An interplay between a generalized-Euler-constant function and the Hurwitz zeta function

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Abstract

For the generalized-Euler-constant function

$$a \mapsto \gamma(a) := \lim_{n\to\infty}\left(\sum_{i=0}^{n-1}\frac{1}{a+i} - \ln\frac{a+n-1}{a}\right)$$

defined on $\mathbb{R}^+$, the expansion $\gamma(a) = \sum_{j=2}^{\infty}\frac{(-1)^j}{j}\zeta(j,a)$, where $\zeta(j,a)$ is the Hurwitz zeta function, is derived and a formula for its numerical computation is presented.

1 Introduction

Recently, [4] and [5], a generalized-Euler-constant-function $a \mapsto \gamma(a)$ has been introduced as the limit of the sequence $n \mapsto \gamma_n(a)$ given as

$$\gamma_n(a) = \sum_{i=0}^{n-1}\frac{1}{a+i} - \ln\frac{a+n-1}{a}\,, \tag{1}$$

where $\gamma(1)$ is the Euler-Mascheroni constant. The author showed that, for $a > 0$, the function $a \mapsto \gamma(a)$ is well defined and strictly decreasing on $\mathbb{R}^+$. Subsequently, several estimates concerning the rate of convergence of the sequence

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$n \mapsto \gamma_n(a)$ were presented. In our contribution we shall reconfirm, using a different method, the existence of the function $\gamma(a)$ by expanding it into an infinite series in terms of the Hurwitz zeta function[1] $\zeta(s,a) := \sum_{i=0}^{\infty}(a+i)^{-s}$. This way we shall obtain a generalization of the well known expansion (see e.g. [3, p. 35])

$$\gamma(1) = \sum_{k=2}^{\infty}(-1)^k \frac{\zeta(k)}{k},$$

where $\zeta(s) = \sum\limits_{k=1}^{\infty} k^{-s}$ $(s > 1)$ is the Riemann zeta-function. Concerning the computational aspects we shall derive an approximation to $\gamma(a)$ in terms of the function $\zeta(s,a)$, assumed to be numerically known. This supposition is not too pretentious since there are known certain algorithms for numerical computation of Hurwitz zeta function $\zeta(s,a)$, especially when s is an integer and a an algebraic number [1]. We also note that $\zeta(s,a)$ is a function built-in *Mathematica* [6], for example.

2 An expansion using Hurwitz zeta function

The identity (1) can be re-formed using the telescoping method as follows

$$\begin{aligned}\gamma_n(a) &= \sum_{i=0}^{n}\left[\frac{1}{a+i} - \ln\left(1+\frac{1}{a+i}\right)\right] - \frac{1}{a+n} + \ln\left(1+\frac{2}{a+n-1}\right) \\ &= \sum_{i=0}^{n}\Big[h_i - \ln(1+h_i)\Big] - h_n + \ln(1+2h_{n-1}), \qquad (2)\end{aligned}$$

where

$$h_i = h_i(a) := \frac{1}{a+i}. \qquad (3)$$

Thus, for positive integers $n \geq m \geq 1$,

$$\begin{aligned}\gamma_n(a) &= \sum_{i=0}^{m-1}\frac{1}{a+i} - \ln\frac{a+m}{a} \\ &\quad + \sum_{i=m}^{n}\Big[h_i - \ln(1+h_i)\Big] - h_n + \ln(1+2h_{n-1}). \qquad (4)\end{aligned}$$

Now, the Hurwitz zeta function can be introduced approximating the logarithmic function. Indeed, according to the identity

$$\frac{1}{1+t} = \sum_{i=0}^{p-1}(-t)^i + \frac{(-t)^p}{1+t}$$

valid for any positive integer p and $t \neq -1$, we have

$$\ln(1+h) = \sum_{i=0}^{p-1}(-1)^i \frac{h^{i+1}}{i+1} + \int_0^h \frac{(-t)^p}{1+t}\,\mathrm{d}t,$$

[1]Hurwitz zeta function is also known as the generalized Riemann zeta function.

for $h > -1$. Thus

$$h - \ln(1+h) = \sum_{j=2}^{p} \frac{(-h)^j}{j} + r_p(h)\,, \tag{5}$$

where

$$0 < \frac{h^{p+1}}{(1+h)(p+1)} < (-1)^{p+1} \cdot r_p(h) < \int_0^h t^p \,\mathrm{d}t = \frac{h^{p+1}}{p+1}\,, \tag{6}$$

for $p \geq 2$ and $h > 0$.

