Cone-Decompositions of the Special Unitary Groups

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Abstract

The Lusternik-Schnirelmann category of a space is a homotopy invariant. Cone-decompositions are used to give an upper bound for Lusternik-Schnirelmann categories of topological spaces. The purpose of this paper is to construct cone-decompositions of the special unitary groups, for which we use a filtration due to Miller. We observe also that Miller's filtration is closely related to a CW-decomposition.

1 Introduction

Throughout this paper, each space is assumed to have the homotopy type of an ANR.

The Lusternik-Schnirelmann category, L-S category for short, of a space is a homotopy invariant defined as follows.

Definition 1.1. Let X be a space. The non-negative integer (or infinity)

$$\min\{\ n \mid X = \bigcup_{k=0}^{n} U_k,\ \text{ and each } U_k \text{ is open and contractible in } X\ \}$$

is denoted by $\mathrm{cat}(X)$ and called the *Lusternik-Schnirelmann category* of X.

To determine the L-S category of a space, we often use a cone-decomposition of the space, which is defined as follows.

Received by the editors November 2007 - In revised form in July 2009.
Communicated by Y. Félix.
2000 *Mathematics Subject Classification* : 55M30, 55P05, 22E15, 57N60.
Key words and phrases : Lusternik-Schnirelmann category, Cone-decomposition, Special unitary Group, Cellular decomposition.

Definition 1.2. Let X be a space. A *cone-decomposition* of X with length m is a sequence of m cofibration sequences $A_k \xrightarrow{i_k} X_k \to X_{k+1}$, $0 \leq k < m$, satisfying $X_0 \simeq *$ and $X_m \simeq X$.

The cone-decomposition gives a homotopy invariant of a space, which is called the cone-length defined as follows.

Definition 1.3. Let X be a space. The non-negative integer (or infinity)

$$\min\{\ m \mid X \text{ has a cone-decomposition with length } m\ \}$$

is called the *cone-length* of X and is denoted by $\mathrm{cl}(X)$

It is well-known that the cone-length gives an upper bound for the L-S category (see [2]). We also use the cup-length (see [4]) for a lower bound for the L-S category. The definition of cup-length is given as follows.

Definition 1.4. Let X be a space. The non-negative integer (or infinity)

$$\max\{\ n \mid \text{there exist multiplicative cohomology theory } h \text{ and } x_1, \ldots, x_n \in \widetilde{h}^*(X) \text{ such that } x_1 \cdots x_n \neq 0\ \}$$

is denoted by $\mathrm{cup}(X)$ and called the *cup-length* of X.

We will mainly use the following inequalities in this paper:

$$\mathrm{cup}(X) \leq \mathrm{cat}(X) \leq \mathrm{cl}(X).$$

The L-S category and the cone-length of $\mathrm{SU}(n)$ are already determined by Singhof in [9] and [10] respectively, and are both equal to $n-1$. We give here a explicit cone-decomposition of $\mathrm{SU}(n)$ with minimal length related with Miller filtration of Stiefel manifolds [7]. A complex Stiefel manifold $\mathrm{V}_{n,m}$ is defined by

$$\mathrm{V}_{n,m} = \{A \text{ is an } n \times m \text{ matrix on } \mathbf{C} \mid A^*A = E_m\},$$

where A^* denotes the transposed conjugate matrix of A and E_m the unit matrix of the unitary group $\mathrm{U}(m)$. We identify the special unitary group $\mathrm{SU}(n)$ with $\mathrm{V}_{n,n-1}$ and the homogeneous space $\mathrm{U}(n)\ /\ \mathrm{U}(n-m) \times \{E_m\}$ with $\mathrm{V}_{n,m}$. A map $p : \mathrm{U}(n) \to \mathrm{V}_{n,m}$ denotes the natural projection. Miller's filtration $\{F_k\mathrm{V}_{n,m}\}_{k=0}^m$ is defined by

$$F_k\mathrm{V}_{n,m} = \{\ V \in \mathrm{V}_{n,m} \mid \dim \mathrm{Ker}(V - E_m^n) \geq m-k\ \},$$

where $E_m^n = p(E_n)$.

The main result of this paper is the following theorem, which gives a cone-decomposition of $\mathrm{V}_{n,n-1}$.

Theorem 1.5. *There exist spaces X_k, A_k $(k = 0, \ldots, n-2)$ and maps $f_k : A_k \to X_k$ $(k = 0, \ldots, n-2)$ satisfying that*

$$X_k \simeq F_k\mathrm{V}_{n,n-1}, \qquad X_k \cup_{f_k} \widetilde{C}A_k \simeq F_{k+1}\mathrm{V}_{n,n-1},$$

where $\widetilde{C}A_k$ denotes the reduced cone over A_k.

Theorem 1.5 gives an alternative proof of Singhof's theorem.

Theorem 1.6 (Singhof)**.**

$$\mathrm{cup}(\mathrm{SU}(n)) = \mathrm{cat}(\mathrm{SU}(n)) = \mathrm{cl}(\mathrm{SU}(n)) = n-1.$$

Proof. We have

$$n-1 \leq \mathrm{cup}(\mathrm{SU}(n)) \leq \mathrm{cat}(\mathrm{SU}(n)) \leq \mathrm{cl}(\mathrm{SU}(n)) \leq n-1$$

by the singular cohomology of $\mathrm{SU}(n)$ and Theorem 1.5. ∎

We will show that relationship between this cone-decomposition and the usual CW-decomposition of the unitary groups given in [11] and [12] (cf section "Preliminaries" for the precise statement of this relation).

The author would like to thank Prof. Mitsutaka Murayama for having numerous discussions and many useful suggestions. He would also like to thank Dr. Tetsu Nishimoto for calling his attention to the topics, Prof. Mamoru Mimura and Prof. Norio Iwase for reading the manuscript and giving him some advice.

2 Preliminaries

Throughout this paper, we regard the unit matrix E_n as the base point of $\mathrm{SU}(n)$. We will introduce some based spaces. We have the following four notations.

Notation 2.1. For each integer $k = 1, \ldots, n$, I^k denotes the k-cube $[0,1]^k$ with base point $0 = (0, \ldots, 0) \in [0,1]^k$.

Notation 2.2. For each integer $k = 1, \ldots, n$, T^k denotes the k-torus $\{ z \in \mathbf{C} \mid |z| = 1 \}^k$ with base point $(1, \ldots, 1) \in \{ z \in \mathbf{C} \mid |z| = 1 \}^k$.

Notation 2.3. For each integer $k = 1, \ldots, n$, let ${\mathrm{V}_{n,k}}^+$ be a space obtained from $\mathrm{V}_{n,k}$ by adding a base point O, the zero $n \times k$ matrix.

Each element of $\mathrm{V}_{n,k}$ is called an (orthonormal) k-frame, which is represented as an $n \times k$-matrix. Especially, each 1-frame is a unit (column) vector.

Notation 2.4. A finite sequence $(m_1, \ldots, m_l)$ of positive integers is a *partition* of k if $m_1 + \cdots + m_l = k$. For each partition $(m_1, \ldots, m_l)$ of k, $\mathrm{F}_{n,k}(m_1, \ldots, m_l)$ denotes the flag manifold

$$\mathrm{V}_{n,k} \;/\; \mathrm{U}(m_1) \times \cdots \times \mathrm{U}(m_l).$$

The flag manifold $\mathrm{F}_{n,k}(\overbrace{1, \ldots, 1}^{k})$ is denoted by $\mathrm{F}_{n,k}$. Observe that the space ${\mathrm{V}_{n,k}}^+ \;/\; \mathrm{U}(1) \times \cdots \times \mathrm{U}(1)$, denoted by ${\mathrm{F}_{n,k}}^+$, is the space obtained from $\mathrm{F}_{n,k}$ by adding a base point $[O]$.

