Coexistence of Unbounded Solutions and Periodic Solutions of a Class of Planar Systems with Asymmetric Nonlinearities*

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Abstract

In this paper we will prove the coexistence of unbounded solutions and periodic solutions for a class of planar systems with asymmetric nonlinearities

$$\begin{cases} u' = v - \alpha u^+ + \beta u^- \\ v' = -\mu u^+ + \gamma u^- - g(u) + p(t), \end{cases}$$

where $g(u)$ is continuous and bounded, $p(t)$ is a continuous 2π-periodic function and $\alpha, \beta \in \mathbb{R}, \mu, \gamma$ are positive constants.

1 Introduction

Recently, many researchers are concerned with the unboundedness of solutions of the following planar system:

$$\begin{cases} u' = v - F(u); \\ v' = -\mu u^+ + \gamma u^- - g(u) + p(t). \end{cases} \tag{1.1}$$

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where μ, γ are positive constants , $u^+ = \max\{u, 0\}$, $u^- = \max\{-u, 0\}$, $g(u)$ is a continuous function and $p(t)$ is a continuous 2π-periodic function.

When $F(u) = \int_0^u f(s)ds$, *Eq*. (1.1) becomes Liénard equations

$$u'' + f(u)u' + \mu u^+ - \gamma u^- + g(x) = p(t), \tag{1.2}$$

which have been studied by Zaihong, Wang and others in [1-3, 18-23]. When $g(x) \equiv 0$, in [22] Wang has shown that the solutions of *Eq*. (1.2) with sufficiently large initial conditions are unbounded either in the past or in the future if

$$\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\gamma}} \in R \setminus Q, F(+\infty) \neq F(-\infty),$$

where F is a primary function of $f(u)$ and $F(\pm\infty) = \lim_{u \to \pm\infty} F(u)$ are not zero and finite. (Here only taking some constant d, replace the conditions in [22] with

$$F(+\infty) - d < 0 < F(-\infty) - d \ \ or \ \ F(-\infty) - d < 0 < F(+\infty) - d.)$$

Wang also prove the equation (1.1) has unbounded solutions provided that

$$\frac{1}{\sqrt{\mu}} + \frac{1}{\sqrt{\gamma}} = \frac{2}{n}$$

and $F(u)$ and $g(u)$ satisfies some limit conditions in [23].

The existence of unbounded solutions for the perturbed forced planar Hamilton system are proved in [7] by A.Fonda and J. Mawhin. The results in [7] can be applied to *Eq*. (1.1), but the obtained existence conditions of unbounded solutions also depend upon the limit conditions

$$\lim_{u \to \pm\infty} \frac{B(u)}{u} = B^{\pm}, B(u) = \int_0^u \int_0^s f(\sigma) d\sigma ds.$$

When $F(u)$ is unbounded , the estimates become difficult and therefore so far few results have been obtained in the literature.

In the present paper , we will mainly discuss the unboundedness and period of solutions of the following interesting planar system:

$$\begin{cases} u' = v - \alpha u^+ + \beta u^-; \\ v' = -\mu u^+ + \gamma u^- - g(u) + p(t). \end{cases} \tag{1.3}$$

where $g(u)$ is a continuous and bounded function and $p(t)$ is a continuous $2\pi - periodic$ function and $\alpha, \beta \in \mathbb{R}, \mu, \gamma$ are positive constants satisfying

$$(H_0): \qquad A := \mu - (\frac{\alpha}{2})^2 > 0, B := \gamma - (\frac{\beta}{2})^2 > 0, \frac{\alpha}{\sqrt{\mu}} + \frac{\beta}{\sqrt{\gamma}} = 0.$$

Let us consider now the autonomous piecewise linear planar system

$$\begin{cases} u' = v - \alpha u^+ + \beta u^-; \\ v' = -\mu u^+ + \gamma u^- \end{cases} \tag{1.4}$$

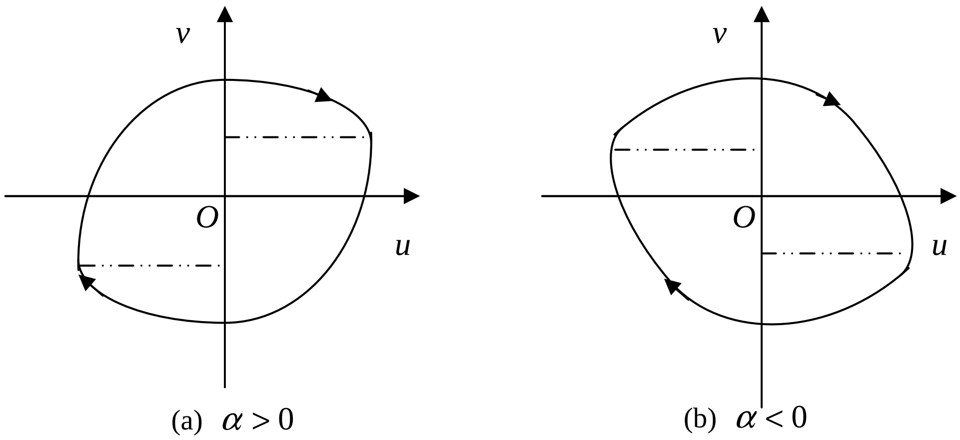

Figure 1: Phase curves of the autonomous piecewise linear planar system

associated to *Eq*. (1.3) with the initial condition $u(0) = 0, v(0) = v > 0$. It is easy to know that every solution of *Eq*. (1.4) is a periodic function with period $T = \frac{\pi}{\sqrt{A}} + \frac{\pi}{\sqrt{B}}$ under the condition (H_0) and every solution of *Eq*. (1.4) moves clockwise in the phase plane (see Fig. 1). However, given a small disturbance as *Eq*. (1.3), the solutions of *Eq*. (1.3), which become complex, may be not periodic solutions even being unbounded solutions.

Based on some estimates for the successor map , we obtain some sufficient conditions for the existence of unbounded solutions and periodic solution for *Eq*. (1.3).