Now, considering (2) and (5), we obtain

$$\begin{aligned}\gamma_n(a) &= \sum_{i=0}^{n} \left[\sum_{j=2}^{p} \frac{(-h_i)^j}{j} + r_p(h_i) \right] - \frac{1}{a+n} + \ln\left(1 + \frac{2}{a+n-1}\right) \\ &= \sum_{j=2}^{p} \frac{(-1)^j}{j} Z_{0,n}(j,a) - \frac{1}{a+n} + \ln\left(1 + \frac{2}{a+n-1}\right) + \rho_{0,n}(a,p),\end{aligned} \tag{7}$$

where

$$Z_{m,n}(s,a) := \sum_{i=m}^{n} \frac{1}{(a+i)^s} \qquad (a > 0,\ n \geq m \geq 0,\ s > 1), \tag{8}$$

and

$$\rho_{m,n}(a,p) := \sum_{i=m}^{n} r_p(h_i) = (-1)^{p+1} \sum_{i=m}^{n} \int_0^{1/(a+i)} \frac{t^p}{1+t}\,\mathrm{d}t, \tag{9}$$

stands for the error term. Obviously, appealing to (6), we have

$$\begin{aligned}0 < (-1)^{p+1} \cdot \rho_{m,n}(a,p) &= \sum_{i=m}^{n} (-1)^{p+1} \cdot r_p(h_i) \\ &< \sum_{i=m}^{n} \frac{h_i^{p+1}}{p+1} = \sum_{i=m}^{n} \frac{1}{(p+1)(a+i)^{p+1}}\,.\end{aligned} \tag{10}$$

Additionally, using the inequality $1/(1+h_i) = 1/\big(1+1/(a+i)\big) \geq \frac{a+m}{a+m+1}$, valid for $i \geq m$, and appealing to (9) and (6), we also estimate

$$\begin{aligned}(-1)^{p+1} \cdot \rho_{m,n}(a,p) &= \sum_{i=m}^{n} (-1)^{p+1} \cdot r_p(h_i) \\ &> \sum_{i=m}^{n} \frac{h_i^{p+1}}{(1+h_i)(p+1)} \geq \tfrac{a+m}{a+m+1} \cdot \sum_{i=m}^{n} \frac{1}{(p+1)(a+i)^{p+1}}\,.\end{aligned} \tag{11}$$

Considering (10) and the convergence of the series $\sum_{i=0}^{\infty} (a+i)^{-(p+1)}$, we see that

$$\rho^*(a,p) := \lim_{n\to\infty} \rho_{0,n}(a,p)$$

exists for $a > 0$ and $p \geq 2$ and the estimate

$$0 < (-1)^{p+1} \rho^*(a,p) < \frac{1}{p+1} \sum_{i=0}^{\infty} \frac{1}{(a+i)^{p+1}} = \frac{\zeta(p+1,a)}{p+1}\,, \tag{12}$$

holds true with $\zeta(s,a)$ being the Hurwitz zeta function,

$$\zeta(s,a) := \lim_{n\to\infty} Z_{0,n}(s,a) = \sum_{i=0}^{\infty} \frac{1}{(a+i)^s} \qquad (a>0,\ s>1). \tag{13}$$

Moreover, referring to (7), the convergence

$$\gamma(a) := \lim_{n\to\infty} \gamma_n(a)$$

is established together with the equality

$$\gamma(a) = \sum_{j=2}^{p} \frac{(-1)^j}{j} \zeta(j,a) + \rho^*(a,p). \tag{14}$$

Hence, letting $p \to \infty$ in (12)–(14) and considering the absolute convergence of the obtained double series, we get the following theorem.

Theorem 1. *The generalized-Euler-constant function $\gamma(a)$ has the expansions*

$$\gamma(a) = \sum_{j=2}^{\infty} \frac{(-1)^j}{j} \zeta(j,a) = \sum_{i=0}^{\infty} \sum_{j=2}^{\infty} \frac{(-1)^j}{j\,(a+i)^j}, \tag{15}$$

for $a>0$, where $\zeta(j,a) = \sum_{i=0}^{\infty}(a+i)^{-j}$ is the generalized Riemann zeta function known also as Hurwitz zeta function.

Using the theorem above, properties of the function $\gamma(a)$ such as the monotonicity, the differentiability and the boundedness, for example, can be studied. However, to estimate $\gamma(a)$ numerically we shall use a slightly different approach.

3 An approximation to $\gamma(a)$

The following theorem gives a useful two-parameter approximation.