Each element of $\mathrm{F}_{n,k}$ is called a k-flag, which is represented as an equivalence class of a k-frame. For example, the k-flag is denoted by $[V]$, where V is a k-frame. For each $V = (\mathbf{v}_1, \ldots, \mathbf{v}_k) \in \mathrm{V}_{n,k}$, $\langle V \rangle$ denotes the subspace of $\mathbf{C}^n$ spanned by the vectors $\mathbf{v}_1, \ldots, \mathbf{v}_k$, which is called a k-plane. Especially, each 1-plane is also called a line.

In the following two definitions and one notation, we will define key maps in this paper.

Definition 2.5. For each integer $k = 1, \dots, n$, we define $\varepsilon = \varepsilon_k : I^k \to T^k$ by

$$\varepsilon_k(x_1, \dots, x_k) = \left(e^{2\pi x_1 \sqrt{-1}}, \dots, e^{2\pi x_k \sqrt{-1}}\right),$$

which is called an *exponential map*.

Definition 2.6. For each integer $k = 1, \dots, n$, we define $\kappa = \kappa_k : T^k \wedge \mathrm{F}_{n,k}{}^+ \to \mathrm{U}(n)$ by

$$\kappa_k((\lambda_1, \dots, \lambda_k) \wedge [\mathbf{v}_1, \dots, \mathbf{v}_k]) = E_n + \sum_{i=1}^{k} (\lambda_i - 1)\mathbf{v}_i \mathbf{v}_i{}^*,$$

which is called a *constructing map*.

Notation 2.7. For abbreviation, the composite map $\kappa \circ (\varepsilon \wedge \mathrm{id}_{\mathrm{F}_{n,k}{}^+}) : I^k \wedge \mathrm{F}_{n,k}{}^+ \to \mathrm{U}(n)$ is denoted by $\kappa\varepsilon : I^k \wedge \mathrm{F}_{n,k}{}^+ \to \mathrm{U}(n)$.

The following equivalence relation is used in Section 3.

Definition 2.8. We define an equivalence relation $\sim$ on $I^k \wedge \mathrm{F}_{n,k}{}^+$ by

$$\begin{aligned} & (x_1, \dots, x_k) \wedge [\mathbf{v}_1, \dots, \mathbf{v}_k] \sim (y_1, \dots, y_k) \wedge [\mathbf{w}_1, \dots, \mathbf{w}_k] \\ \iff & (x_1, \dots, x_k) \wedge [\mathbf{v}_1, \dots, \mathbf{v}_k] = (y_1, \dots, y_k) \wedge [\mathbf{w}_1, \dots, \mathbf{w}_k] \\ & \text{or} \quad \sum_{x_i = r} \mathbf{v}_i \mathbf{v}_i{}^* = \sum_{y_j = r} \mathbf{w}_j \mathbf{w}_j{}^*, \quad \text{for each } r \in [0,1], \end{aligned}$$

where $(x_1, \dots, x_k) \wedge [\mathbf{v}_1, \dots, \mathbf{v}_k]$ and $(y_1, \dots, y_k) \wedge [\mathbf{w}_1, \dots, \mathbf{w}_k]$ belong to $I^k \wedge \mathrm{F}_{n,k}{}^+$.

Since the relation $\sim$ is compatible with the map $\kappa\varepsilon : I^k \wedge \mathrm{F}_{n,k}{}^+ \to \mathrm{U}(n)$, a new map $\widetilde{\kappa\varepsilon} = (\kappa\varepsilon \,/ \sim) : (I^k \wedge \mathrm{F}_{n,k}{}^+ \,/ \sim) \to \mathrm{U}(n)$ is induced.

We will define the angle formed by each k-frame and each unit vector and state the properties of angles. Let V be a k-frame and $\mathbf{u}$ a unit vector. The definition of angle is necessary to understand what is going on in the proof of Theorem 1.5.

The orthogonal projection onto the k-plane $\langle V \rangle$ is represented as the idempotent Hermite matrix VV^*. The value of VV^* at $\mathbf{u}$ is $VV^*\mathbf{u}$. The angle formed by the k-plane $\langle V \rangle$ and the line $\langle \mathbf{u} \rangle$ is given as the one formed by two vectors $VV^*\mathbf{u}$ and $\mathbf{u}$. The inner product of $VV^*\mathbf{u}$ and $\mathbf{u}$ is $(VV^*\mathbf{u})^*\mathbf{u} = \mathbf{u}^*VV^*\mathbf{u} = \|V^*\mathbf{u}\|^2$. Then

$$0 \leq \mathbf{u}^*VV^*\mathbf{u} \leq 1.$$

We define angles as follows.

Definition 2.9. The real number $\cos^{-1}\sqrt{\mathbf{u}^*VV^*\mathbf{u}}$ is denoted by $\mathrm{agl}(V, \mathbf{u})$ and called the *angle* formed by V and $\mathbf{u}$.

Remark 1. We can define the angle formed by a quaternionic k-frame and a quaternionic unit vector, in the same manner as Definition 2.9.

From the definition, we can easily show the following proposition.

Proposition 2.10. *Let V' be a k-frame and $\mathbf{u}'$ a unit vector. Suppose that $\langle V \rangle = \langle V' \rangle$ and $\langle \mathbf{u} \rangle = \langle \mathbf{u}' \rangle$. Then* $\mathrm{agl}(V, \mathbf{u}) = \mathrm{agl}(V', \mathbf{u}')$.

It follows from Proposition 2.10 that the angle of a k-frame and a unit vector induce the one of the k-plane and the line as well as of the k-flag and the 1-flag. Definition 2.9 and Proposition 2.10 can be extended to $O \in \mathrm{V}_{n,k}{}^{+}$ by $\mathrm{agl}(O, \mathbf{u}) = \frac{\pi}{2}$ for each unit vector $\mathbf{u}$.

We consider rotating $\langle V \rangle$ to a k-plane including $\mathbf{e}_1$ where $(\mathbf{e}_1, \ldots, \mathbf{e}_n) = E_n$. We suppose that $\mathrm{agl}(V, \mathbf{e}_1) \neq 0, \frac{\pi}{2}$. Let θ_V denote $\mathrm{agl}(V, \mathbf{e}_1)$, and

$$\mathbf{w}(V) = \frac{VV^*\mathbf{e}_1 - (\mathbf{e}_1{}^*VV^*\mathbf{e}_1)\mathbf{e}_1}{\|VV^*\mathbf{e}_1 - (\mathbf{e}_1{}^*VV^*\mathbf{e}_1)\mathbf{e}_1\|}.$$

The vector $\mathbf{w}(V)$ is perpendicular to $\mathbf{e}_1$ and

$$\frac{VV^*\mathbf{e}_1}{\|VV^*\mathbf{e}_1\|} = \mathbf{e}_1 \cos\theta_V + \mathbf{w}(V)\sin\theta_V.$$

For each $t \in I$, we define $\mathbf{w}_t(V)$ by

$$\mathbf{w}_t(V) = \mathbf{e}_1 \cos((1-t)\theta_V) + \mathbf{w}(V)\sin((1-t)\theta_V).$$

