Inverse problem for symmetric P-symmetric matrices with a submatrix constraint *

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Abstract

For a fixed generalized reflection matrix P, i.e., $P^T = P, P^2 = I$, and $P \neq \pm I$, then a matrix A is called a symmetric P-symmetric matrix if $A = A^T$ and $(PA)^T = PA$. This paper is mainly concerned with finding the least squares symmetric P-symmetric solutions to the matrix inverse problem $AX = B$ with a submatrix constraint, where X and B are given matrices of suitable size. By applying the generalized singular value decomposition and the canonical correlation decomposition, an analytical expression of the least squares solutions is derived basing on the Projection Theorem in Hilbert inner products spaces. Moreover, in the corresponding solution set, the analytical expression of the unique minimum-norm solution is described in detail.

1 Introduction

Let $R^{n\times m}$, $OR^{n\times n}$, $SR^{n\times n}$, $ASR^{n\times n}$ denote the set of all $n \times m$ real matrices, the set of all $n \times n$ orthogonal matrices, the set of all $n \times n$ real symmetric matrices, the set of all $n \times n$ real anti-symmetric matrices, respectively; The symbols, A^+ and $\|A\|$ denote the Moore-Penrose generalized inverse, the Frobenius norm respectively. For two matrices $A = (a_{ij}), B = (b_{ij}) \in R^{n\times m}$, $A * B = (a_{ij}b_{ij}) \in R^{n\times m}$

*Research supported by National Natural Science Foundation of China (10571047 and 10861005), and Provincial Natural Science Foundation of Guangxi (0991238).

Received by the editors November 2008 - In revised form in May 2009.

Communicated by A. Bultheel.

2000 *Mathematics Subject Classification :* 15A57; 15A24.

Key words and phrases : Inverse problem; symmetric P-symmetric matrix; least squares solutions; generalized singular value decomposition; canonical correlation decomposition.

denotes their Hadamard product, $\langle A, B\rangle = tr(B^T A)$ represents their inner product, where $tr(.)$ denotes the trace of the corresponding matrix. Then we can easily see that $R^{n\times m}$ is a Hilbert inner product space equipped with the Frobnius norm of matrices, which is induced from the inner product $\langle A, B\rangle = tr(B^T A)$.

Matrix inverse problem: given three sets of real $n \times n$ matrices S, real $n-$vectors $x_1, \dots x_m$, and $n-$vectors $b_1, \cdots b_m$, $m \leq n$, find a real $n \times n$ matrix $A \in \mathscr{L}$ such that

$$Ax_i = b_i, \quad i = 1, 2, \dots m.$$

Let $X = (x_1, x_2, \dots x_m)$, $B = (b_1, b_2, \dots b_m)$, then the above relation can be written as

$$AX = B$$

If $B = X\Lambda$, $\Lambda = diag(\lambda_1, \lambda_1, \dots \lambda_m)$, where $\lambda_1, \lambda_1, \dots \lambda_m$ are numbers, then the above problem is called the inverse eigenvalue problem.

The prototype of those problems initially arose in the design of Hopfield neural networks [9, 10]. It is applied in various areas, such as the discrete analogue of inverse Sturm-Liouville problem [7], vibration design [22], and structural design [8].

For decades, many authors have been devoted to the study of matrix inverse problem associated with several kinds of different sets $\mathscr{L}$, and we refer to [11, 14, 15, 20]. However, we should point out that the matrices X and B occurring in practice are usually obtained from experiment and they may not satisfy the solvability condition. Therefore , we need further study the least-squares solutions for the problem above which is associated with several kinds of different sets $\mathscr{L}$, for instance, general matrices, symmetric matrices, symmetric nonnegative definite matrices and so on.

Let $P \in R^{n\times n}$ be a fixed generalized reflection matrix, i.e.,$P^T = P, P^2 = I$, and $P \neq \pm I$, then a matrix A is called a symmetric P-symmetric matrix if $A = A^T$ and $(PA)^T = PA$, or symmetric P-skew symmetric matrix if $A = A^T$ and $(PA)^T = -PA$. The set of all symmetric P-(skew)symmetric matrices is denoted by $SR_P^{n\times n}(SAR_P^{n\times n})$. In particular, if J is the flip matrix with ones on the secondary diagonal and zeros elsewhere, then a symmetric J-symmetric matrix is bi-symmetric, i.e., $a_{ij} = a_{ji} = a_{n-i+1,n-j+1}$, $1 \leq i, j \leq n$, while a symmetric J-skew symmetric matrix is symmetric and skew-antisymmetric, i.e.,$a_{ij} = a_{ji} = -a_{n-i+1,n-j+1}$, $1 \leq i, j \leq n$. Bi-symmetric matrices, such as symmetric Toeplitz matrices and autocovariance matrices, have practical application in information theory, linear system theory and numerical analysis. If $P = I_n$, then $SR_P^{n\times n}$ is a symmetric matrix set and $SAR_P^{n\times n}$ is trivial due to the fact that $SR^{n\times n} \cap ASR^{n\times n} = 0$.