Theorem 2. *For real $a>0$ and for integers (parameters) $m \ge 1$ and $p \ge 2$ we have*

$$\gamma(a) = \sigma_m(a,p) + \rho_m^*(a,p), \tag{16}$$

where

$$\sigma_m(a,p) = \sum_{j=2}^{p} \frac{(-1)^j}{j} \zeta(j,a) + \sum_{i=0}^{m-1} \left(\frac{1}{a+i} - \sum_{j=2}^{p} \frac{(-1)^j}{j\,(a+i)^j} \right) - \ln\frac{a+m}{a} \tag{17}$$

and

$$\frac{a+m}{a+m+1} \cdot \frac{1}{p(p+1)(a+m)^p} < (-1)^{p+1} \rho_m^*(a,p) < \frac{1}{p(p+1)(a+m-1)^p}. \tag{18}$$

Proof. Using (4), (5) and (8), we get

$$\gamma_n(a) = \sum_{i=0}^{m-1} \frac{1}{a+i} - \ln\frac{a+m}{a} + \sum_{j=2}^{p} \frac{(-1)^j}{j} Z_{m,n}(j,a) - \frac{1}{a+n} + \ln\left(1+\frac{2}{a+n-1}\right) + \rho_{m,n}(a,p), \tag{19}$$

where, according to (10),

$$\rho_m^*(a,p) := \lim_{n\to\infty} \rho_{m,n}(a,p) \tag{20}$$

exists for $p \geq 2$. Referring to (10) and (11), the estimates

$$\frac{a+m}{a+m+1} \cdot \sum_{i=m}^{\infty} \frac{1}{(p+1)(a+i)^{p+1}} < (-1)^{p+1}\rho_m^*(a,p) < \sum_{i=m}^{\infty} \frac{1}{(p+1)(a+i)^{p+1}}. \tag{21}$$

are seen to hold true. Consequently, letting $n \to \infty$ in (19), the relations (16)–(17) follow.

Since, for $b > 0$ and $s > 1$, the function $x \mapsto (b+x)^{-s}$ is strictly decreasing on $\mathbb{R}^+$, the estimates

$$Z_{m,n}(s,b) = \sum_{i=m}^{n} \frac{1}{(b+i)^s} > \int_m^{n+1} \frac{1}{(b+x)^s}\,dx = \frac{1}{s-1}\left[\frac{1}{(b+m)^{s-1}} - \frac{1}{(b+n+1)^{s-1}}\right], \tag{22}$$

$$Z_{m,n}(s,b) = \sum_{i=m}^{n} \frac{1}{(b+i)^s} < \int_{m-1}^{n} \frac{1}{(b+x)^s}\,dx = \frac{1}{s-1}\left[\frac{1}{(b+m-1)^{s-1}} - \frac{1}{(b+n)^{s-1}}\right] \tag{23}$$

hold true for integers $n \geq m \geq 1$ and for real $b > 0$ and $s > 1$.

Obviously, the relations (21)–(23) imply the estimates (18). ∎

Now, using Theorem 2, the constant $\gamma(a)$ can be computed quite accurately. Namely, according to (18), we have, for $a > 0$,

$$\begin{aligned} -2.1 \times 10^{-3} &< \rho_{10}^*(a,2) < -1.6 \times 10^{-3}, \\ 0.9 \times 10^{-5} &< \rho_{20}^*(a,3) < 1.3 \times 10^{-5}, \\ -2.6 \times 10^{-41} &< \rho_{100}^*(a,19) < -3.2 \times 10^{-41}. \end{aligned}$$

Even for small m or p, Theorem 2 gives a useful estimate for $\gamma(a)$. For example, setting $m = p = 2$ in it, we obtain the next corollary.

Corollary 2.1. *For $a > 0$ the following estimates hold*

$$\gamma(a) > \gamma^*(a) := \left[\frac{1}{a} + \frac{1}{a+1} - \ln\left(1+\frac{2}{a}\right)\right] + \frac{1}{2(a+2)} - \frac{1}{6(a+1)^2} \tag{24}$$

$$\gamma(a) < \gamma^{**}(a) := \left[\frac{1}{a} + \frac{1}{a+1} - \ln\left(1+\frac{2}{a}\right)\right] + \frac{1}{2(a+1)} - \frac{1}{6(a+2)(a+3)}. \tag{25}$$

Consequently, $\lim_{a\downarrow 0}\gamma(a) = \infty$ *and* $\lim_{a\to\infty}\gamma(a) = 0.$

Proof. Using (17), we calculate

$$\begin{aligned}\sigma_2(a,2) =& \frac{1}{2}\zeta(2,a) + \frac{1}{a} + \frac{1}{a+1} - \frac{1}{2a^2} - \frac{1}{2(a+1)^2} - \ln\left(1+\frac{2}{a}\right)\\ =& \left[\frac{1}{a} + \frac{1}{a+1} - \ln\left(1+\frac{2}{a}\right)\right] + \frac{1}{2}\sum_{i=2}^{\infty}\frac{1}{(a+i)^2} \qquad (a>0).\end{aligned} \tag{26}$$