2 the Successor Map

The successor map was used successfully in studying bounded perturbations of oscillators in [4, 11, 13]. At first, we will need to prove the definition of the successor map S well, then give some estimates for the successor map based on phase plane analysis.

Let $(u(t;\tau_0,v_0),\ v(t;\tau_0,v_0))$ be the unique solution of *Eq*. (1.3) satisfying the initial conditions

$$u(\tau_0;\tau_0,v_0) = 0,\ v(t;\tau_0,v_0) = v_0 > 0.$$

Denote by τ_1 is the next zero of $u(t;\tau_0,v_0)$ to τ_0, that is ,

$$\tau_1 > \tau_0,\ u(\tau_1;\tau_0,v_0) = 0,\ u(t;\tau_0,v_0) \neq 0, \text{ for } t \in (\tau_0,\tau_1).$$

We also use the notation $v_1 = v(\tau_1;\tau_0,v_0)$. If v_1 is finite, then the successor map can be defined by

$$S : (\tau_0,v_0) \mapsto (\tau_1,v_1).$$

All along this paper, the iteration of the successor map is defined by $S^n(\tau_0,v_0) = (\tau_n,v_n)$ and denote by

$$P_n = S^{2n} : (\tau_0,v_0) \mapsto (\tau_{2n},v_{2n}).$$

Therefore,

$$\prod_1(S^n(\tau_0,v_0)) = \tau_n,\ \prod_2(S^n(\tau_0,v_0)) = v_n.$$

We also assume that the limits

$$(H_1): \qquad \lim_{u\to+\infty} g(u) = g(+\infty),\ \lim_{u\to-\infty} g(u) = g(-\infty)$$

exist and are finite.

Lemma 2.1. *Every solution* $(u(t;\tau_0,v_0), v(t;\tau_0,v_0))$ *of Eq.* (1.3) *moves around the origin on the phase plane and meets* $v-$ *axis again in a finite time for* $v_0 >> 1$. *Moreover,* S *is well defined and one to one for* $v_0 >> 1$.

Proof. Using polar coordinates

$$u = r\cos\theta, v = r\sin\theta,\ (u,v) \neq (0,0)$$

on right-half phase plane, we have the polar form associated to *Eq.* (1.3)

$$\begin{cases} \theta' = -\mu\cos^2\theta + \alpha\sin\theta\cos\theta - \sin^2\theta - \frac{1}{r}(g(r\cos\theta) - p(t))\cos\theta; \\ r' = -\alpha r\cos^2\theta + r\sin\theta\cos\theta(1-\mu) - (g(r\cos\theta) - p(t))\sin\theta \end{cases} \tag{2.5}$$

Let $(\theta(t;\tau_0,\theta_0),\ r(t;\tau_0,\theta_0))$ be the solution of Eq. (2.5) with the initial conditions

$$\theta_0 = \theta(\tau_0;\tau_0,\theta_0) = \frac{\pi}{2},\ r_0 = r(\tau_0;\tau_0,\theta_0).$$

Using the Gronwall inequality to

$$\left|\frac{dr}{dt}\right| \leq Mr, \text{ where } M \text{ is a certain positive constant },$$

we can prove that for any $T > 0$, there is $v_0 > 0$, such that

$$r_0 e^{-MT} \leq r(t) \leq r_0 e^{MT}, \text{ for } |t - t_0| \leq T \text{ and } r_0 > v_0. \tag{2.6}$$

From(H_0) and inequalities (2.5)-(2.6), we have obtained

$$\begin{aligned} \frac{d\theta}{dt} &= \mu\cos^2\theta + \alpha\sin\alpha\sin\theta - \sin^2\theta \\ &\quad + \frac{1}{r}(g(r\cos\theta) - p(t))\cos\theta < 0, \theta \in (-\frac{\pi}{2},\frac{\pi}{2}), r_0 >> 1, \end{aligned}$$

because that

$$\Delta = (\alpha\sin\theta)^2 - 4\mu\sin^2\theta < 0, \theta \in (-\frac{\pi}{2},\frac{\pi}{2}).$$

Therefore, there is a finite time τ_1 such that

$$\theta(\tau_1;\tau_0,v_0) = -\frac{\pi}{2}.$$

Similar discussion can be got on the left-half phase plane, thus the conclusion of the first part of the lemma is proved. The uniqueness of the solution for the initial value problem guarantees that S is continuous and one to one. ∎

Moreover, we have

Lemma 2.2. *Assume* $\tau_1 = \prod_1(S(\tau_0, v_0))$ *and* (H_0) *holds, then*

$$\tau_1 = \tau_0 + \frac{\pi}{\sqrt{A}} + o(\frac{1}{v_0}), \text{ for } v_0 >> 1,$$

where $o(\frac{1}{v_0}) \to 0$ *for* $v_0 \to +\infty$ *as usual.*

Proof. From the first formula of the equality (2.5), we have

$$\begin{aligned} \tau_1 - \tau_0 &= \int_{\theta(\tau_0)}^{\theta(\tau_1)} \frac{d\theta}{-\mu\cos^2\theta + \alpha\sin\theta\cos\theta - \sin^2\theta + o(\frac{1}{v_0})} \\ &= \int_{\theta(\tau_0)}^{\theta(\tau_1)} \frac{d\theta}{-\mu\cos^2\theta + \alpha\sin\theta\cos\theta - \sin^2\theta} + o(\frac{1}{v_0}) \\ &= \frac{\pi}{\sqrt{A}} + o(\frac{1}{v_0}). \end{aligned}$$

∎

By using some phase plane analysis, we have more delicate estimates via the following lemma.