The symmetric P-symmetric matrices were initially considered by Zhou, Hu and Zhang, associated with matrix equations and inverse eigenvalue problems, see[25, 26]. Peng [18] has investigated the symmetric P-symmetric solution to the matrix equation

$$A^T X A = B,$$

which arose in an inverse problem of structural modification or the dynamic behavior of a structure. However the inverse problem for symmetric P-symmetric matrices with a submatrix constraint has not been discussed. The inverse problem with a submatrix constraint comes from a practical subsystem expansion problem. Researchers have great interest in studying a variety of inverse problem under submatrices constraint in recent years. For example, Deift and Nanda [4] discussed an inverse eigenvalue problem of a tridiagonal matrix under a submatrix constraint; Peng and Hu [16] considered an inverse eigenpair problem of a Jacobi matrix under a leading principal submatrix constraint; Peng and Hu [17] studied a inverse problem of bi-symmetric matrices with a leading principal submatrix constraint, for more we refer the reader to [6, 12, 24]. To our knowledge, there is no result about the least-squares solutions of matrix inverse problem for symmetric P-symmetric matrices with a submatrix constraint. In this paper , we will mainly discuss this problem.

Throughout, we always assume that P is a fixed generalized reflection matrix. The problem studied in this paper can be described as follows

Problem I. Given matrices $X, B \in R^{n\times m}$ and $A_0 \in SR^{q\times q}$. Let

$$\Gamma = \left\{ A \in SR_P^{n\times n} \,|\, \|AX - B\| = \min \right\},$$

find $\tilde{A} \in \Gamma$ such that

$$\|\tilde{A}([1:q]) - A_0\| = \min_{A\in\Gamma} \|A([1:q]) - A_0\|.$$

$A([1:q])$ is the principal submatrix of A lying in the first q rows and columns.

Problem II. Let S_E be the solution set of Problem I. Find $\hat{A} \in S_E$ such that

$$\|\hat{A}\| = \min_{\bar{A}\in S_E} \|\bar{A}\|. \tag{1.1}$$

We remark that when $q = 0$, Problem I is reduced to the inverse problem for symmetric P-symmetric matrices discussed by [5] and [25]. When $q = n$, A_0 is the best approximation of the matrix $\tilde{A} \in \Gamma$. In this paper, we consider the general case when $0 < q < n$. Problem II is in fact to find the minimum-norm solution of the solution set of Problem I.

The paper is organized as follows. After introducing some necessary concepts of two matrix-factorization techniques in Section 2 and some useful preliminary results in Section 3, we will derive an analytical expression for the solution of Problem I in section 4. The expressions for the unique solution of Problem II is obtained in Section 5. At last, in Section 6, we use some brief conclusions to end the paper.

2 Two matrix-factorization techniques

As a preliminary, we briefly state the concepts of generalized singular value decomposition (GSVD) and canonical correlation decomposition (CCD), which are essential tools for deriving the solution of Problem I.

Let $N_1 \in R^{q\times(r-r_1)}$, $N_2 \in R^{q\times(s-r_2)}$, then the GSVD of the matrix pair $(N_1,\ N_2)$ is given by

$$N_1^T = Q_1\Omega_1 M,\ \ N_2^T = Q_2\Omega_2 M, \tag{2.1}$$

where $M \in R^{q\times q}$ is a non-singular matrix, $Q_1 \in OR^{(r-r_1)\times(r-r_1)}$, $Q_2 \in OR^{(s-r_2)\times(s-r_2)}$, and

$$\Omega_1 = \begin{pmatrix} I_f & & & \vdots & \\ & S_1 & & \vdots & O \\ & & O_1 & \vdots & \end{pmatrix} \begin{matrix} f \\ g \\ m_1 \end{matrix} ,\quad \Omega_2 = \begin{pmatrix} O_2 & & & \vdots & \\ & S_2 & & \vdots & O \\ & & I_{t-r-s} & \vdots & \end{pmatrix} \begin{matrix} m_2 \\ g \\ t-f-g \end{matrix},$$
$$\quad f \quad g \quad t-g-f \quad q-t \qquad\qquad f \quad g \quad t-g-f \quad q-t$$

where $m_1 = r - r_1 - f - g, m_2 = s - r_2 - t + f$, with the diagonal matrices S_1 and S_2 being given by

$$S_1 = diag(\mu_1, \mu_2, \cdots \mu_g) > 0 \text{ and } S_2 = diag(\lambda_1, \lambda_2, \cdots \lambda_g) > 0.$$

Here

$$t = rank\,[N_1,\ N_2]\,,\ \ f = t - rank(N_2),\ \ g = rank(N_1) + rank(N_2) - t,$$

O, O_1, O_2 are zero matrices of suitable sizes.

We further partition the non-singular matrix M as

$$M^{-1} = (M_1, M_2, M_3, M_4), \tag{2.2}$$

where $M_1 \in R^{q\times f}, M_2 \in R^{q\times g}, M_3 \in R^{q\times(t-f-g)}, M_4 \in R^{q\times(q-t)}$.

The CCD of the matrix pair $(N_1,\ \ N_2)$ is given by

$$N_1 = H(\Upsilon_1, 0)E_1^{-1}\ \ \ N_2 = H(\Upsilon_2, 0)E_2^{-1}, \tag{2.3}$$

where $E_1 \in R^{(r-r_1)\times(r-r_1)}, E_2 \in R^{(s-r_2)\times(s-r_2)}$ are non-singular matrices, $H \in OR^{q\times q}$, and

$$\Upsilon_1 = \begin{pmatrix} I_{r_0} & 0 & 0 \\ 0 & S_N & 0 \\ 0 & 0 & 0 \\ 0 & 0 & 0 \\ 0 & C_N & 0 \\ 0 & 0 & I_{t_0} \end{pmatrix},\ \ \Upsilon_2 = \begin{pmatrix} I_h \\ 0 \end{pmatrix},$$

are block matrices, with the diagonal matrices S_N and C_N given by

$$S_N = diag(\alpha_1, \alpha_2, \cdots \alpha_{s_0}) > 0, \text{ and } C_N = diag(\beta_1, \beta_2, \cdots \beta_{s_0}) > 0$$