Then

$$\mathbf{w}_0(V) = \frac{VV^*\mathbf{e}_1}{\|VV^*\mathbf{e}_1\|}, \quad \mathbf{w}_1(V) = \mathbf{e}_1.$$

If $\mathrm{agl}(V, \mathbf{e}_1) = 0$, then we define $\mathbf{w}_t(V)$ by $\mathbf{w}_t(V) = \mathbf{e}_1$. For unit real vectors $\mathbf{a}, \mathbf{b}$ such that $\mathbf{a} \neq -\mathbf{b}$, the rotation $T(\mathbf{a}, \mathbf{b}) \in \mathrm{O}(n)$ which maps $\mathbf{a}$ to $\mathbf{b}$, and leaves everything perpendicular to $\mathbf{a}$ and $\mathbf{b}$ fixed is defined by

$$T(\mathbf{a}, \mathbf{b})\mathbf{v} = \mathbf{v} - \frac{(\mathbf{a}+\mathbf{b})^*\mathbf{v}}{(\mathbf{a}+\mathbf{b})^*\mathbf{b}}(\mathbf{a}+\mathbf{b}) + 2(\mathbf{a}^*\mathbf{v})\mathbf{b}, \qquad (\text{for each } \mathbf{v} \in \mathbf{R}^n)$$

(see Milnor and Stasheff [8], Section 6). The similar construction extended to $\mathbf{C}^n$ is necessary for CW-decompositions of the complex Grassmann manifolds. To use the idea, for each $t \in [0,1]$, we define a matrix $\rho(t, V) \in \mathrm{U}(n)$ by

$$\rho(t, V)\mathbf{v} = \mathbf{v} - \frac{(\mathbf{w}_t(V)+\mathbf{e}_1)^*\mathbf{v}}{(\mathbf{w}_t(V)+\mathbf{e}_1)^*\mathbf{e}_1}(\mathbf{w}_t(V)+\mathbf{e}_1) + 2(\mathbf{w}_t(V)^*\mathbf{v})\mathbf{e}_1 \qquad (\text{for each } \mathbf{v} \in \mathbf{C}^n).$$

The matrix $\rho(t, V) \in \mathrm{U}(n)$ maps $\mathbf{w}_t(V)$ to $\mathbf{e}_1$, and fixes everything perpendicular to $\mathbf{w}_t(V)$ and $\mathbf{e}_1$. Especially, $\rho(1, V)$ is the identity translation.

For each matrix $A = (a_{ij})$, the norm $\|A\|$ is defined by $\|A\| = \sum_{i,j} |a_{ij}|^2$.

Lemma 2.11. *For each k-frame $V \in \mathrm{V}_{n,k}$ and $1 \geq s \geq t \geq 0$,*

$$\|\rho(s, V)V - V\| \leq \|\rho(t, V)V - V\|.$$

Proof. We take an arbitrary vector $\mathbf{v} \in \langle V \rangle$, which is represented by

$$\mathbf{v} = \alpha\mathbf{w}_0(V) + \mathbf{v}', \qquad (\alpha \in \mathbf{C},\ \mathbf{v}' \perp \mathbf{w}_0(V)).$$

Then $\mathbf{v}' \perp \mathbf{e}_1$. Hence

$$\|\rho(t, V)\mathbf{v} - \mathbf{v}\| = \|\alpha\rho(t, V)\mathbf{w}_0(V) - \alpha\mathbf{w}_0(V)\| = |\alpha|\|\mathbf{w}_{1-t}(V) - \mathbf{w}_0(V)\|.$$

Since $\|\mathbf{w}_{1-s}(V) - \mathbf{w}_0(V)\| \leq \|\mathbf{w}_{1-t}(V) - \mathbf{w}_0(V)\|$, we obtain

$$\|\rho(s,V)\mathbf{v} - \mathbf{v}\| \leq \|\rho(t,V)\mathbf{v} - \mathbf{v}\|.$$

If we represent V as $(\mathbf{v}_1, \dots, \mathbf{v}_k) \in \mathrm{V}_{n,k}$ and substitute $\mathbf{v}_1, \dots, \mathbf{v}_k$ for $\mathbf{v}$, then

$$\begin{aligned}\|\rho(s,V)V - V\|^2 = \sum_{i=1}^{k} \|\rho(s,V)\mathbf{v}_i - \mathbf{v}_i\|^2 &\leq \sum_{i=1}^{k} \|\rho(t,V)\mathbf{v}_i - \mathbf{v}_i\|^2 \\ &= \|\rho(t,V)V - V\|^2. \end{aligned}$$ ■

We will use Lemma 2.11 for a proof in Section 4.

We recall a CW-decomposition of $\mathrm{V}_{n,m}$. In the case $k = 1$, the constructing map $\kappa : T^1 \wedge \mathrm{F}_{n,1}{}^{+} \to \mathrm{U}(n)$ is used to construct a CW-decomposition of the unitary group in [11] and [12]. A CW-decomposition of $\kappa(T^1 \wedge \mathrm{F}_{n,1}{}^{+})$ is given by

$$e^0 \cup \left(\bigcup_{n \geq n_1 > 0} e^{2n_1 - 1} \right).$$

The CW-decomposition of $\mathrm{V}_{n,m}$ described in the following theorem (see Steenrod [11]).

Theorem 2.12. *The Stiefel manifold* $\mathrm{V}_{n,m}$ *has a CW-decomposition*

$$p(e^0) \cup \bigcup_{j=1}^{m} \left(\bigcup_{n \geq n_j > n_{j-1} > \cdots > n_1 > n-m} p\left(e^{2n_j - 1} e^{2n_{j-1} - 1} \cdots e^{2n_1 - 1}\right) \right).$$

We will prove the following theorem in Section 3, which describes the relationship between Miller's filtration and the CW-decomposition.

Theorem 2.13. *The* 0*-th filter* $F_0\mathrm{V}_{n,m}$ *is equal to* $p(e^0)$*, and the k-th filter* $F_k\mathrm{V}_{n,m}$ *for each* $k = 1, \dots, m$ *has a CW-decomposition*

$$p(e^0) \cup \bigcup_{j=1}^{k} \left(\bigcup_{n \geq n_j > n_{j-1} > \cdots > n_1 > n-m} p\left(e^{2n_j - 1} e^{2n_{j-1} - 1} \cdots e^{2n_1 - 1}\right) \right).$$

Remark 2. One can generalize and verify Theorem 2.13 in the case $\mathbf{F} = \mathbf{R}, \mathbf{H}$.

We will see the relationship between cells and angles.

We take a unitary matrix $U \in F_k\mathrm{U}(n) \setminus F_{k-1}\mathrm{U}(n)$, and suppose that U belongs to a cell $e^{2n_k - 1} e^{2n_{k-1} - 1} \cdots e^{2n_1 - 1}$. Then the matrix U is represented by

$$U = (E_n + (\mu_k - 1)\mathbf{w}_k {\mathbf{w}_k}^*) \cdots (E_n + (\mu_1 - 1)\mathbf{w}_1 {\mathbf{w}_1}^*),$$

where $(\mu_1, \dots, \mu_k) \in (T^1 \setminus \{1\})^k$ and $\mathbf{w}_i \in \mathbf{C}^{n_i} \setminus \mathbf{C}^{n_i - 1}$. From the spectral resolution, it is also represented by

$$U = E_n + \sum_{i=1}^{k} (\lambda_i - 1)\mathbf{v}_i {\mathbf{v}_i}^*,$$

where $(\lambda_1, \ldots, \lambda_k) \in (T^1 \setminus \{1\})^k$ and $[V] = [\mathbf{v}_1, \ldots, \mathbf{v}_k] \in \mathrm{F}_{n,k}$. The k-plane $\langle V \rangle$ is the direct sum of the eigenspaces of all eigenvalues which are not equal to 1. We have that $\langle \mathbf{v}_1, \ldots, \mathbf{v}_k \rangle = \langle \mathbf{w}_1, \ldots, \mathbf{w}_k \rangle$, since their orthogonal complement is equal to the eigenspaces of U with eigenvalue 1. By using angles, one can show the following lemma which states that $n_1 = 1$.