Lemma 2.3. *Assume* (H_0) *holds,then we have*

$$\tau_1 = \tau_0 + \frac{\pi}{\sqrt{A}} + \frac{1}{v_0}\varphi(\tau_0)e^{-\alpha\tau_0} + o(\frac{1}{{v_0}^2}), \tag{2.7}$$

$$|v_1| = e^{-\frac{\alpha}{2\sqrt{A}}\pi}(v_0 - e^{-\alpha\tau_0}\varphi'(\tau_0)) + o(\frac{1}{v_0}), \tag{2.8}$$

where

$$\varphi(\tau_0) = -\int_{\tau_0}^{\tau_0+\frac{\pi}{\sqrt{A}}} \frac{1}{\sqrt{A}} \sin\sqrt{A}(s-\tau_0)(g(u(s)) - p(s))e^{\frac{\alpha}{2}(s+\tau_0)}ds.$$

Proof. By using variation of the constant formula , we have , for $t \in [\tau_0, \tau_1]$, that

$$\begin{aligned} u(t) &= \frac{v_0}{\sqrt{A}}e^{\frac{\alpha}{2}(\tau_0-t)}\sin\sqrt{A}(t-\tau_0) \\ &\quad + \int_{\tau_0}^{t} \frac{1}{\sqrt{A}}\sin\sqrt{A}(s-t)(g(u(s)) - p(s))e^{\frac{\alpha}{2}(s-t)}ds, \\ v(t) &= \frac{v_0}{\sqrt{A}}e^{\frac{\alpha}{2}(\tau_0-t)}(\sqrt{A}\cos\sqrt{A}(t-\tau_0) + \frac{\alpha}{2}\sin\sqrt{A}(t-\tau_0)) \\ &\quad + \int_{\tau_0}^{t}(-cos\sqrt{A}(s-t) + \frac{\alpha}{2\sqrt{A}}\sin\sqrt{A}(s-t))(g(u(s)) - p(s))e^{\frac{\alpha}{2}(s-t)}ds. \end{aligned}$$

Note that $u(\tau_1) = 0$. It follows that

$$\begin{aligned} \frac{v_0}{\sqrt{A}}e^{\frac{\alpha}{2}(\tau_0-\tau_1)}\sin\sqrt{A}(\tau_1-\tau_0) &= -\int_{\tau_0}^{\tau_1}\frac{1}{\sqrt{A}}\sin\sqrt{A}(s-\tau_1) \\ &\qquad (g(u(s)) - p(s))e^{\frac{\alpha}{2}(s-\tau_1)}ds. \end{aligned}$$

From lemma 2.2, we obtain

$$\frac{1}{\sqrt{A}}e^{-\frac{\alpha}{2}(\frac{\pi}{\sqrt{A}}+o(\frac{1}{v_0}))}\sin\sqrt{A}(\frac{\pi}{\sqrt{A}}+o(\frac{1}{v_0})) = -\frac{1}{v_0}\int_{\tau_0}^{\tau_0+\frac{\pi}{\sqrt{A}}+o(\frac{1}{v_0})}\frac{1}{\sqrt{A}}sin\sqrt{A}(s-\tau_0) (g(u(s))-p(s))e^{\frac{\alpha}{2}(s-\tau_0)}ds.$$

Therefore, it follows that

$$o(\frac{1}{v_0}) = -\frac{1}{v_0}\int_{\tau_0}^{\tau_0+\frac{\pi}{\sqrt{A}}}\frac{1}{\sqrt{A}}\sin\sqrt{A}(s-\tau_0)(g(u(s))-p(s))e^{\frac{\alpha}{2}(s-\tau_0)}ds+o(\frac{1}{{v_0}^2}),$$

which, together with lemma 2.2, yields (2.7). Similarly, note that $v(\tau_1) = v_1$. It follows that

$$\begin{aligned} v_1 &= \frac{v_0}{\sqrt{A}}e^{\frac{\alpha}{2}(\tau_0-\tau_1)}(\sqrt{A}\cos\sqrt{A}(\tau_1-\tau_0)+\frac{\alpha}{2}\sin\sqrt{A}(\tau_1-\tau_0)\\ &\quad+\int_{\tau_0}^{\tau_1}(-\cos\sqrt{A}(s-\tau_1)+\frac{\alpha}{2\sqrt{A}}\sin\sqrt{A}(s-\tau_1))\\ &\quad(g(u(s))-p(s))e^{\frac{\alpha}{2}(s-\tau_1)}ds.\end{aligned}$$

Since $e^{\frac{\alpha}{2}(\tau_0-\tau_1)}, cos\sqrt{A}(\tau_0-\tau_1), \frac{\alpha}{2}sin\sqrt{A}(\tau_0-\tau_1)$ can be expressed in the form by (2.7),

$$\begin{aligned} e^{\frac{\alpha}{2}(\tau_0-\tau_1)} &= e^{-\frac{\alpha}{2\sqrt{A}}\pi}(1-\frac{1}{2v_0}\alpha\varphi(\tau_0)e^{-\alpha\tau_0})+o(\frac{1}{{v_0}^2}),\\ \cos\sqrt{A}(\tau_0-\tau_1) &= -\cos(\sqrt{A}e^{-\alpha\tau_0}\varphi(\tau_0)\frac{1}{v_0}) = -1+o(\frac{1}{{v_0}^2}),\\ \frac{\alpha}{2}\sin\sqrt{A}(\tau_0-\tau_1) &= -\sin(\frac{\sqrt{A}}{v_0})e^{-\alpha\tau_0}\varphi(\tau_0)\frac{1}{v_0}) = o(\frac{1}{v_0}),\end{aligned}$$

we have

$$\begin{aligned} |v_1| = -v_1 &= e^{-\frac{\alpha}{2\sqrt{A}}\pi}(v_0+\int_{\tau_0}^{\tau_1}(-\cos\sqrt{A}(s-\tau_1)\\ &\quad+\frac{\alpha}{2\sqrt{A}}\sin\sqrt{A}(s-\tau_1))(g(u(s))-p(s))e^{\frac{\alpha}{2}(s-\tau_1)}ds. \quad \blacksquare\end{aligned}$$

Denote by τ_2 is the next zero of $u(t;\tau_0,v_0)$ to τ_0, that is,

$$\tau_2 > \tau_1,\ u(\tau_2;\tau_0,v_0) = 0,\ u(t;\tau_0,v_0) \neq 0,\ \text{ for } t \in (\tau_1,\tau_2).$$

Similarly, we can obtain some delicate estimates for τ_2, v_2 by the following lemma.