Using (22)–(23), we estimate

$$\frac{1}{a+2} < \sum_{i=2}^{\infty}\frac{1}{(a+i)^2} < \frac{1}{a+1} \qquad (a>0) \tag{27}$$

and, appealing to (18), also

$$-\frac{1}{6(a+1)^2} < \rho_2^*(a,2) < -\frac{1}{6(a+2)(a+3)} \qquad (a>0). \tag{28}$$

The relations (26)–(28) verify the corollary. ■

Figure 1 shows the graph of the function $\gamma(a)$ and the graphs of its lower and upper bounds $\gamma^*(a)$ and $\gamma^{**}(a)$.

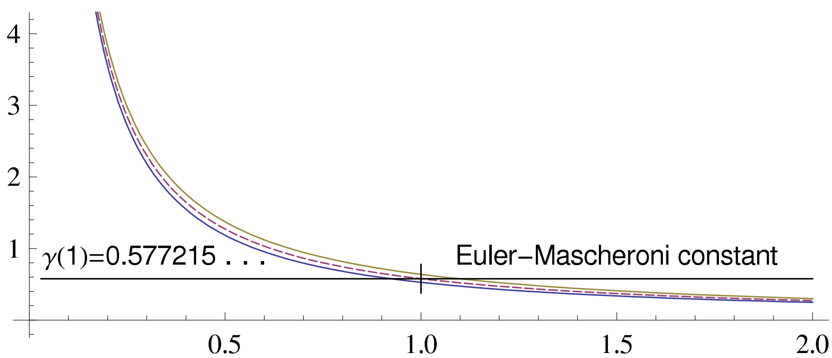

Figure 1: The graph of the function $\gamma(a)$ (dashed line) between its bounds; $\gamma^*(a)$ and $\gamma^{**}(a)$.

The relative error $E(a)$ of the approximation $\gamma(a) \approx \gamma^{**}(a)$,

$$E(a) := \frac{\gamma(a) - \gamma^{**}(a)}{\gamma^{**}(a)},$$

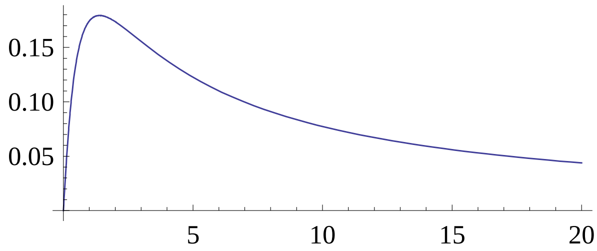

Figure 2: The graph of the absolute relative error of the approximation $\gamma(a) \approx \gamma^{**}(a)$.

is absolutely less than 20% as it is evident from Figure 2 showing the graph of the function $a \mapsto \big(\gamma^{**}(a) - \gamma^{*}(a)\big)/\gamma^{**}(a) > |E(a)|$.

Corollary 2.2. *For real $a > 0$ and for integers $n \geq m \geq 1$, we have*

$$\sum_{i=m}^{n} \frac{1}{a+i} = \sum_{j=2}^{p} \frac{(-1)^j}{j} Z_{m,n}(j,a) + \ln\frac{a+n+1}{a+m} + \rho_{m,n}(a,p), \tag{29}$$

where $\rho_{m,n}(a,p)$ can be estimated using (10)–(11) *and* (22)–(23).

Proof. The corollary follows directly from (19) and (1). ∎

References

[1] E.A. Karatsuba, *Fast computation of the values of the Hurwitz zeta function and Dirichlet L-series*, (Russian) Problemy Peredachi Informatsii 34(1998), no. 4, 62–75; translation in Problems Inform. Transmission 34(1998), no. 4, 342–353.

[2] R. Kreminski, *Newton-Cotes integration for approximating Stieltjes (generalized Euler) constants*, Math. Comp. 72(2003), no. 243, 1379–1397.

[3] W. Magnus, F. Oberhettinger and R.P. Soni; *Formulas and theorems for the special functions of mathematical physics*, 3rd/ed, Springer-Verlag, Berlin-Heidelberg-New York, 1966.

[4] A. Sîntămărian, *A generalization of Euler's constant*, Numer. Algor. 46(2007), 141–151.

[5] A. Sîntămărian, *Some inequalities regarding a generalization of Euler's constant*, J. Ineq. Pure Appl. Math. 9(2008)2, art. 46, 7 pp.

[6] S. Wolfram, *Mathematica*, version 6.0, Wolfram Research, Inc., 1988–2008.

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