Here

$$h = rank(N_2),\ r_0 = rank(N_1) + rank(N_2) - rank\,[N_1,\ N_2]\,,\ s_0 = rank(N_1^T N_2) - r_0.$$

We further partition the orthogonal matrix H as

$$H = (H_1, H_2, H_3, H_4, H_5, H_6), \tag{2.4}$$

where $H_1 \in R^{n\times r_0}, H_2 \in R^{n\times s_0}, H_3 \in R^{n\times(h-r_0-s_0)}, H_4 \in R^{n\times(q-h-r_0-s_0)}$, $H_5 \in R^{n\times s_0}, H_6 \in R^{n\times t_0}$.

3 Preliminary results

Let r and s be respectively the dimensions of the eigenspaces of P associated with the eigenvalues $\lambda = 1$ and $\lambda = -1$; thus $r, s > 1$ and $r + s = n$. Since a generalized reflection matrix is diagonalizable and $P \neq \pm I$. Let

$$P_1 = [p_1, \ldots, p_r] \in R^{n\times r} \text{ and } P_2 = [q_1, \ldots, q_s] \in R^{n\times s},$$

where $p_1, \ldots, p_r$ and $q_1, \ldots, q_s$ are orthonormal bases for the eigenspaces. P_1 and P_2 can be found by applying the Gram-Schmidt process to the columns of $I + P$ and $I - P$, respectively.

The following lemma characterizes the class of symmetric P-symmetric matrices, which are the special case of [[14], Theorem 1].

Lemma 1. *$A \in R^{n\times n}$ is symmetric P-symmetric if and only if*

$$A = [P_1 \;\; P_2]\begin{bmatrix} A_1 & 0 \\ 0 & A_2 \end{bmatrix}\begin{bmatrix} P_1^T \\ P_2^T \end{bmatrix}, \tag{3.1}$$

where

$$A_1 = P_1^T A P_1 \in SR^{r\times r} \text{ and } A_2 = P_2^T A P_2 \in SR^{s\times s}. \tag{3.2}$$

The following lemma from [21] is a directly use for our mainly results.

Lemma 2. (Projection Theorem) *Let $\mathbb{X}$ be a finite dimensional inner product space, $\mathbb{M}$ be a subspace of $\mathbb{X}$, and $\mathbb{M}^\perp$ be the orthogonal complement subspace of $\mathbb{M}$. For a given $x \in \mathbb{X}$, there always exists an $m_0 \in \mathbb{M}$ such that $\|x - m_0\| \leq \|x - m\|, \;\; \forall m \in \mathbb{M}$, where $\|.\|$ is the norm associated with the inner product defined in $\mathbb{X}$. Moreover, $m_0 \in \mathbb{M}$ is the unique minimization vector in $\mathbb{M}$ if and only if $(x - m_0)\perp\mathbb{M}$ i.e, $(x - m_0) \in \mathbb{M}^\perp$.*

To derive the solutions of Problem I, we need to characterize the elements in the set Γ. In order to do that, let the SVDs of $P_1^T X$, $P_2^T X$ be, respectively,

$$P_1^T X = U_1 \begin{pmatrix} \Sigma_1 & 0 \\ 0 & 0 \end{pmatrix} V_1^T, \quad P_2^T X = U_2 \begin{pmatrix} \Sigma_2 & 0 \\ 0 & 0 \end{pmatrix} V_2^T, \tag{3.3}$$

where $U_1 = (U_{11}\,, U_{12}) \in OR^{r\times r}$ and $V_1 = (V_{11}\,, V_{12}) \in OR^{m\times m}$, $U_2 = (U_{21}\,, U_{22})$ $\in OR^{s\times s}$ and $V_2 = (V_{21}\,, V_{22}) \in OR^{m\times m}$; $\Sigma_1 = diag(\sigma_1, \sigma_2 \ldots \sigma_{r_1})$, and $\Sigma_2 = diag(\gamma_1, \gamma_2 \ldots \gamma_{r_2})$ are diagonal matrices with positive diagonal entries; $U_{11} \in R^{r\times r_1}$, $V_{11} \in R^{m\times r_1}$, $U_{21} \in R^{s\times r_2}$, $U_{11} \in R^{m\times r_2}$, here $r_i = \operatorname{rank}(P_i^T X)$ (i=1, 2).

From [5] and [25], we know that the matrices A in the set Γ have the following expression

$$A = [P_1 \;\; P_2]\begin{pmatrix} A_{11}^{(0)} + U_{12}G_1U_{12}^T & 0 \\ 0 & A_{22}^{(0)} + U_{22}G_2U_{22}^T \end{pmatrix}\begin{bmatrix} P_1^T \\ P_2^T \end{bmatrix}, \tag{3.4}$$

where

$$A_{11}^{(0)} = U_1 \begin{pmatrix} \Psi_1 * (U_{11}^T P_1^T B V_{11}\Sigma_1 + \Sigma_1 V_{11}^T B^T P_1 U_{11}) & \Sigma_1^{-1} V_{11}^T B^T P_1 U_{12} \\ U_{12}^T P_1^T B V_{11}\Sigma_1^{-1} & 0 \end{pmatrix} U_1^T, \tag{3.5}$$