Lemma 2.14. *$n_1 = 1$ if and only if $\mathrm{agl}(V, \mathbf{e}_1) = 0$.*

3 Proofs of theorems

Using the four lemmas stated below, we will verify Theorem 1.5 in this section. Proofs of the lemmas will be given in Section 4.

For each $k = 0, \ldots, m$, we hold

$$\kappa(T^k \wedge \mathrm{F}_{n,k}{}^+) = \kappa(T^1 \wedge \mathrm{F}_{n,1}{}^+)^k = F_k\mathrm{U}(n), \quad p(F_k\mathrm{U}(n)) = F_k\mathrm{V}_{n,m}.$$

They are proved at Lemmas 4.4 and 4.6.

We can see that $\widetilde{\kappa\varepsilon}(I^{k+1} \wedge \mathrm{F}_{n,k+1}{}^+/\sim) = F_{k+1}\mathrm{U}(n)$ from Lemma 4.4 and that $p(F_{k+1}\mathrm{U}(n)) = F_{k+1}\mathrm{V}_{n,n-1}$ from Lemma 4.6 mentioned later. We define a subspace $B'_k \subset (I^{k+1} \wedge \mathrm{F}_{n,k+1}{}^+/\sim)$ by

$$B'_k = \left\{ [\mathbf{x} \wedge [V]] \in (I^{k+1} \wedge \mathrm{F}_{n,k+1}{}^+/\sim) \mid \frac{\pi}{2}\max(\mathbf{x}) \geq \mathrm{agl}(V, \mathbf{e}_1) \right\},$$

and $Y_k \subset F_{k+1}\mathrm{U}(n), X_k \subset F_{k+1}\mathrm{V}_{n,n-1}$ by

$$Y_k = p^{-1}(F_k\mathrm{V}_{n,n-1}) \cup \widetilde{\kappa\varepsilon}(B'_k), \quad X_k = F_k\mathrm{V}_{n,n-1} \cup p(\widetilde{\kappa\varepsilon}(B'_k)).$$

The subspace $(\widetilde{\kappa\varepsilon})^{-1}(p^{-1}(F_k\mathrm{V}_{n,n-1}))$ of B'_k is denoted by B_k. For the two maps

$$(B'_k, B_k) \xrightarrow{\widetilde{\kappa\varepsilon}} (Y_k, p^{-1}(F_k\mathrm{V}_{n,n-1})) \xrightarrow{p} (X_k, F_k\mathrm{V}_{n,n-1})$$

given, there hold the following two lemmas.

Lemma 3.1. *For each $k = 0, \ldots, n-2$, the maps $\widetilde{\kappa\varepsilon}$ and p in the sequence*

$$(B'_k, B_k) \xrightarrow{\widetilde{\kappa\varepsilon}} (Y_k, p^{-1}(F_k\mathrm{V}_{n,n-1})) \xrightarrow{p} (X_k, F_k\mathrm{V}_{n,n-1})$$

are relative homeomorphisms.

Lemma 3.2. *For each $k = 0, \ldots, n-2$, the space B_k is a deformation retract of B'_k, that is,*

$$(B'_k, B_k) \simeq (B_k, B_k) \ \ \mathrm{rel}\ B_k.$$

We obtain $F_k\mathrm{V}_{n,n-1} \simeq X_k$ from Lemmas 3.1 and 3.2.

We define a subspace $A'_{k+1} \subset (I^{k+1} \wedge \mathrm{F}_{n,k+1}{}^+/\sim)$ by

$$A'_{k+1} = \left\{ [\mathbf{x} \wedge [V]] \in (I^{k+1} \wedge \mathrm{F}_{n,k+1}{}^+/\sim) \mid \frac{\pi}{2}\max(\mathbf{x}) \leq \mathrm{agl}(V, \mathbf{e}_1) \right\}$$

and $A_k \subset {A'}_{k+1}$ by
$$A_k = (\widetilde{\kappa\varepsilon})^{-1}(Y_k).$$
For the two maps
$$({A'}_{k+1}, A_k) \xrightarrow{\widetilde{\kappa\varepsilon}} (F_{k+1}\mathrm{U}(n), Y_k) \xrightarrow{p} (F_{k+1}\mathrm{V}_{n,n-1}, X_k)$$
given, there hold the following two lemmas.

Lemma 3.3. *For each $k = 0, \ldots, n-2$, the maps $\widetilde{\kappa\varepsilon}$ and p in the sequence*
$$({A'}_{k+1}, A_k) \xrightarrow{\widetilde{\kappa\varepsilon}} (F_{k+1}\mathrm{U}(n), Y_k) \xrightarrow{p} (F_{k+1}\mathrm{V}_{n,n-1}, X_k)$$
are relative homeomorphisms.

Lemma 3.4. *For each $k = 0, \ldots, n-2$, we have*
$$(\widetilde{C}A_k, A_k) \simeq ({A'}_{k+1}, A_k) \quad \mathrm{rel}\ A_k$$
where the base point of A_k is $[(0, \ldots, 0) \wedge [V]]$.

We define a map $f_k : A_k \to X_k$ by $f_k = p \circ \widetilde{\kappa\varepsilon}$. Then it follows from Lemmas 3.3 and 3.4 that
$$F_{k+1}\mathrm{V}_{n,n-1} \approx X_k \cup_{f_k} {A'}_{k+1} \simeq X_k \cup_{f_k} \widetilde{C}A_k.$$
We will prove Theorem 2.13 by using Lemmas 4.4 and 4.6 in Section 4.

Proof of Theorem 2.13. It is clear that $F_0\mathrm{V}_{n,m} = p(e^0)$.

For each $k = 1, \ldots, m$, $F_k\mathrm{V}_{n,m} = p(F_k\mathrm{U}(n)) = p(\kappa(T^1 \wedge \mathrm{F}_{n,1}{}^+)^k)$ from Lemmas 4.4 and 4.6. It is already shown in Steenrod [11] that the space $p(\kappa(T^1 \wedge \mathrm{F}_{n,1}{}^+)^k)$ has the CW-decomposition
$$p(e^0) \cup \bigcup_{j=1}^{k} \left(\bigcup_{n \geq n_j > n_{j-1} > \cdots > n_1 > n-m} p(e^{2n_j-1}e^{2n_{j-1}-1} \cdots e^{2n_1-1}) \right). \qquad \blacksquare$$

4 Proofs of the lemmas

In this section, we will prove the lemmas.

Notation 4.1. For each $[V] \in \mathrm{F}_{n,k+1}{}^+$, α_V denotes $\frac{2}{\pi}\mathrm{agl}(V, \mathbf{e}_1)$.

Notation 4.2. Let $\mathbf{1}$ denote $(1, \ldots, 1) \in I^{k+1}$.