Lemma 2.4. *Assume* (H_0) *holds, then we have*

$$\tau_2 = \tau_1+\frac{\pi}{\sqrt{B}}+\frac{1}{v_1}\psi(\tau_1)e^{-\beta\tau_1}+o(\frac{1}{{v_1}^2}), \tag{2.9}$$

$$|v_2| = e^{-\frac{\beta}{2\sqrt{B}}\pi}(|v_1|+e^{-\beta\tau_1}\psi'(\tau_1))+o(\frac{1}{v_1}), \tag{2.10}$$

where

$$\psi(\tau_1) = -\int_{\tau_1}^{\tau_1+\frac{\pi}{\sqrt{B}}}\frac{1}{\sqrt{B}}\sin\sqrt{B}(s-\tau_1)(g(u(s))-p(s))e^{\frac{\beta}{2}(s+\tau_1)}ds.$$

Note that

$$\tau_2 = \Pi_1 S^2(\tau_0, v_0) \quad , v_2 = \Pi_2 S^2(\tau_0, v_0), \quad (\tau_2, v_2) = P(\tau_0, v_0).$$

Lemma 2.5. *Assume that* $(H_0), (H_1)$ *holds, the mapping* $P : (\tau_0, v_0) \mapsto (\tau_2, v_2)$ *can be expressed in the form:*

$$\begin{cases} \tau_2 &= \tau_0 + \frac{\pi}{\sqrt{A}} + \frac{\pi}{\sqrt{B}} \\ & \quad + \frac{1}{v_0}[(1 + e^{\frac{\alpha}{2\sqrt{A}}\pi})(\frac{1}{\gamma}g(-\infty) - \frac{1}{\gamma}g(+\infty)) + \mu_1(\tau_0)] + o(\frac{1}{v_0{}^2}), \\ v_2 &= v_0 + (1 + e^{\frac{\alpha}{2\sqrt{A}}\pi})(\frac{\alpha}{\mu}g(+\infty) - \frac{\beta}{\gamma}g(-\infty)) + \mu_2(\tau_0) + o(\frac{1}{v_0}). \end{cases} \tag{2.11}$$

for $v_0 >> 1$, *where*

$$\begin{aligned} \mu_1(\tau_0) &= \int_0^{\frac{\pi}{\sqrt{A}}} \frac{1}{\sqrt{A}} \sin\sqrt{A}t e^{\frac{\alpha}{2}t} p(t + \tau_0) dt \\ & \quad + \int_{\frac{\pi}{\sqrt{A}}}^{\frac{\pi}{\sqrt{A}} + \frac{\pi}{\sqrt{B}}} \frac{1}{\sqrt{B}} \sin\sqrt{B}(\frac{\pi}{\sqrt{A}} - t) e^{\frac{\beta}{2}t} p(t + \tau_0) dt, \\ \mu_2(\tau_0) &= \int_0^{\frac{\pi}{\sqrt{A}}} (-\frac{\alpha}{2\sqrt{A}} \sin\sqrt{A}t + \cos\sqrt{A}t) e^{\frac{\alpha}{2}t} p(t + \tau_0) dt \\ & \quad + \int_{\frac{\pi}{\sqrt{A}}}^{\frac{\pi}{\sqrt{A}} + \frac{\pi}{\sqrt{B}}} (-\frac{\beta}{2\sqrt{B}} \sin\sqrt{B}(t - \frac{\pi}{\sqrt{A}}) \\ & \quad + \cos\sqrt{B}(t - \frac{\pi}{\sqrt{A}})) e^{-(\frac{\beta\pi}{2\sqrt{A}} + \frac{\beta\pi}{2\sqrt{B}}) + \frac{\beta}{2}t} p(t + \tau_0) dt. \end{aligned}$$

Proof. From lemma 2.2 and lemma 2.4, we know that

$$\begin{aligned} \frac{1}{v_1} &= \frac{1}{|v_1|} = -\frac{1}{e^{-\frac{\alpha\pi}{2\sqrt{A}}}(v_0 - e^{\alpha\tau_0}\varphi'(\tau_0)) + o(\frac{1}{v_0})} \\ &= -e^{\frac{\alpha\pi}{2\sqrt{A}}} \frac{1}{v_0} \cdot \frac{1}{1 - e^{\alpha\tau_0}\frac{1}{v_0}\varphi'(\tau_0) + o(\frac{1}{v_0{}^2})} \\ &= -\frac{1}{v_0} e^{\frac{\alpha\pi}{2\sqrt{A}}} + o(\frac{1}{v_0{}^2}), \\ e^{-\beta\tau_1}\varphi(\tau_1) &= e^{-\beta(\tau_0 + \frac{\pi}{\sqrt{A}})} \int_{\tau_0 + \frac{\pi}{\sqrt{A}}}^{\tau_0 + \frac{\pi}{\sqrt{A}} + \frac{\pi}{\sqrt{B}}} (\frac{1}{\sqrt{B}} \sin\sqrt{B}(s - \tau_0 - \frac{\pi}{\sqrt{A}}) \\ & \quad (-g(u(s)) + p(s)) e^{\frac{\beta}{2}(s + \tau_0 + \frac{\pi}{\sqrt{A}})} ds + (\frac{1}{v_0}) \\ &= e^{-\frac{\beta\pi}{2\sqrt{A}}} \int_{\frac{\pi}{\sqrt{A}}}^{\frac{\pi}{\sqrt{A}} + \frac{\pi}{\sqrt{B}}} \frac{1}{\sqrt{B}} \sin\sqrt{B}(t - \frac{\pi}{\sqrt{A}}) \\ & \quad (-g(u(t + \tau_0)) + p(t + \tau_0)) e^{\frac{\beta}{2}t} dt + (\frac{1}{v_0}). \end{aligned}$$