$$A_{22}^{(0)} = U_2 \begin{pmatrix} \Psi_2 * (U_{21}^T P_2^T B V_{21} \Sigma_2 + \Sigma_2 V_{21}^T B^T P_2 U_{21}) & \Sigma_2^{-1} V_{21}^T B^T P_2 U_{22} \\ U_{22}^T P_2^T B V_{21} \Sigma_2^{-1} & 0 \end{pmatrix} U_2^T. \tag{3.6}$$

with

$$\Psi_1 = (\psi_{ij}^{(1)}) \in R^{r_1 \times r_1}, \ \psi_{ij}^{(1)} = \frac{1}{\sigma_i^2 + \sigma_j^2}, \quad 1 \le i, j \le r_1,$$
$$\Psi_2 = (\psi_{ij}^{(2)}) \in R^{r_2 \times r_2}, \ \psi_{ij}^{(2)} = \frac{1}{\gamma_i^2 + \gamma_j^2}, \quad 1 \le i, j \le r_2.$$

and $G_1 \in SR^{(r-r_1)\times(r-r_1)}$ and $G_2 \in SR^{(s-r_2)\times(s-r_2)}$ are arbitrary symmetric matrices.

4 General expression of the solutions to Problem I

In this section, we derive an analytical expression for the solution of Problem I. Obviously, solving Problem I is equivalent to find $A \in \Gamma$ such that $A([1:q])$ is the best approximation leading principal submatrix of A_0, i.e., find $A \in \Gamma$ such that

$$\|(I_q, 0)A(I_q, 0)^T - A_0\| = \min, \tag{4.1}$$

where $(I_q, 0) \in R^{q\times n}$. We further partition

$$(I_q, 0)[P_1, P_2] = (D_1, D_2), \quad D_1 \in R^{q\times r}, \quad D_2 \in R^{q\times s}. \tag{4.2}$$

Then for $A \in \Gamma$, we have

$$\begin{aligned} &\|(I_q, 0)A(I_q, 0)^T - A_0\| \\ &= \left\| (I_q, 0)[P_1, P_2] \begin{pmatrix} A_{11}^{(0)} + U_{12} G_1 U_{12}^T & 0 \\ 0 & A_{22}^{(0)} + U_{22} G_2 U_{22}^T \end{pmatrix} \begin{bmatrix} P_1^T \\ P_2^T \end{bmatrix} (I_q, 0)^T - A_0 \right\| \\ &= \left\| (D_1, D_2) \begin{pmatrix} A_{11}^{(0)} + U_{12} G_1 U_{12}^T & 0 \\ 0 & A_{22}^{(0)} + U_{22} G_2 U_{22}^T \end{pmatrix} \begin{pmatrix} D_1^T \\ D_2^T \end{pmatrix} - A_0 \right\| \\ &= \|D_1 U_{12} G_1 U_{12}^T D_1^T + D_2 U_{22} G_2 U_{22}^T D_2^T - (A_0 - D_1 A_{11}^{(0)} D_1^T - D_2 A_{22}^{(0)} D_2^T)\|. \end{aligned}$$

Denote

$$D_1 U_{12} = N_1, \quad D_2 U_{22} = N_2, \quad A_0 - D_1 A_{11}^{(0)} D_1^T - D_2 A_{22}^{(0)} D_2^T = W. \tag{4.3}$$

Then $\|(I_q, 0)A(I_q, 0)^T - A_0\| = \min_{A\in\Gamma}$ is equivalent to find the least squares symmetric solution (G_1, G_2) with respect to the inconsistent matrix equation

$$N_1 G_1 N_1^T + N_2 G_2 N_2^T = W. \tag{4.4}$$

Therefore, the remainder of this section is devoted the solution of the following equivalent least squares problem:

Problem A: Given matrix $N_1 \in R^{q\times(r-r_1)}$, $N_2 \in R^{q\times(s-r_2)}$ and $W \in R^{q\times q}$, find $\widetilde{G}_1$, $\widetilde{G}_2$ such that

$$\|N_1 \widetilde{G}_1 N_1^T + N_2 \widetilde{G}_2 N_2^T - W\| = \min_{\substack{G_1 \in SR^{(r-r_1)\times(r-r_1)} \\ G_2 \in SR^{(k-r_2)\times(k-r_2)}}} \|N_1 G_1 N_1^T + N_2 G_2 N_2^T - W\|.$$

Actually *Problem A* has been investigated by [23], and the explicit solutions was obtained by using the canonical correlation decomposition(CCD). However the optimal approximation solution to a given matrix pair $(G_1^*,\ G_2^*)$ cannot be obtained in the corresponding solution set, and the difficulty is due to that the invariance of the Frobenius norm only holds for orthogonal matrices, but does not hold non-singular matrices that appear in CCD used in [23]. For the purpose of overcoming the above mentioned difficulty, another expression of the general solution of *Problem A* is derived by adopting a new approach, which is not only favorable for finding the optimal approximation solution to a given matrix pair $(G_1^*,\ G_2^*)$ in the solution set of *Problem A* , but it is also useful for finding the minimum-norm solution in the S_E. Our approach is based upon the Projection Theorem in Hilbert products spaces, as well as GSVD and CCD of matrix pairs, and can be essentially divided into three parts:

part 1: Find a least squares solution $(\widetilde{G}_1^{(0)}, \widetilde{G}_2^{(0)})$ of *Problem A* by using CCD.

part 2: By utilizing the solution $(\widetilde{G}_1^{(0)}, \widetilde{G}_2^{(0)})$ and the Projection Theorem in Hilbert spaces, we transform *Problem A* to a problem of finding the symmetric solutions of a consistent matrix equation.

part 3: Find the symmetric solutions of this consistent matrix equation by using GSVD.