To prove Lemma 3.2, we recall that
$$\begin{aligned} {B'}_k &= \{\ [\mathbf{x} \wedge [V]] \in I^{k+1} \wedge \mathrm{F}_{n,k+1}{}^+/\sim\ \mid\ \max(\mathbf{x}) \geq \alpha_V\ \}, && (1) \\ B_k &= \{\ [\mathbf{x} \wedge [V]] \in {B'}_k \mid \min(\mathbf{x}) = 0 \text{ or } \max(\mathbf{x}) = 1 \text{ or } \alpha_V = 0\ \}. && (2) \end{aligned}$$

The equality (2) is a result obtained by applying Lemma 2.14 to the definition of B_k.

Proof of Lemma 3.2. We will show a partition of ${B'}_k$ into three closed subspaces ${B'}_k^1$, ${B'}_k^2$ and ${B'}_k^3$, and an analogous partition of B_k into three closed subspaces ${B_k}^1, {B_k}^2$ and ${B_k}^3$, and construct three homotopies

$$h^1 : I \times {B'}_k^1 \to {B'}_k^1, \qquad h^2 : I \times {B'}_k^2 \to {B'}_k^2, \qquad h^3 : I \times {B'}_k^3 \to {B'}_k^3.$$

We define ${B'}_k^1$ and ${B_k}^1$ respectively by

$$\begin{aligned} {B'}_k^1 &= \{ \, [\mathbf{x} \wedge [V]] \in {B'}_k \mid \alpha_V \leq 1 - \max(\mathbf{x}), \ \alpha_V \leq 2\min(\mathbf{x}) \, \}, \\ {B_k}^1 &= {B'}_k^1 \cap B_k = \{ \, [\mathbf{x} \wedge [V]] \in {B'}_k^1 \mid \alpha_V = 0 \, \}. \end{aligned}$$

The homotopy $h^1 : I \times {B'}_k^1 \to {B'}_k^1$ is defined as follows. For each $(t, [\mathbf{x} \wedge [V]]) \in I \times {B'}_k^1$,

$$h^1(t, [\mathbf{x} \wedge [V]]) = \left[\left(t\mathbf{x} + (1-t)\,(\max(\mathbf{x}) + \alpha_V)\, \frac{2\mathbf{x} - \alpha_V \mathbf{1}}{2\max(\mathbf{x}) - \alpha_V} \right) \wedge [\, \rho\,(t, V)\, V] \right]$$

if $2\max(\mathbf{x}) \neq \alpha_V$ and

$$h^1(t, [\mathbf{x} \wedge [V]]) = [\mathbf{0} \wedge [V]]$$

if $2\max(\mathbf{x}) = \alpha_V$.

We define ${B'}_k^2$ and ${B_k}^2$ respectively by

$$\begin{aligned} {B'}_k^2 &= \{ \, [\mathbf{x} \wedge [V]] \in {B'}_k \mid 1 - \max(\mathbf{x}) \leq 2\min(\mathbf{x}), \ 1 - \max(\mathbf{x}) \leq \alpha_V \, \}, \\ {B_k}^2 &= {B'}_k^2 \cap B_k = \{ \, [\mathbf{x} \wedge [V]] \in {B'}_k^1 \mid \max(\mathbf{x}) = 1 \, \}. \end{aligned}$$

The homotopy $h^2 : I \times {B'}_k^2 \to {B'}_k^2$ is defined as follows. For each $(t, [\mathbf{x} \wedge [V]]) \in I \times {B'}_k^2$,

$$\begin{aligned} h^2(t, [\mathbf{x} \wedge [V]]) &= \left[\left(t\mathbf{x} + (1-t) \frac{2\mathbf{x} + \max(\mathbf{x})\mathbf{1} - \mathbf{1}}{3\max(\mathbf{x}) - 1} \right) \right. \\ &\qquad \left. \wedge \left[\rho \left(t + (1-t) \left(1 - \frac{1 - \max(\mathbf{x})}{\alpha_V} \right), V \right) V \right] \right] \end{aligned}$$

if $\alpha_V \neq 0, 1$, and

$$h^2(t, [\mathbf{x} \wedge [V]]) = [\mathbf{x} \wedge [V]]$$

if $\alpha_V = 0, 1$. Continuity of the map h^2 follows from Lemma 2.11.

We define ${B'}_k^3$ and ${B_k}^3$ respectively by

$$\begin{aligned} {B'}_k^3 &= \{ \, [\mathbf{x} \wedge [V]] \in {B'}_k \mid 2\min(\mathbf{x}) \leq \alpha_V, \ 2\min(\mathbf{x}) \leq 1 - \max(\mathbf{x}) \, \}, \\ {B_k}^3 &= {B'}_k^3 \cap B_k = \{ \, [\mathbf{x} \wedge [V]] \in {B'}_k^3 \mid \min(\mathbf{x}) = 0 \, \}. \end{aligned}$$

The homotopy $h^3 : I \times {B'}_k^3 \to {B'}_k^3$ is defined as follows. For each $(t, [\mathbf{x} \wedge [V]]) \in I \times {B'}_k^3$,

$$\begin{aligned} h^3(t, [\mathbf{x} \wedge [V]]) &= \left[\left(t\mathbf{x} + (1-t)(\max(\mathbf{x}) + 2\min(\mathbf{x})) \frac{\mathbf{x} - \min(\mathbf{x})\mathbf{1}}{\max(\mathbf{x}) - \min(\mathbf{x})} \right) \right. \\ &\qquad \left. \wedge \left[\rho \left(t + (1-t) \left(1 - \frac{2\min(\mathbf{x})}{\alpha_V} \right), V \right) V \right] \right] \end{aligned}$$

if $\max(\mathbf{x}) \neq \min(\mathbf{x})$ and $\alpha_V \neq 0, 1$, and

$$h^3(t, [\mathbf{x} \wedge [V]]) = [\mathbf{x} \wedge [V]]$$

if $\max(\mathbf{x}) = \min(\mathbf{x})$ or $\alpha_V = 0, 1$. Continuity of the map h^3 follows too from Lemma 2.11.

We define a homotopy $h : I \times B'_k \to B'_k$ by

$$h(t, [\mathbf{x} \wedge [V]]) = \begin{cases} h^1(t, [\mathbf{x} \wedge [V]]) & \text{if } [\mathbf{x} \wedge [V]] \in {B'_k}^1, \\ h^2(t, [\mathbf{x} \wedge [V]]) & \text{if } [\mathbf{x} \wedge [V]] \in {B'_k}^2, \\ h^3(t, [\mathbf{x} \wedge [V]]) & \text{if } [\mathbf{x} \wedge [V]] \in {B'_k}^3. \end{cases}$$

The homotopy h is well-defined and hence we obtain that

$$(B'_k, B_k) \simeq (B_k, B_k) \quad \text{rel } B_k.$$

■

To prove Lemma 3.4 we recall that

$$\begin{aligned} A'_{k+1} &= \{ [\mathbf{x} \wedge [V]] \in I^{k+1} \wedge {\mathrm{F}_{n,k+1}}^+ / \sim \mid \max(\mathbf{x}) \leq \alpha_V \}, \\ A_k &= \{ [\mathbf{x} \wedge [V]] \in A'_{k+1} \mid \min(\mathbf{x}) = 0 \text{ or } \max(\mathbf{x}) = \alpha_V \}. \end{aligned}$$

Notation 4.3. Let $\mathbf{c}_V$ denote $\frac{\alpha_V}{2}\mathbf{1}$.