Since (H_1) holds, by the lebesgue dominated convergence theorem, it is not hard to prove that (2.9) becomes

$$\begin{aligned}\tau_2 &= \tau_0+\frac{\pi}{\sqrt{A}}+\frac{\pi}{\sqrt{B}}+\frac{1}{v_0}\int_0^{\frac{\pi}{\sqrt{A}}}\frac{1}{\sqrt{A}}\sin\sqrt{A}t(-g(+\infty)\\ &+ p(t+\tau_0))e^{\frac{\alpha}{2}t}dt+\int_{\frac{\pi}{\sqrt{A}}}^{\frac{\pi}{\sqrt{A}}+\frac{\pi}{\sqrt{B}}}\frac{1}{\sqrt{B}}\sin\sqrt{B}(t-\frac{\pi}{\sqrt{A}})(-g(-\infty)\\ &+ p(t+\tau_0))e^{\frac{\alpha-\beta}{2\sqrt{A}}+\frac{\alpha}{2}t}dt+o(\frac{1}{{v_0}^2}),\end{aligned}$$

for $v_0 >> 1$.
Since

$$\frac{\alpha}{\sqrt{A}}+\frac{\beta}{\sqrt{B}}=0,$$

we calculate

$$\int_0^{\frac{\pi}{\sqrt{A}}}\frac{1}{\sqrt{A}}\sin\sqrt{A}te^{\frac{\alpha}{2}t}dt=\frac{1}{\mu}(1+e^{\frac{\alpha}{2\sqrt{A}}\pi}),$$

$$\begin{aligned}\int_{\frac{\pi}{\sqrt{A}}}^{\frac{\pi}{\sqrt{A}}+\frac{\pi}{\sqrt{B}}}\frac{1}{\sqrt{B}}\sin\sqrt{B}(t-\frac{\pi}{\sqrt{A}})e^{\frac{\alpha-\beta}{2\sqrt{A}}+\frac{\beta}{2}t}dt &= -\frac{1}{\gamma}e^{\frac{\alpha}{2\sqrt{A}}\pi}(1+e^{\frac{\beta}{2\sqrt{B}}\pi})\\ &= -\frac{1}{\gamma}(1+e^{\frac{\alpha}{2\sqrt{A}}\pi}).\end{aligned}$$

Then it follows the first formula of the lemma 2.5.
From (2.10), using the similar method above, we have

$$\begin{aligned}v_2 &= v_0+\int_{\tau_0}^{\tau_0+\frac{\pi}{\sqrt{A}}}(-\cos\sqrt{A}(s-\tau_0)+\frac{\alpha}{2\sqrt{A}}\sin\sqrt{A}(s-\tau_0))(g(u(s))\\ &\quad -p(s))e^{\frac{\alpha}{2}(s-\tau_0)}ds+e^{\frac{\beta\pi}{2\sqrt{B}}}\int_{\tau_1}^{\tau_1+\frac{\pi}{\sqrt{B}}}(-\cos\sqrt{B}(s-\tau_1)\\ &\quad +\frac{\beta}{2\sqrt{B}}\sin\sqrt{B}(s-\tau_1))(g(u(s))-p(s))e^{\frac{\alpha}{2}(s-\tau_1)}ds+o(\frac{1}{v_0})\\ &= v_0+\int_0^{\frac{\pi}{\sqrt{A}}}(-\cos\sqrt{A}t+\frac{\alpha}{2\sqrt{A}}\sin\sqrt{A}t)(g(+\infty)\\ &\quad -p(t+\tau_0))e^{\frac{\alpha}{2}t}dt+e^{-(\frac{\beta}{2\sqrt{A}}\pi+\frac{\beta}{2\sqrt{B}}\pi)}\int_{\frac{\pi}{\sqrt{A}}}^{\frac{\pi}{\sqrt{A}}+\frac{\pi}{\sqrt{B}}}(-cos\sqrt{B}(t-\frac{\pi}{\sqrt{A}})\\ &\quad +\frac{\beta}{2\sqrt{B}}\sin\sqrt{B}(t-\frac{\pi}{\sqrt{A}}))(g(-\infty)-p(t+\tau_0))e^{\frac{\alpha}{2}t}ds+o(\frac{1}{v_0})\\ &= v_0+\frac{\alpha}{\mu}(1+e^{(\frac{\alpha}{2\sqrt{A}}\pi})g(+\infty)-\frac{\beta}{\gamma}e^{-(\frac{\beta}{2\sqrt{B}}\pi}(1+e^{(\frac{\beta}{2\sqrt{B}}\pi})g(-\infty)+\mu_2(\tau_0)+o(\frac{1}{v_0})\\ &= v_0+(1+e^{\frac{\alpha}{2\sqrt{A}}\pi})(\frac{\alpha}{\mu}g(+\infty)-\frac{\beta}{\gamma}g(-\infty))+\mu_2(\tau_0)+o(\frac{1}{v_0}),\end{aligned}$$

for $v_0 >> 1$. Thus the lemma 2.5 is proved. ∎

3 Resonance

3.1 Resonance: Periodic and Unbounded Solutions

Assume that

$$\frac{T}{2\pi} = \frac{p}{q} \in \mathbb{Q},\ p,q \in \mathbb{Z}^+,\ T = \frac{\pi}{\sqrt{A}} + \frac{\pi}{\sqrt{B}},$$

and the fraction $\frac{p}{q}$ is in the reduced form, that is ,

$$\frac{1}{\sqrt{A}} + \frac{1}{\sqrt{B}} \in \mathbb{Q}.$$

J.M.Alonso and R.Qrtega in [2] have studied the dynamics of a class of mappings defined on the plane, which have an asymptotic expression

$$\begin{cases} \theta_1 = \theta + 2\pi\frac{p}{q} + \frac{1}{r}\widetilde{\mu}_1(\theta) + o(\frac{1}{r^2}) \\ r_1 = r + \widetilde{\mu}_2(\theta) + o(\frac{1}{r}), r \to +\infty, \end{cases} \tag{3.12}$$

