \geq \frac{1}{q}\|\dot{u}_{1k}\|_q^q + \frac{1}{p}\|\dot{u}_{2k}\|_p^p +$$
$$+\frac{1}{\mu}\int_0^T F_1(t, \lambda\bar{u}_{1k}, \lambda\bar{u}_{2k})dt - \int_0^T F_1(t, -\tilde{u}_{1k}(t), -\tilde{u}_{2k}(t))dt +$$
$$+\int_0^T \big[F_2(t, u_{1k}(t), u_{2k}(t)) - F_2(t, \bar{u}_{1k}, \bar{u}_{2k})\big]dt + \int_0^T F_2(t, \bar{u}_{1k}, \bar{u}_{2k})dt \geq$$
$$\geq \frac{1}{q}\|\dot{u}_{1k}\|_q^q - \tilde{c}_{13}\|\dot{u}_{1k}\|_q + \frac{1}{p}\|\dot{u}_{2k}\|_p^p - \tilde{c}_{23}\|\dot{u}_{2k}\|_p - \big[a_1(\|\tilde{u}_{1k}\|_\infty) +$$
$$+a_2(\|\tilde{u}_{2k}\|_\infty)\big]\int_0^T b(t)dt + \int_0^T F_2(t, \bar{u}_{1k}, \bar{u}_{2k})dt + \frac{1}{\mu}\int_0^T F_1(t, \lambda\bar{u}_{1k}, \lambda\bar{u}_{2k})dt$$

for all k. It follows from (vi) and the boundedness of $(\tilde{u}_{1k}, \tilde{u}_{2k})$ that $(\bar{u}_{1k}, \bar{u}_{2k})$ is bounded. Hence φ has a bounded minimizing sequence (u_{1k}, u_{2k}). Now Theorem 2 follows from Theorem 1.1 in [1] and Corollary 6.

Proof of Theorem 3. From (vii) and Sobolev's inequality it follows like in the proof of Theorem 1 that

$$\varphi(u_1, u_2) \geq \frac{1}{q}\|\dot{u}_1\|_q^q + \frac{1}{p}\|\dot{u}_2\|_p^p + \int_0^T \big((h_1(t), h_2(t)), (u_1(t), u_2(t))\big)dt + \int_0^T \gamma(t)dt +$$
$$+\int_0^T F_2(t, \bar{u}_1, \bar{u}_2)dt + \int_0^T \big[F_2(t, u_1(t), u_2(t)) - F_2(t, \bar{u}_1, \bar{u}_2)\big]dt \geq$$
$$\geq \frac{1}{2q}\|\dot{u}_1\|_q^q - \tilde{c}_{11}\|\dot{u}_1\|_q^{\alpha_1+1} - \tilde{c}_{13}\|\dot{u}_1\|_q - \tilde{c}_{12}|\bar{u}_1|^{q'\alpha_1} +$$
$$+\frac{1}{2p}\|\dot{u}_2\|_p^p - \tilde{c}_{21}\|\dot{u}_2\|_p^{\alpha_2+1} - \tilde{c}_{23}\|\dot{u}_2\|_p - \tilde{c}_{22}|\bar{u}_2|^{p'\alpha_2} +$$
$$+\int_0^T F_2(t, \bar{u}_1, \bar{u}_2)dt + \int_0^T \gamma(t)dt - \|\tilde{u}_1\|_\infty \int_0^T |h_1(t)|dt - \|\tilde{u}_2\|_\infty \int_0^T |h_2(t)|dt \geq$$

$$\geq \frac{1}{2q}\|\dot{u}_1\|_q^q - \tilde{c}_{11}\|\dot{u}_1\|_q^{\alpha_1+1} - \tilde{c}_{14}\|\dot{u}_1\|_q +$$

$$+\frac{1}{2p}\|\dot{u}_2\|_p^p - \tilde{c}_{21}\|\dot{u}_2\|_p^{\alpha_2+1} - \tilde{c}_{24}\|\dot{u}_2\|_p - c-$$

$$-\max(\tilde{c}_{12},\tilde{c}_{22})\big(|\bar{u}_1|^{q'\alpha_1} + |\bar{u}_2|^{p'\alpha_2}\big) + \int_0^T F_2(t,\bar{u}_1,\bar{u}_2)dt =$$

$$= \frac{1}{2q}\|\dot{u}_1\|_q^q - \tilde{c}_{11}\|\dot{u}_1\|_q^{\alpha_1+1} - \tilde{c}_{14}\|\dot{u}_1\|_q +$$

$$+\frac{1}{2p}\|\dot{u}_2\|_p^p - \tilde{c}_{21}\|\dot{u}_2\|_p^{\alpha_2+1} - \tilde{c}_{24}\|\dot{u}_2\|_p - c+$$

$$+\big(|\bar{u}_1|^{q'\alpha_1} + |\bar{u}_2|^{p'\alpha_2}\big)\left[\frac{1}{|\bar{u}_1|^{q'\alpha_1} + |\bar{u}_2|^{p'\alpha_2}}\int_0^T F_2(t,\bar{u}_1,\bar{u}_2)dt - \max(\tilde{c}_{12},\tilde{c}_{22})\right]$$

for all $(u_1,u_2) \in W$ and some positive constants $\tilde{c}_{11}, \tilde{c}_{14}, \tilde{c}_{21}, \tilde{c}_{24}$. Now follows like in the proof of Theorem 1 that φ is coercive by (ix), which completes the proof.

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Department of Mathematics and Informatics,
University of Oradea, University Street 1,
410087 Oradea, Romania
email:dpasca@uoradea.ro