It is clear that $[\mathbf{x} \wedge [V]] \in A_k$ if and only if $\|\mathbf{x} - \mathbf{c}_V\|_\infty = \frac{\alpha_V}{2}$. The proof of Lemma 3.4 is essentially the same as the one in the paper Kadzisa [5], Section 4.

Proof of Lemma 3.4. We define a homotopy equivalence $\varphi : \widetilde{C}A_k \to A'_{k+1}$ by

$$\varphi(t \wedge [\mathbf{x} \wedge [V]]) = [t\mathbf{x} \wedge [V]],$$

and its homotopy inverse $\psi : A'_{k+1} \to \widetilde{C}A_k$ by

$$\psi[\mathbf{x} \wedge [V]] = \frac{2}{\alpha_V}\|\mathbf{x} - \mathbf{c}_V\|_\infty \wedge \left[\left(\frac{\alpha_V}{2} \frac{\mathbf{x} - \mathbf{c}_V}{\|\mathbf{x} - \mathbf{c}_V\|_\infty} + \mathbf{c}_V \right) \wedge [V] \right].$$

The map ψ is well-defined and continuous. Then a homotopy $\eta : I \times \widetilde{C}A_k \to \widetilde{C}A_k$ from $\psi \circ \varphi$ to the identity map is defined by

$$\eta(s, t \wedge [\mathbf{x} \wedge [V]]) = \psi[(t\mathbf{x} + s(1-t)\mathbf{c}_V) \wedge [V]],$$

and a homotopy $\zeta : I \times A'_{k+1} \to A'_{k+1}$ from $\varphi \circ \psi$ to the identity map is defined by

$$\zeta(s, [\mathbf{x} \wedge [V]]) = \left[\left((1-s) \left(\mathbf{x} - \mathbf{c}_V + \frac{2}{\alpha_V}\|\mathbf{x} - \mathbf{c}_V\|_\infty \mathbf{c}_V \right) + s\mathbf{x} \right) \wedge [V] \right].$$

For each $s \in I$ and each $[\mathbf{x} \wedge [V]] \in A_k$, we see that

$$\varphi(1 \wedge [\mathbf{x} \wedge [V]]) = [\mathbf{x} \wedge [V]], \qquad \psi[\mathbf{x} \wedge [V]] = 1 \wedge [\mathbf{x} \wedge [V]],$$

and

$$\eta(s, 1 \wedge [\mathbf{x} \wedge [V]]) = 1 \wedge [\mathbf{x} \wedge [V]], \qquad \zeta(s, [\mathbf{x} \wedge [V]]) = [\mathbf{x} \wedge [V]].$$

Therefore

$$(\widetilde{C}A_k, A_k) \simeq (A'_{k+1}, A_k) \quad \text{rel } A_k.$$

■

We will prove the following three lemmas for the CW-decompositions of Miller's filtration.

Lemma 4.4. *For each $k = 1, \dots, n$, there holds that*

$$\kappa(T^k \wedge \mathrm{F}_{n,k}{}^+) = \kappa(T^1 \wedge \mathrm{F}_{n,1}{}^+)^k = F_k\mathrm{U}(n).$$

Proof. It is clear that $\kappa(T^k \wedge \mathrm{F}_{n,k}{}^+) \subset \kappa(T^1 \wedge \mathrm{F}_{n,1}{}^+)^k$, since

$$E_n + \sum_{i=1}^{k}(\lambda_i - 1)\mathbf{v}_i\mathbf{v}_i{}^* = (E_n + (\lambda_1 - 1)\mathbf{v}_1\mathbf{v}_1{}^*) \cdots (E_n + (\lambda_k - 1)\mathbf{v}_k\mathbf{v}_k{}^*)$$

for each $(\lambda_1, \dots, \lambda_k) \wedge [\mathbf{v}_1, \dots, \mathbf{v}_k] \in T^k \wedge \mathrm{F}_{n,k}{}^+$.

We will show that $\kappa(T^1 \wedge \mathrm{F}_{n,1}{}^+)^k \subset F_k\mathrm{U}(n)$. Take $U \in \kappa(T^1 \wedge \mathrm{F}_{n,1}{}^+)^k$ and suppose that U is described as $(E_n + (\lambda_1 - 1)\mathbf{v}_1\mathbf{v}_1{}^*) \cdots (E_n + (\lambda_k - 1)\mathbf{v}_k\mathbf{v}_k{}^*)$, where $\lambda_1, \dots, \lambda_k \in T^1$ and $\mathbf{v}_1, \dots, \mathbf{v}_k \in \mathrm{V}_{n,1}$. There exists an $(n-k)$-frame $(\mathbf{u}_1, \dots, \mathbf{u}_{n-k})$ of the orthogonal complement of the space $\langle \mathbf{v}_1, \dots, \mathbf{v}_k \rangle$ spanned by $\mathbf{v}_1, \dots, \mathbf{v}_k$. The matrix U belongs to the filter $F_k\mathrm{U}(n)$, since $U\mathbf{u}_i = \mathbf{u}_i$ for all $i = 1, \dots, n-k$. Consequently we have $\kappa(T^1 \wedge \mathrm{F}_{n,1}{}^+)^k \subset F_k\mathrm{U}(n)$.

We will show that $F_k\mathrm{U}(n) \subset \kappa(T^k \wedge \mathrm{F}_{n,k}{}^+)$. Take $U \in F_k\mathrm{U}(n)$. There exists an orthonormal basis $\{\mathbf{v}_1, \dots, \mathbf{v}_n\}$ whose elements are eigenvectors of U. We may suppose that the eigenvalues of $\mathbf{v}_{k+1}, \dots, \mathbf{v}_n$ are 1, since the dimension of the eigenspace of eigenvalue 1 is greater than or equal to $n-k$. For each $i = 1, \dots, k$, a scalar λ_i denotes the eigenvalue of $\mathbf{v}_i$. Hence

$$U = E_n + \sum_{i=1}^{k}(\lambda_i - 1)\mathbf{v}_i\mathbf{v}_i{}^*$$

and $U \in \kappa(T^k \wedge \mathrm{F}_{n,k}{}^+)$. Consequently we have $F_k\mathrm{U}(n) \subset \kappa(T^k \wedge \mathrm{F}_{n,k}{}^+)$. Therefore we have $\kappa(T^k \wedge \mathrm{F}_{n,k}{}^+) = \kappa(T^1 \wedge \mathrm{F}_{n,1}{}^+)^k = F_k\mathrm{U}(n)$. ∎

Lemma 4.5. *To every m-frame $V \in \mathrm{V}_{n,m}$, there exists a matrix $U \in F_m\mathrm{U}(n)$ such that $p(U) = V$.*

Proof. Take an m-frame $V \in \mathrm{V}_{n,m}$. There exists a matrix $U' \in \mathrm{U}(n)$ satisfying $p(U') = V$. By Theorem 2.12, there exist scalars $\lambda_1, \dots, \lambda_n \in T^1$ and vectors $\mathbf{v}_i \in \mathrm{V}_{i,1} \quad (i = 1, \dots, n)$ such that

$$U' = (E_n + (\lambda_n - 1)\mathbf{v}_n\mathbf{v}_n{}^*) \cdots (E_n + (\lambda_1 - 1)\mathbf{v}_1\mathbf{v}_1{}^*).$$