where $\widetilde{\mu}_1(\theta), \widetilde{\mu}_2(\theta)$ are continuous and 2π-periodic functions. They prove the existence of orbits that go to infinity in the future provide that there exists $\omega \in R$ such that

$$\widetilde{\mu}_2(\omega) = 0, \widetilde{\mu}_1(\omega) < 0, \widetilde{\mu}_1(\theta)(\theta - \omega) > 0,\ \ for\ \theta \neq \omega\ \ and\ \ |\theta - \omega|\ is\ small. \tag{3.13}$$

or in the past provided that there exists $\omega \in R$ such that

$$\widetilde{\mu}_2(\omega) < 0, \widetilde{\mu}_1(\omega) = 0, \widetilde{\mu}_2(\theta)(\theta - \omega) > 0,\ \ for\ \theta \neq \omega\ \ and\ \ |\theta - \omega|\ is\ small. \tag{3.14}$$

The periodicity of $p(t)$ leads that

$$S(\tau_0 + 2\pi, v_0) = S(\tau_0, v_0) + (2\pi, 0).$$

Let

$$\widetilde{\mu}_1(\theta) = (1 + e^{\frac{\alpha}{2\sqrt{A}}\pi})(\frac{1}{\gamma}g(-\infty) - \frac{1}{\gamma}g(+\infty)) + \mu_1(\theta),$$

$$\widetilde{\mu}_2(\theta) = (1 + e^{\frac{\alpha}{2\sqrt{A}}\pi})(\frac{\alpha}{\mu}g(+\infty) - \frac{\beta}{\gamma}g(-\infty)) + \mu_2(\theta)$$

and we just see v_0 as a radius r and τ_0 as an angle θ. Then the successor map P has an asymptotic expression as (3.12).

With the application of the propositions of the planar mapping (3.12) , we can easily get the conclusion below.

Theorem 3.1. *Assume that conditions* $(H_0), (H_1)$ *hold and* $\frac{1}{\sqrt{A}} + \frac{1}{\sqrt{B}} \in \mathbb{Q}$. *If for some* $\omega \in \mathbb{R}$ *and the successor map such that* (3.13) *or* (3.14), *then there exists a constant* $\Gamma > 0$ *such that for* $v_0 > \Gamma$, *solution* $(u(t;\tau_0,v_0), v(t;\tau_0,v_0))$ *of Eq.(1.3) with* $u(\tau_0;\tau_0,v_0) = 0, v(\tau_0;\tau_0,v_0) = v_0$ *is unbounded either in the future or in the past.*

We also can have the existence of periodic solution under some conditions.

Theorem 3.2. *Assume that conditions* $(H_0), (H_1)$ *hold and* $\frac{1}{\sqrt{A}} + \frac{1}{\sqrt{B}} \in \mathbb{Q}$. *Then Eq. (1.3) possesses at least one* 2π*-periodic solution provided that either the function*

$$\kappa(\frac{g(-\infty)}{\gamma} - \frac{g(+\infty)}{\mu}) + \mu_1(\tau_0),\ \tau_0 \in \mathbb{R}$$

or the function

$$\kappa(\alpha\frac{g(+\infty)}{\gamma} - \beta\frac{g(-\infty)}{\mu}) + \mu_2(\tau_0),\ \tau_0 \in \mathbb{R}$$

has a constant sign, where $\kappa = 1 + e^{\frac{\alpha}{2\sqrt{A}}\pi}$.

The proof of Theorem 3.2 is similar to that for Theorem 2.3 in [21].

3.2 Twist Map and Periodic Solution

We will use the twist property of the successor map based on the estimates in section 2. Denote by $(\tau_m, v_m) = S^m(\tau_0, v_0)$.Since $\mu_1(\tau), \mu_2(\tau)$ are continuous and $2\pi-$periodic functions, let

$$\begin{aligned} P_1 &= \min_{\tau\in[0,2\pi]} \kappa(\mu^{-1}g(+\infty) - \gamma^{-1}g(-\infty)) - \mu_1(\tau), \\ P_2 &= \max_{\tau\in[0,2\pi]} |\kappa(\mu^{-1}\alpha g(+\infty) - \gamma^{-1}\beta g(-\infty)) - \mu_2(\tau)|. \end{aligned}$$

From lemma 2.5, we know that there is $\Gamma > 0$ such that

$$\begin{aligned} ||v_2| - |v_0|| &< P_2, \\ \tau_2 - \tau_0 &< 2\frac{q}{p}\pi - \frac{P_1}{v_0},\ for\ |v_0| > \Gamma. \end{aligned}$$

Assume that there exists positive constant $\alpha_0 > 1$ such that

$$(H_2):\quad 0 < P_1 \le 2\pi\alpha_0 P_2.$$

Taking a large enough integer m such that $v_0 \in [\Gamma + (mp-1)P_2, (\alpha_0 mp + 1)P_2]$, we have

$$\begin{aligned} \tau_2 pm - \tau_0 &< 2pm\frac{q}{p}\pi - P_1(\frac{1}{v_0} + \frac{1}{v_1} + \cdots + \frac{1}{v_{2(mp-1)}}) \\ &< 2qm\pi - \frac{P_1}{P_2}(\frac{1}{v_0 P_2^{-1}} + \frac{1}{v_0 P_2^{-1} + 1} + \cdots + \frac{1}{v_0 P_2^{-1} + mp - 1}) \\ &= 2qm\pi - \frac{P_1}{P_2}\sum_{j=0}^{mp-1} \frac{1}{v_0 P_2^{-1} + j}. \end{aligned}$$

Since

$$\sum_{j=0}^{mp-1} \frac{1}{v_0 P_2^{-1} + j} > \sum_{j=0}^{mp-1} \frac{1}{\alpha_0 mp + 1 + j} \to ln(1 + \frac{1}{\alpha_0}),\ as\ m \to +\infty,$$

we obtain

$$\tau_{2mp} - \tau_0 < 2mq\pi - \frac{P_1}{P_2}\frac{1}{\alpha_0} < 2(mq-1)\pi.$$

On the other hand, from Lemma 2.2, for any fixed $m \in \mathbb{N}$,we have $\Gamma^* > 0$, such that

$$\tau_{2mp} - \tau_0 > 2(mq-1)\pi,\ for\ |v_0| \geq \Gamma^*.$$

Hence, taking $\Gamma_* \in [\Gamma + (mp-1)P_2, (\alpha_0 mp + 1)P_2]$, we have proved that the successor map S^{2mp} has a twist property on annulus $S^1 \times [\Gamma_*, \Gamma^*]$.