Next, we first transform the least-squares problem with respect to the matrix equation (4.4) to the problem of finding the symmetric solution of a consistent matrix equation by applying the Projection Theorem. This technique is precisely described in the following theorem.

Theorem 1. *Given matrices $N_1 \in R^{q\times(r-r_1)}$ and $N_2 \in R^{q\times(s-r_2)}$ and $W \in SR^{q\times q}$, let $(\widetilde{G}_1^{(0)}, \widetilde{G}_2^{(0)})$ be one of the solutions of Problem A , and define*

$$W_0 = N_1\widetilde{G}_1^{(0)}N_1^T + N_2\widetilde{G}_2^{(0)}N_2^T. \tag{4.5}$$

Then the matrix equation

$$N_1G_1N_1^T + N_2G_2N_2^T = W_0, \tag{4.6}$$

is consistent over the symmetric matrices, and its symmetric solution set is the same as the least square symmetric solution set of inconsistent matrix equation (4.4).

Proof. : Let

$$\mathcal{S} = \{Z|Z = N_1G_1N_1^T + N_2G_2N_2^T, G_1 \in SR^{(r-r_1)\times(r-r_1)}, G_2 \in SR^{(s-r_2)\times(s-r_2)}\} \tag{4.7}$$

then $\mathcal{S}$ is obviously a linear subspace of $SR^{q\times q}$. Because $(\widetilde{G}_1^{(0)}, \widetilde{G}_2^{(0)})$ is a least squares solution of the inconsistent matrix equation (4.4), from (4.5) we see that $W_0 \in \mathcal{S}$ and

$$\begin{aligned}\|W_0 - W\| &= \|N_1\widetilde{G}_1^{(0)}N_1^T + N_2\widetilde{G}_2^{(0)}N_2^T - W\| \\ &= \min_{G_1,G_2}\|N_1G_1N_1^T + N_2G_2N_2^T - W\| \\ &= \min_{Z\in\mathcal{S}}\|Z - W\|.\end{aligned}$$

Now, from Lemma 2 (Projection Theorem), we have

$$(W_0 - W) \perp \mathcal{S}, \ (W_0 - W) \in \mathcal{S}^{\perp}.$$

For $G_1 \in SR^{(r-r_1)\times(r-r_1)}, G_2 \in SR^{(s-r_2)\times(s-r_2)}$, we know that

$$(N_1G_1N_1^T + N_2G_2N_2^T - W_0) \in \mathcal{S}$$

it follows that

$$\begin{aligned} \|N_1G_1N_1^T + N_2G_2N_2^T - W\|^2 &= \|(N_1G_1N_1^T + N_2G_2N_2^T - W_0) + (W_0 - W)\|^2 \\ &= \|N_1G_1N_1^T + N_2G_2N_2^T - W_0\|^2 + \|W_0 - W\|^2, \end{aligned}$$

which implies that the conclusion of this theorem holds. ■

Form Theorem 4.1, we can easily see that *Problem A* is equivalent to the problem of finding the symmetric solutions of the consistent matrix equation (4.6), and the key is to find W_0. The crux of finding W_0 is to derive a least square solution of the matrix equation (4.4). In order to do that, we denote

$$H^TWH = (W_{ij})_{6\times 6}, \quad W_{ij} = H_i^TWH_j, \ \ i,j = 1,2,\cdots 6 \tag{4.8}$$

where the matrices $H_i (i = 1,2\cdots 6)$ are given by (2.4), and W is given by (4.3). Based on the CCD of the matrix pair (N_1, N_2), the following lemma gives such a matrix W_0.

Lemma 3. *The matrix W_0, which corresponds to a least squares solution $(\widetilde{G}_1^0\ , \widetilde{G}_2^0)$ of the matrix equation* (4.4) *and satisfies* (4.5)*, is given by*

$$W_0 = H\begin{pmatrix} W_{11} & W_{12} & W_{13} & 0 & W_{15} & W_{16} \\ W_{12}^T & W_{22} & W_{23} & 0 & S_N\bar{Y}_{22}C_N & S_N\bar{Y}_{23} \\ W_{13}^T & W_{23}^T & W_{33} & 0 & 0 & 0 \\ 0 & 0 & 0 & 0 & 0 & 0 \\ W_{15}^T & C_N\bar{Y}_{22}S_N & 0 & 0 & C_N\bar{Y}_{22}C_N & C_N\bar{Y}_{23} \\ W_{16}^T & \bar{Y}_{23}S_N & 0 & 0 & \bar{Y}_{23}^TC_N & W_{66} \end{pmatrix}H^T, \tag{4.9}$$

where

$$\bar{Y}_{22} = \Phi * (S_NW_{25}C_N + C_NW_{25}S_N + C_NW_{55}C_N), \ \ \bar{Y}_{23} = S_NW_{26} + C_NW_{56}, \tag{4.10}$$

with

$$\Phi = (\phi_{ij} \in R^{s_0\times s_0}), \quad \phi_{ij} = \frac{1}{1-\alpha_i^2\alpha_j^2}, i,j = 1,2,...,s_0.$$