Define a matrix U by

$$U = U'(E_n + (\overline{\lambda_1} - 1)\mathbf{v}_1\mathbf{v}_1{}^*) \cdots (E_n + (\overline{\lambda_{n-m}} - 1)\mathbf{v}_{n-m}\mathbf{v}_{n-m}{}^*).$$

Since we have $(E_n + (\lambda_i - 1)\mathbf{v}_i\mathbf{v}_i{}^*)(E_n + (\overline{\lambda_i} - 1)\mathbf{v}_i\mathbf{v}_i{}^*) = E_n$ for all $i = 1, \dots, n$, we have

$$U = (E_n + (\lambda_n - 1)\mathbf{v}_n\mathbf{v}_n{}^*) \cdots (E_n + (\lambda_{n-m+1} - 1)\mathbf{v}_{n-m+1}\mathbf{v}_{n-m+1}{}^*).$$

The matrix U belongs to $F_m\mathrm{U}(n)$ by Lemma 4.4. We obtain that

$$(E_n + (\overline{\lambda_1} - 1)\mathbf{v}_1\mathbf{v}_1{}^*) \cdots (E_n + (\overline{\lambda_{n-m}} - 1)\mathbf{v}_{n-m}\mathbf{v}_{n-m}{}^*) \in \mathrm{U}(n-m) \times \{E_m\},$$

which implies that $p(U) = p(U') = V$. ∎

Lemma 4.6. *For each $k = 0, \dots, m$, there holds*

$$p(F_k\mathrm{U}(n)) = F_k\mathrm{V}_{n,m}.$$

Proof. We will show that $p(F_k\mathrm{U}(n)) \subset F_k\mathrm{V}_{n,m}$. Take a matrix $U \in F_k\mathrm{U}(n)$. The eigenspace of 1 of U is denoted by W. Then $\dim W \geq n-k$. Thus we have $\dim(W \cap \langle \mathbf{e}_{n-m+1}, \dots, \mathbf{e}_n \rangle) \geq m-k$. Consequently there exists an $(m-k)$-frame $(\mathbf{v}_1, \dots, \mathbf{v}_{m-k})$ in the space $W \cap \langle \mathbf{e}_{n-m+1}, \dots, \mathbf{e}_n \rangle$. The transposed matrix of E^n_m is denoted by E^m_n. Then the matrix $(E^m_n\mathbf{v}_1, \dots, E^m_n\mathbf{v}_{m-k})$ is an $(m-k)$-frame in the space $\mathbf{C}^m$, since $\mathbf{v}_1, \dots, \mathbf{v}_{m-k} \in \langle \mathbf{e}_{n-m+1}, \dots, \mathbf{e}_n \rangle$. One has

$$p(U)E^m_n\mathbf{v}_i = UE^n_mE^m_n\mathbf{v}_i = U\mathbf{v}_i = \mathbf{v}_i = E^n_mE^m_n\mathbf{v}_i$$

for all $i = 1, \dots, m-k$. Thus we have $\dim \mathrm{Ker}(p(U) - E^n_m) \geq m-k$, that is, $p(U) \in F_k\mathrm{V}_{n,m}$. Therefore $p(F_k\mathrm{U}(n)) \subset F_k\mathrm{V}_{n,m}$.

We will show that $F_k\mathrm{V}_{n,m} \subset p(F_k\mathrm{U}(n))$. Take a matrix $V \in F_k\mathrm{V}_{n,m}$. There exists an $(m-k)$-frame $(\mathbf{u}_{k+1}, \dots, \mathbf{u}_m) \in \mathrm{V}_{m,m-k}$ such that $V\mathbf{u}_i = E^n_m\mathbf{u}_i$ for all $i = k+1, \dots, m$. Adding unit vectors $\mathbf{u}_1, \dots, \mathbf{u}_k \in \mathbf{C}^m$ to them, we obtain an orthonormal basis $\{\mathbf{u}_1, \dots, \mathbf{u}_m\}$ of $\mathbf{C}^m$. Define U_1 and V_1 respectively by

$$U_1 = (\mathbf{u}_1, \dots, \mathbf{u}_m), \qquad V_1 = \begin{pmatrix} E_{n-m} & O \\ O & U_1{}^{-1} \end{pmatrix} VU_1.$$

Then we have $V_1(\mathbf{e}_{k+1}, \dots, \mathbf{e}_m) = E^n_m(\mathbf{e}_{k+1}, \dots, \mathbf{e}_m)$. Hence there exists a matrix $V_2 \in \mathrm{V}_{n-m+k,k}$ satisfying that

$$V_1 = \begin{pmatrix} V_2 & O \\ O & E_{m-k} \end{pmatrix}.$$

It follows from Lemma 4.5 that there exists a matrix $U_2 \in F_k\mathrm{U}(n-m+k)$ such that $V_2 = p^{n-m+k}_k(U_2)$, where $p^{n-m+k}_k : \mathrm{U}(n-m+k) \to \mathrm{V}_{n-m+k,k}$ is the natural projection. Then the dimension of the eigenspace of 1 of U_2 is equal to or greater than $n-m$. Define a matrix U by

$$U = \begin{pmatrix} E_{n-m} & O \\ O & U_1 \end{pmatrix} \begin{pmatrix} U_2 & O \\ O & E_{m-k} \end{pmatrix} \begin{pmatrix} E_{n-m} & O \\ O & U_1{}^{-1} \end{pmatrix}.$$

The matrix U belongs to the filter $F_k\mathrm{U}(n)$, since the matrix $\begin{pmatrix} U_2 & O \\ O & E_{m-k} \end{pmatrix}$ belongs to $F_k\mathrm{U}(n)$ and $\begin{pmatrix} E_{n-m} & O \\ O & U_1 \end{pmatrix}$ is a unitary matrix. Thus we have $p(U) = V$, which implies $V \in p(F_k\mathrm{U}(n))$. Thus we have shown $p(F_k\mathrm{U}(n)) = F_k\mathrm{V}_{n,m}$. ∎

For each $k = 0, 1, \dots, n-2$, we see that the space $F_k\mathrm{V}_{n,n-1}$ has the CW-decomposition by Theorem 2.13 and that it is homotopy equivalent to X_k by Lemmas 3.1 and 3.2. Consequently we obtain that the space X_k has the homotopy type of an ANR.

We will show that the following lemma.

Lemma 4.7. *The space A_k is a compact Hausdorff ANR.*

We recall the following theorem which is used for the proof of Lemma 4.7.

Theorem 4.8. *If $h : (X, A) \to (Y, B)$ is a relative homeomorphism, where X, A, B are compact ANR's and Y is a Hausdorff space, then Y is also an ANR.*

(For a proof of Theorem 4.8 see Hu [3].) Theorem 4.8 implies the following corollary.

Corollary 4.9. *If $h : (X, A) \to (Y, B)$ is a relative homeomorphism, where X, A, B are compact Hausdorff ANR's, then Y is also a compact Hausdorff ANR.*

Proof. The space Y is homeomorphic to the adjunction space

$$B \cup_{h|A} X.$$

The space Y is compact, since X and B are compact.

We will show that the space Y is a Hausdorff space. Take two different points in $B \cup_{h|A} X$. If the points belong to B then they are separated by two disjoint open subset, since B is a Hausdorff space and since (X, A) is an NDR-pair. If the points belong to $X \setminus A$ then they are separated by two disjoint open subset, since $X \setminus A$ is an open subset of the Hausdorff space X. If a point belongs to B and the other belongs to $X \setminus A$, then they are separated by two disjoint open subset, since X is a regular space and since A is a closed subset of X. Hence the space Y is a Hausdorff space.