Therefore, we obtained the following theorem.

Theorem 3.3. *Assume that* $(H_0), (H_2)$ *hold and* $\frac{1}{\sqrt{A}} + \frac{1}{\sqrt{B}} = \frac{q}{p}$, *where* $\frac{q}{p}$ *is the reduced form,* $p, q \in \mathbb{Z}^+$. *Then Eq. (1.3) has one* 2π*−periodic solution.*

Proof. We meet all the assumptions of the Dingweiyue's twist theorem(see Theorem 3.1 in [24]). ■

4 Nonresonance

4.1 Nonresonance: Unbounded Solutions

Assume that

$$\frac{T}{2\pi} \in \mathbb{R}\backslash\mathbb{Q},\ T = \frac{\pi}{\sqrt{A}} + \frac{\pi}{\sqrt{B}},$$

that is ,

$$\frac{1}{\sqrt{A}} + \frac{1}{\sqrt{B}} \in \mathbb{R}\backslash\mathbb{Q}.$$

Given $\sigma > 0$, let us define

$$E_\sigma = \{(\theta, \rho) \in \mathbb{R} \times (0, +\infty) : \rho > \sigma\}.$$

Assume that the mapping $P : E_\sigma \to \mathbb{R}^2$ is a one-to-one and continuous mapping, whose lift (also denoted by P) can be expressed in the form

$$P : \begin{cases} \theta_1 = \theta + 2\widetilde{\alpha}\pi + \frac{1}{\rho}\widetilde{\mu}_1(\theta) + \rho H(\theta, \rho); \\ \rho_1 = \rho + \widetilde{\mu}_2(\theta) + G(\theta, \rho), \rho \to +\infty. \end{cases} \tag{4.15}$$

where

$$\alpha \in \mathbb{R}^+\backslash\mathbb{Q},\ \widetilde{\mu}_1, \widetilde{\mu}_2 : S^1 \mapsto S^1 areLipschitzcontinuous \tag{4.16}$$

with $S^1 = \mathbb{R} \setminus 2\pi\mathbb{Z}$,H and G are continuous functions,which are 2π-periodic in θ and satisfy

$$\rho|H(\theta, \rho)| + |G(\theta, \rho)| \to 0 \tag{4.17}$$

uniformly with respect to $\theta \in R$ as $\rho \to +\infty$.

Given a point $(\theta_0, \rho_0) \in E_\sigma$, denote by $\{(\theta_j, \rho_j)\}$ the orbit of the mapping P through the point (θ_0, ρ_0). That is to say $(\theta_j + 1, \rho_j + 1) = P((\theta_j, \rho_j))$.

The following abstract propositions taken from [22] will be applied to prove our theorem.

Proposition 4.1. *Assume that the condition* (4.16), (4.17) *hold and*

$$\int_0^{2\pi} \widetilde{\mu_2}(\theta)d\theta > 0.$$

Then there exists $R_0 > \sigma$ *such that if* $\rho_0 \geq R_0$*, the orbit* $\{(\theta_j, \rho_j)\}$ *exists in the future and satisfies*

$$\lim_{j\to+\infty} \rho_j = +\infty.$$

Proposition 4.2. *Assume that the condition* (4.16), (4.17) *hold and*

$$\int_0^{2\pi} \widetilde{\mu_2}(\theta)d\theta < 0.$$

Then there exists $R_0 > \sigma$ *such that if* $\rho_0 \geq R_0$*, the orbit* $\{(\theta_j, \rho_j)\}$ *exists in the future and satisfies*

$$\lim_{j\to-\infty} \rho_j = +\infty.$$

Now we are ready to state one of our main results. Denote by

$$\overline{p} = \int_0^{2\pi} p(t)dt.$$

Theorem 4.3. *Assume that* $(H_0), (H_1)$ *hold and* $\frac{1}{\sqrt{A}} + \frac{1}{\sqrt{B}} \in \mathbb{R}\backslash\mathbb{Q}$.

(i)If

$$\beta(g(+\infty) - \overline{p}) < \alpha(g(-\infty) - \overline{p}),$$

then there exists a constant $\Gamma > 0$ *such that, for* $v_0 \geq \Gamma$*, the solution* $(u(t;\tau_0, v_0), v(t;\tau_0, v_0))$ *of Eq.* (1.3) *with* $u(\tau_0;\tau_0, v_0) = 0, v(\tau_0;\tau_0, v_0) = v_0$ *satisfies*

$$\lim_{t\to+\infty}(|u(t;\tau_0, v_0)|^2 + |v(t;\tau_0, v_0)|^2) = +\infty.$$

(ii)If

$$\beta(g(+\infty) - \overline{p}) > \alpha(g(-\infty) - \overline{p}),$$

then there exists a constant $\Gamma > 0$ *such that, for* $v_0 \geq \Gamma$*, the solution* $(u(t;\tau_0, v_0), v(t;\tau_0, v_0))$ *of Eq.* (1.3) *with* $u(\tau_0;\tau_0, v_0) = 0, v(\tau_0;\tau_0, v_0) = v_0$ *satisfies*

$$\lim_{t\to-\infty}(|u(t;\tau_0, v_0)|^2 + |v(t;\tau_0, v_0)|^2) = +\infty.$$

Moreover, Eq. (1.3) *has at least one* 2π*-periodic solution.*

Proof. We prove the result on the unboundedness of solutions with large initial energy in the first case; the other case can be handled similarly.