Proof. : From Theorem 3.1 in [23] we know that the least squares solution of the matrix equation (4.4) can be given by using the CCD of matrix pair (N_1, N_2), and have the following form:

$$\begin{aligned} \widetilde{G}_1 &= E_1\begin{pmatrix} W_{11}-Z_{11} & W_{15}S_N^{-1} & W_{16} & Y_{14} \\ S_N^{-1}W_{15}^T & \bar{Y}_{22} & \bar{Y}_{23} & Y_{24} \\ W_{16}^T & \bar{Y}_{23}^T & W_{66} & Y_{34} \\ Y_{14}^T & Y_{24}^T & Y_{34}^T & Y_{44} \end{pmatrix}E_1^T, \\ \widetilde{G}_2 &= E_2\begin{pmatrix} Z_{11} & W_{12}-W_{15}C_NS_N^{-1} & W_{13} & Z_{14} \\ W_{12}^T - S_N^{-1}C_NW_{15}^T & W_{22}-S_N\bar{Y}_{22}S_N & W_{23} & Z_{24} \\ W_{13}^T & W_{23}^T & W_{33} & Z_{34} \\ Z_{14}^T & Z_{24}^T & Z_{34}^T & Z_{44} \end{pmatrix}E_2^T, \end{aligned} \tag{4.11}$$

where $\bar{Y}_{22} \in R^{s_0\times s_0}$ and $\bar{Y}_{23} \in R^{s_0\times t_0}$ are defined by (4.10); $Y_{44} \in R^{f_1\times f_1}$, $Z_{11} \in R^{r_0\times r_0}$, and $Z_{44} \in R^{f_2\times f_2}$ are arbitrary symmetric matrices; and $Y_{14} \in R^{r_0\times f_1}, Y_{24} \in R^{s_0\times f_1}, Y_{34} \in R^{t_0\times f_1}$ and $Z_{14}^{r_0\times f_2}, Z_{24}^{s_0\times f_2}, Z_{34}^{(h-r_0-s_0)\times f_2}$ are arbitrary matrices, where $f_1 = r - r_1 - r_0 - s_0 - t_0, f_2 = k - r_2 - h$.

By inserting the matrices N_1 and N_2 in (2.3) and the matrices $\widetilde{G}_1$ and $\widetilde{G}_2$ in (4.11) into Eq.(4.5), we can immediately get (4.9) by straightforward computations. ∎

Remark 1. Evidently, (4.9) shows that the matrix W_0 given in Lemma 3 is unique and only dependent on the matrices N_1, N_2 and W, but is independent on the least squares solution $(\widetilde{G}_1^0, \widetilde{G}_2^0)$ of matrix equation (4.4). The characteristics is in accordance with the Projection Theorem in the Hilbert space. In fact W_0 is the orthogonal projection of W onto subspace $\mathcal{S}$ defined by (4.7).

From Theorem 1 and Lemma 3, we know that the least squares symmetric solution set the matrix equation (4.4) is the same as the symmetric solution set of the consistent matrix equation (4.6), with the matrix W_0 being given by (4.9), and from [3], the solution of (4.6) can be obtained by using the GSVD of the matrix pair (N_1, N_2). So we have the following conclusion.

Theorem 2. *Let matrices N_1, N_2 and W be given in Problem A. Denote*

$$M^{-T}W_0M^{-1} = (\widehat{W}_{ij})_{4\times 4}, \ \text{with } \widehat{W}_{ij} = M_i^TW_0M_j, \ i,j = 1,2,3,4, \tag{4.12}$$

where $M_i (i,j = 1,2,3,4)$ are given by (2.2) and W_0 is given by (4.9). Then the expressions of the solutions $(\widetilde{G}_1, \widetilde{G}_2)$ of Problem A are as follows:

$$\begin{aligned} \widetilde{G}_1 &= Q_1 \begin{pmatrix} \widehat{W}_{11} & \widehat{W}_{12}S_1^{-1} & Y_{13} \\ S_1^{-1}\widehat{W}_{12}^T & S_1^{-1}(\widehat{W}_{22} - S_2Z_{22}S_2)S_1^{-1} & Y_{23} \\ Y_{13}^T & Y_{23}^T & Y_{33} \end{pmatrix} Q_1^T, \\ \widetilde{G}_2 &= Q_2 \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{12}^T & Z_{22} & S_2^{-1}\widehat{W}_{23} \\ Z_{13}^T & \widehat{W}_{23}^TS_2^{-1} & \widehat{W}_{33} \end{pmatrix} Q_2^T, \end{aligned} \tag{4.13}$$

where $Y_{33}^{m_1\times m_1}$, $Z_{11}^{m_2\times m_2}$, $Z_{22}^{g\times g}$ are arbitrary symmetric matrices, $Y_{13}^{f\times m_1}, Y_{23}^{g\times m_1}, Z_{12}^{m_2\times g}$, $Z_{13}^{m_2\times (t-f-g)}$ are arbitrary matrices.

So, we know that the solution $(\widetilde{G}_1, \widetilde{G}_2)$ of *Problem A* can be expressed by (4.13). From the discussion above, after substituting $(\widetilde{G}_1, \widetilde{G}_2)$ in (4.13) into (3.4), we can derive the following conclusion:

Theorem 3. *Given matrices $X, B \in R^{n\times m}$ and $A_0 \in SR^{q\times q}$. The solution set S_E of Problem I can be expressed as*

$$S_E = \left\{ \tilde{A} | \tilde{A} = [P_1, P_2] \begin{pmatrix} A_{11}^{(0)} + U_{12}\widetilde{G}_1U_{12}^T & 0 \\ 0 & A_{22}^{(0)} + U_{22}\widetilde{G}_2U_{22}^T \end{pmatrix} \begin{bmatrix} P_1^T \\ P_2^T \end{bmatrix} \right\}, \tag{4.14}$$

where $\widetilde{G}_1$ and $\widetilde{G}_2$ are given by (4.13).