Therefore the space Y is an ANR from Theorem 4.8. ■

We will prove Lemma 4.7.

Proof of Lemma 4.7. The space A_k is equal to

$$\{\ [\mathbf{x} \wedge [V]] \in A'_{k+1} \mid \min(\mathbf{x}) = 0 \text{ or } \max(\mathbf{x}) = \alpha_V\ \},$$

which is decomposed by using some flag manifolds and simplices. We defined in Section 2 the flag manifold $\mathrm{F}_{n,k+1}(m_1, \ldots, m_{l+1})$ with a partition $(m_1, \ldots, m_{l+1})$ of $k+1$. For each $l = 0, \ldots, k$, the $(l+1)$-dimensional simplex Δ^{l+1} is defined by

$$\Delta^{l+1} = \{\ (x_1, \ldots, x_{l+1}) \in \mathbf{R}^{l+1} \mid 0 \leq x_1 \leq \cdots \leq x_{l+1} \leq 1\}.$$

A subspace

$$\{\ (x_1, \ldots, x_{l+1}) \in \Delta^{l+1} \mid x_1 = 0 \text{ or } x_{l+1} = 1\ \}$$

is denoted by D^l, where the boundary ∂D^l is defined by

$$\partial D^l = \{\ (x_1, \ldots, x_{l+1}) \in D^l \mid x_i = x_{i+1} \text{ for some } i = 1, \ldots, l\ \}.$$

For each $l = 0, \ldots, k$, we define a map

$$r_{k,l+1} : D^l \times \bigcup_{m_1+\cdots+m_{l+1}=k+1} \mathrm{F}_{n,k+1}(m_1, \ldots, m_{l+1}) \to A_k$$

by

$$r_{k,l+1}((x_1, \ldots, x_{l+1}), [V]) = [\alpha_V(\overbrace{x_1, \ldots, x_1}^{m_1}, \overbrace{x_2, \ldots, x_2}^{m_2}, \ldots, \overbrace{x_{l+1}, \ldots, x_{l+1}}^{m_{l+1}}) \wedge [V]]$$

for each $((x_1, \dots, x_{l+1}), [V]) \in D^l \times \bigcup_{m_1+\cdots+m_{l+1}=k+1} \mathrm{F}_{n,k+1}(m_1, \dots, m_{l+1})$. The map $r_{k,l+1}$ is well-defined. The image of $r_{k,l+1}$ is denoted by $R_{k,l+1}$. Define a subspace $R_{k,0}$ of A_k by

$$R_{k,0} = \{ \, [(0, \dots, 0) \wedge [V]] \in A_k \mid \alpha_V = 0 \, \} \, .$$

Then we obtain a filtration

$$R_{k,0} \subset R_{k,1} \subset \cdots \subset R_{k,k+1} = A_k.$$

We will prove by induction that the space A_k is a compact Hausdorff ANR. The space $R_{k,0}$ is a compact Hausdorff ANR, since it consists of the single element $[(0, \dots, 0) \wedge [V]]$.

We will show that if the space $R_{k,l}$ is a compact Hausdorff ANR then so is the space $R_{k,l+1}$ for each $l = 0, \dots, k$. Suppose that $R_{k,l}$ is a compact Hausdorff ANR. The space $R_{k,l+1}$ is equal to the adjunction space

$$R_{k,l} \cup_r \left(D^l \times \bigcup_{m_1+\cdots+m_{l+1}=k+1} \mathrm{F}_{n,k+1}(m_1, \dots, m_{l+1}) \right),$$

where r denotes the restriction of $r_{k,l+1}$ to the $(r_{k,l+1})^{-1}(R_{k,l})$. The space

$$D^l \times \bigcup_{m_1+\cdots+m_{l+1}=k+1} \mathrm{F}_{n,k+1}(m_1, \dots, m_{l+1})$$

is a compact Hausdorff ANR, since it is a finite disjoint union of compact manifolds.

We observe that the space $(r_{k,l+1})^{-1}(R_{k,l})$ is a compact Hausdorff ANR. The space $(r_{k,l+1})^{-1}(R_{k,l})$ is equal to

$$\left(\partial D^l \times \bigcup_{m_1+\cdots+m_{l+1}=k+1} \mathrm{F}_{n,k+1}(m_1, \dots, m_{l+1}) \right) \cup \left(D^l \times \bigcup_{m_1+\cdots+m_{l+1}=k+1} \{ \, [V] \in \mathrm{F}_{n,k+1}(m_1, \dots, m_{l+1}) \mid \alpha_V = 0 \, \} \right).$$

The space

$$\{ \, [V] \in \mathrm{F}_{n,k+1}(m_1, \dots, m_{l+1}) \mid \alpha_V = 0 \, \}$$

is a compact Hausdorff ANR, since it is a deformation retract of an open subset

$$\{ \, [V] \in \mathrm{F}_{n,k+1}(m_1, \dots, m_{l+1}) \mid \alpha_V \neq 1 \, \}$$

of the the flag manifold $\mathrm{F}_{n,k+1}(m_1, \dots, m_{l+1})$. If $l = 0$ then $\partial D^l = \emptyset$. Consequently the space $(r_{k,1})^{-1}(R_{k,0})$ is a compact Hausdorff ANR. We suppose that $l > 0$. It is clear that the space

$$\partial D^l \times \bigcup_{m_1+\cdots+m_{l+1}=k+1} \mathrm{F}_{n,k+1}(m_1, \dots, m_{l+1})$$

is a compact Hausdorff ANR. The spaces

$$D^l \times \bigcup_{m_1+\cdots+m_{l+1}=k+1} \{\, [V] \in \mathrm{F}_{n,k+1}(m_1,\ldots,m_{l+1}) \mid \alpha_V = 0 \,\},$$
$$\partial D^l \times \bigcup_{m_1+\cdots+m_{l+1}=k+1} \{\, [V] \in \mathrm{F}_{n,k+1}(m_1,\ldots,m_{l+1}) \mid \alpha_V = 0 \,\}$$

are compact Hausdorff ANRs. Hence the space $(r_{k,l+1})^{-1}(R_{k,l})$ is a compact Hausdorff ANR from Corollary 4.9.

We have shown by Corollary 4.9 that $R_{k,l+1}$ is a compact Hausdorff ANR.

Therefore the space A_k is a compact Hausdorff ANR by induction. ■

Concluding Remark. We have already known the cone-decomposition $\{A_k \xrightarrow{i_k} X_k \to X_{k+1}\}_{k=0}^{n-1}$ of the unitary group $\mathrm{U}(n)$ such that $F_k\mathrm{U}(n) \simeq X_k$ in [5]. Miller's filtration of Stiefel manifolds are closely related to Morse-Bott functions of them defined by Frankel [1]. By using Frankel's Morse-Bott function, we can construct cone-decompositions $\{A_k \xrightarrow{i_k} X_k \to X_{k+1}\}_{k=0}^{m-1}$ of the complex Stiefel manifold $\mathrm{V}_{n,m}$ such that $F_k\mathrm{V}_{n,m} \simeq X_k$. For each real and quaternionic Stiefel manifolds containing all the orthonormal m-frames in $\mathbf{R}^n$ and $\mathbf{H}^n$ respectively, where $0 < m \le \frac{n}{2}$, we obtain a similar result to the above, but not for the case of rotation groups and symplectic groups. We can also expand the above method into some symmetric Riemannian spaces. These further results will be written in [6].

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