Assume $\beta(g(+\infty) - \overline{p}) < \alpha(g(-\infty) - \overline{p})$. We can apply Proposition 4.1 to the successor mapping $P = S^2$, where

$$\widetilde{\mu_2}(\tau_0) = \kappa(\mu^{-1}\alpha g(+\infty) - \gamma^{-1}\beta g(-\infty)) + \mu^2(\tau_0),\ \kappa = 1 + e^{\frac{\alpha}{2\sqrt{A}}\pi}.$$

We obtain

$$\begin{aligned}\int_0^{2\pi}\widetilde{\mu}_2(\tau_0)d\tau_0 &= 2\pi\kappa(\mu^{-1}\alpha g(+\infty)-\gamma^{-1}\beta g(-\infty))+\overline{p}\int_0^{\frac{\pi}{\sqrt{A}}}(-\frac{\alpha}{2\sqrt{A}}\sin\sqrt{A}t\\ &\quad +\cos\sqrt{A}t)e^{\frac{\alpha}{2}t}dt+\overline{p}\int_{\frac{\pi}{\sqrt{A}}}^{\frac{\pi}{\sqrt{A}}+\frac{\pi}{\sqrt{B}}}(-\frac{1}{2\sqrt{B}}\sin\sqrt{B}(t-\frac{\pi}{\sqrt{A}})+\\ &\quad \cos\sqrt{B}(t-\frac{\pi}{\sqrt{A}}))e^{-(\frac{\beta\pi}{2\sqrt{A}}+\frac{\beta\pi}{2\sqrt{B}})+\frac{\beta}{2}t}dt\\ &= 2\pi\kappa(\mu^{-1}\alpha(g(+\infty)-\overline{p})-\gamma^{-1}\beta(g(-\infty)-\overline{p}))\\ &= 2\pi\kappa\beta\alpha^{-1}\gamma^{-1}(\beta(g(+\infty)-\overline{p})-\alpha(g(-\infty)-\overline{p}))>0.\end{aligned}$$

Therefore, there exists $\Gamma > 0$ such that, if $V_0 \geq \Gamma$, then the orbit (τ_j, v_j) exists in the future and satisfies $\lim_{j\to+\infty} v_j = +\infty$. With the lemma 2.5 ,we obtain the proof of the first part of the theorem.

Since $\widetilde{\alpha} = \frac{1}{\sqrt{A}} + \frac{1}{\sqrt{B}}$ is irrational and in view of the expression for the successor mapping P as (2.11), it follows that (τ_2, v_2) and (τ_0, v_0) never lie on the ray $\tau_2 = \tau_0$ for v_0 large enough. So the successor mapping meets all the assumptions of the Poincar-Bohl Theorem. Therefore P has at least one fixed point. Consequently, *Eq*. (1.3) has at least a 2π-periodic solution. Thus we have finished the proof of Theorem 4.3. ∎

Corollary 4.4. *Assume that* $(H_0), (H_1)$ *hold and* $\frac{1}{\sqrt{A}} + \frac{1}{\sqrt{B}} \in \mathbb{R}\backslash\mathbb{Q}$.

If such that

$$\max\{g(+\infty), g(-\infty)\} < \overline{p},$$

or

$$\min\{g(+\infty), g(-\infty)\} > \overline{p},$$

then there exists a constant $\Gamma > 0$ *such that, for* $v_0 \geq \Gamma$, *every solution* $(u(t;\tau_0,v_0), v(t;\tau_0,v_0))$ *of Eq.(1.3) with* $u(\tau_0;\tau_0,v_0) = 0, v(\tau_0;\tau_0,v_0) = v_0$ *is unbounded either in the future or in the past.*

We consider a particular case for *Eq*. (1.3). With similar discussion we can have the theorem below.

Theorem 4.5. *Assume that* (H_0) *hold and* $\frac{1}{\sqrt{A}} + \frac{1}{\sqrt{B}} \in \mathbb{R}\backslash\mathbb{Q}$,

$$\overline{p} = \int_0^{2\pi} p(t)dt \neq 0$$

and

$$\lim_{u\to+\infty} g(u) = \lim_{u\to-\infty} g(u) = 0,$$

(i)If $\alpha < \beta$, *then there existence a constant* $\Gamma > 0$ *such that, for* $v_0 \geq \Gamma$, *the solution* $(u(t;\tau_0,v_0), v(t;\tau_0,v_0))$ *of Eq*. (1.3) *with* $u(\tau_0;\tau_0,v_0) = 0, v(\tau_0;\tau_0,v_0) = v_0$ *satisfies*

$$\lim_{t\to+\infty}(|u(t;\tau_0,v_0)|^2 + |v(t;\tau_0,v_0)|^2) = +\infty.$$

(ii)If $\alpha > \beta$, then there existence a constant $\Gamma > 0$ such that, for $v_0 \geq \Gamma$, the solution $(u(t;\tau_0, v_0), v(t;\tau_0, v_0))$ of Eq.(1.3) with $u(\tau_0;\tau_0, v_0) = 0, v(\tau_0;\tau_0, v_0) = v_0$ satisfies

$$\lim_{t\to-\infty}(|u(t;\tau_0, v_0)|^2 + |v(t;\tau_0, v_0)|^2) = +\infty.$$

Moreover, Eq. (1.3) has at least one 2π-periodic solution.

It is obvious that when $g(u) \equiv 0$, *Eq.* (1.3) has become

$$\begin{cases} u' = v - \alpha u^+ + \beta u^-; \\ v' = -\mu u^+ + \gamma u^- + p(t). \end{cases} \tag{4.18}$$

Owing to Theorem 4.5, the following result can be obtained.

Corollary 4.6. *Assume that (H_0) hold and $\frac{1}{\sqrt{A}} + \frac{1}{\sqrt{B}} \in \mathbb{R}\backslash\mathbb{Q}$.*

If $\alpha \neq 0, \overline{p} \neq 0$, then all solutions of Eq. (4.18) with large initial energy are unbounded either in the future or in the past.

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