The solvability conditions of the matrix equation $AX = B$ over symmetric P-symmetric matrices with a submatrix constraint , that is, the set

$$S_A = \{A | A \in SR_P^n,\ AX = B,\ A([1:q]) = A_0\}$$

is non-empty, from Theorem 1 and Theorem 2, we can obtain the equivalent solvability conditions, which are described as follows.

Corollary 4.1. *Given $X, B \in R^{n\times m}$ and $A_0 \in SR^{q\times q}$. The set S_A is non-empty if and only if*

$$\begin{aligned} &(a)\quad X^TP_iP_i^TB = B^TP_iP_i^TX,\ \ P_i^TB(P_i^TX)^+P_i^TX = P_i^TB,\ \ i = 1,2,\\ &(b)\quad (W_{14}^T,\ W_{24}^T,\ W_{34}^T) = 0,\ (W_{44},\ W_{45},\ W_{46}) = 0,\ (W_{35},\ W_{36}) = 0, \\ &(c)\quad S_N\bar{Y}_{22}C_N = W_{25},\ \ C_N\bar{Y}_{22}C_N = W_{55} \end{aligned} \tag{4.15}$$

where the matrix blocks W_{ij} (i,j=1,2,...,6) are determined by (4.12). *When the conditions above all hold, the set S_A is the same as S_E in* (4.14).

Proof. : Evidently, the set S_A is non-empty if and only if there exists a matrix $A \in SR_P^n$ such that

$$\|AX - B\| = 0, \quad \|A([1:q]) - A_0\| = 0.$$

From Lemma 1 , we know $\|AX - B\| = 0$ is equivalent to

$$\|A_1P_1^TX - P_1^TB\| = 0,\ \ \|A_2P_2^TX - P_2^TB\| = 0,$$

and from [19], the above equalities are hold if and only if

$$X^TP_iP_i^TB = B^TP_iP_i^TX,\ \ P_i^TB(P_i^TX)^+P_i^TX = P_i^TB,\ \ i = 1,2.$$

From Theorem 1, we have

$$\begin{aligned} \|A([1:q]) - A_0\|^2 &= \|N_1G_1N_1^T + N_2G_2N_2^T - W\|^2 \\ &= \|(N_1G_1N_1^T + N_2G_2N_2^T - W_0) + (W_0 - W)\|^2 \\ &= \|N_1G_1N_1^T + N_2G_2N_2^T - W_0\|^2 + \|W_0 - W\|^2. \end{aligned}$$

Since the matrix equation $N_1G_1N_1^T + N_2G_2N_2^T = W_0$ is consistent over symmetric matrix space, then $\|A([1:q]) - A_0\| = 0$ if and only if

$$W_0 = W.$$

From (4.8) and (4.9), we know that $W_0 = W$ if and only if

$$(W_{14}^T,\ W_{24}^T,\ W_{34}^T) = 0,\ (W_{44},\ W_{45},\ W_{46}) = 0,\ (W_{35},\ W_{36}) = 0$$

and

$$S_N\bar{Y}_{22}C_N = W_{25},\ \ C_N\bar{Y}_{22}C_N = W_{55},\ \ S_N\bar{Y}_{23} = W_{26},\ \ C_N\bar{Y}_{23} = W_{56}.$$

Furthermore, for all $W_{26}, W_{56} \in R^{s_0\times t_0}$,it is easy to know that $S_N\bar{Y}_{23} = W_{26}$ and $C_N\bar{Y}_{23} = W_{56}$ from $\bar{Y}_{23} = S_NW_{26} + C_NW_{56}$. Therefore, from the discussions above, the set S_A is non-empty if and only if (4.15) holds and S_A can be expressed by (4.14). ∎

5 The solution of Problem II

It is easy to verify that the solution set S_E is nonempty and is a closed convex subset of the Hilbert space $R^{n\times n}$. From the best approximation Theorem [1] that there exists a unique matrix $\hat{A} \in S_E$ satisfying (1.1). From Theorem 3, we can obtain the analytical expression of the solution $\hat{A}$ of Problem II.

Theorem 4. *Given matrices $X, B \in R^{n\times m}$, $A_0 \in SR^{q\times q}$, then the solution $\hat{A}$ of problem II can be expressed as*

$$\hat{A} = [P_1, P_2] \begin{pmatrix} A_{11}^{(0)} + U_{12}\hat{G}_1 U_{12}^T & 0 \\ 0 & A_{22}^{(0)} + U_{22}\hat{G}_2 U_{22}^T \end{pmatrix} \begin{bmatrix} P_1^T \\ P_2^T \end{bmatrix}, \tag{5.1}$$

where

$$\hat{G}_1 = Q_1 \begin{pmatrix} \widehat{W}_{11} & \widehat{W}_{12}S_1^{-1} & 0 \\ S_1^{-1}\widehat{W}_{12}^T & S_1^{-1}(\widehat{W}_{22} - S_2\breve{Z}_{22}S_2)S_1^{-1} & 0 \\ 0 & 0 & 0 \end{pmatrix} Q_1^T,$$

$$\hat{G}_2 = Q_2 \begin{pmatrix} 0 & 0 & 0 \\ 0 & \breve{Z}_{22} & S_2^{-1}\widehat{W}_{23} \\ 0 & \widehat{W}_{23}^T S_2^{-1} & \widehat{W}_{33} \end{pmatrix} Q_2^T.$$

with

$$\breve{Z}_{22} = K * (S_2\widehat{W}_{22}S_2) \tag{5.2}$$

and

$$K = (\kappa_{ij}) \in R^{g\times g}, \quad \kappa_{ij} = \frac{1}{\mu_i^2\mu_j^2 + \lambda_i^2\lambda_j^2}, \quad i, j = 1, 2, \ldots, g$$

Proof. : From Theorem 3, we know that the solution set S_E of Problem I is given by (4.14). For all $\tilde{A} \in S_E$, if follows from (4.14) and (4.13) that

$$\begin{aligned}
\|\tilde{A}\|^2 &= \left\| \begin{bmatrix} P_1^T \\ P_2^T \end{bmatrix} \tilde{A}[P_1, P_2] \right\|^2 = \left\| \begin{pmatrix} A_{11}^{(0)} + U_{12}\widetilde{G}_1 U_{12}^T & 0 \\ 0 & A_{22}^{(0)} + U_{22}\widetilde{G}_2 U_{22}^T \end{pmatrix} \right\|^2 \\
&= \left\| A_{11}^{(0)} + U_1 \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{G}_1 \end{pmatrix} U_1^T \right\|^2 + \left\| A_{22}^{(0)} + U_2 \begin{pmatrix} 0 & 0 \\ 0 & \widetilde{G}_2 \end{pmatrix} U_2^T \right\|^2 \\
&= \|A_{11}^{(0)}\|^2 + \|\widetilde{G}_1\|^2 + \|A_{22}^{(0)}\|^2 + \|\widetilde{G}_2\|^2 \\
&= \|A_{11}^{(0)}\|^2 + \|Q_1^T\widetilde{G}_1Q_1\|^2 + \|A_{22}^{(0)}\|^2 + \|Q_2^T\widetilde{G}_2Q_2\|^2 \\
&= \left\| \begin{pmatrix} \widehat{W}_{11} & \widehat{W}_{12}S_1^{-1} & Y_{13} \\ S_1^{-1}\widehat{W}_{12}^T & S_1^{-1}(\widehat{W}_{22} - S_2Z_{22}S_2)S_1^{-1} & Y_{23} \\ Y_{13}^T & Y_{23}^T & Y_{33} \end{pmatrix} \right\|^2 \\
&\quad + \left\| \begin{pmatrix} Z_{11} & Z_{12} & Z_{13} \\ Z_{12}^T & Z_{22} & S_2^{-1}\widehat{W}_{23} \\ Z_{13}^T & \widehat{W}_{23}^T S_2^{-1} & \widehat{W}_{33} \end{pmatrix} \right\|^2 + \|A_{11}^{(0)}\|^2 + \|A_{22}^{(0)}\|^2
\end{aligned}$$

Thus, $\|\hat{A}\|^2 = \min\limits_{\tilde{A}\in S_E} \|\tilde{A}\|^2$ if and only if

$$Y_{i3} = 0, \quad Z_{1i} = 0, \quad i = 1, 2, 3, \tag{5.3}$$

and

$$f(Z_{22}) = \|S_1^{-1}(\widehat{W}_{22} - S_2Z_{22}S_2)S_1^{-1}\|^2 + \|Z_{22}\|^2 = \min, \ \forall Z_{22} \in SR^{g\times g} \tag{5.4}$$

Let $Z_{22} = [z_{ij}] \in SR^{g\times g}$, $\widehat{W}_{22} = [w_{ij}] \in SR^{g\times g}$. Form (5.4) we have

$$f(Z_{22}) = \sum_{i=j=1}^{n} (z_{ii}^2 + \left(\frac{1}{\mu_i}w_{ii}\frac{1}{\mu_i} - \frac{\lambda_i}{\mu_i}z_{ii}\frac{\lambda_i}{\mu_i}\right)^2)+ \\ 2\sum_{1\le i<j\le n}\left[z_{ij}^2 + \left(\frac{1}{\mu_i}w_{ij}\frac{1}{\mu_j} - \frac{\lambda_i}{\mu_i}z_{ij}\frac{\lambda_j}{\mu_j}\right)^2\right]$$

Clearly, $f(Z_{22})$ is a differentiable function of $\frac{1}{2}g(g-1)$ variables $z_{ij}(1 \le i < j \le n)$. According to the necessary condition of function which is minimizing at a point, function $f(Z_{22})$ attains the smallest value at

$$z_{ij} = \frac{\lambda_i\lambda_j w_{ij}}{\mu_i^2\mu_j^2 + \lambda_i^2\lambda_j^2} \tag{5.5}$$

Let $K = [\frac{1}{\mu_i^2\mu_j^2+\lambda_i^2\lambda_j^2}] \in R^{g\times g}$, then the solution $\check{Z}_{22}$ of (5.4) can be expressed by (5.2). Now , after substituting (5.2) and (5.3) into (4.14) we immediately get (5.1). ■

6 Conclusion

In this paper, we have considered the least squares symmetric P-symmetric solutions to the matrix inverse problem $AX = B$ with a submatrix constraint. First we have introduced some preliminary results, then we have converted this least squares problem to a equivalent least squares problem, i.e., *Problem A.* trickily. Then by applying the generalized singular value decomposition and the canonical correlation decomposition and basing on the projection theorem, we have obtained an analytical expression for the solutions of corresponding problem, we have also derived the necessary and sufficient conditions under which the corresponding inverse problem is consistent. Moreover, we have given the analytical expression of the unique minimum-norm solution of the solution set S_E.

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