Convolution equations on spaces of quasi-nuclear functions of a given type and order

Vinícius V. Fávaro*

Abstract

In this article we prove existence and approximation results for convolution equations on the spaces of $(s;(r,q))$-quasi-nuclear mappings of a given type and order on a Banach space E. As special case this yields results for partial differential equations with constant coefficients for entire functions on finite-dimensional complex Banach spaces. We also prove division theorems for $(s;m(r,q))$-summing functions of a given type and order, that are essential to prove the existence and approximation results.

1 Introduction

In 1955-1956 Malgrange [14] proved an existence theorem for convolution equations on the Fréchet space of entire functions $\mathcal{H}(\mathbb{C}^n)$ with the compact-open topology. In this case, a convolution equation is an equation of the form $\mathcal{O}f = g$ where $\mathcal{O}$ is a convolution operator on $\mathcal{H}(\mathbb{C}^n)$, that is, a continuous linear operator on $\mathcal{H}(\mathbb{C}^n)$ that commutes with all directional derivatives. An illustrative example is the following differential partial equation with constant coefficients

$$P(D)f = g,$$

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where $P\colon \mathbb{C}^n \longrightarrow \mathbb{C}$ is a polynomial given by

$$P(z_1, \ldots, z_n) = \alpha_0 + \sum_{k=1}^{m} \sum_{j_1+\ldots+j_n=k} \alpha_{j_1,\ldots,j_m} z_1^{j_1} \ldots z_n^{j_n} \tag{1}$$

and $P(D)$ is the linear operator defined on $\mathcal{H}(E)$, E a finite-dimensional Banach space, obtained by replacing in (1) the z_j^h by the h-th partial derivative of order h in the direction of some nonzero vector v_j in E.

Malgrange [14] also proved an approximation theorem for solutions of the associate homogeneous equation by solutions of polynomial-exponential type. Motivated by these results Martineau [15] in 1967 proved existence and approximation theorems for convolution equations on spaces of entire functions on $\mathbb{C}^n$ of a given type and a given order.

The next natural step in this line of investigation is the consideration of convolution operators in the space $\mathcal{H}(E)$ of the entire functions on a Banach space E. So, [14, 15] can be regarded as starting points of a series of related results for convolution operators on spaces of holomorphic functions on complex Banach spaces (see Gupta [12] 1969, Dineen [7] 1971, Dwyer III [10] 1971 and [9] 1976, Boland [1] 1974, Colombeau-Matos [5] 1980, Colombeau-Perrot [6] 1980, Matos-Nachbin [22] 1981 and Matos [16] 1980, [17] 1984, [18] 1986 and [21] 2007).

In order to describe the aim of this paper, we recall that the usual approach to prove existence and approximation results for convolution equations on a Fréchet subspace $\mathcal{F}$ of $\mathcal{H}(E)$ considers the following three steps:

(A) To characterize the topological dual $\mathcal{F}'$ of $\mathcal{F}$, through an isomorphism, called Fourier-Borel transformation, as a subspace $\mathcal{S}$ of functions of exponential type on E'.

(B) To prove a division result on $\mathcal{S}$, that is, if $fg = h$, $g \neq 0$, $g, h \in \mathcal{S}$, $f \in \mathcal{H}(E')$, then it is possible to show that $f \in \mathcal{S}$.

(C) To manipulate the results obtained in (A) and (B) in order to show that each convolution operator $\mathcal{O}$ is of the form $\mathcal{O}f = T * f$ for some $T \in \mathcal{F}'$ and all $f \in \mathcal{F}$. After that, Functional Analysis methods, including the Hahn-Banach Theorem, and a Dieudonné-Schwartz Theorem are used in order to prove the existence and approximation theorems for the convolution equations.

The development of the theory of absolutely summing mappings between Banach spaces (see, for instance, Diestel-Jarchow-Tonge [8], Piestch [27, 28], Matos [19, 20] , Botelho [2], Pellegrino [24, 25], Botelho-Pellegrino [3], Pérez-García-Villanueva [26], Cilia-Gutiérrez [4] and references therein) motivated Matos [21] to consider, in step (A), several Fréchet spaces of quasi-nuclear entire functions as $\mathcal{F}$ and identify the image of the corresponding Fourier-Borel transformations as spaces of absolutely summing exponential type functions on E'. Then he proceeds to prove steps (B) and (C) in order to get the existence and approximation theorems.

Motivated by these procedures we have introduced in [11] the spaces of $(s;(r,q))$-quasi-nuclear functions of a given type and order and the spaces of $(s;m(r,q))$-summing functions of a given type and order and proved that the range of the Fourier-Borel transforms of these spaces are algebraically identical to the spaces of the $(s';m(r',q'))$-summing functions of a given type and order defined in E'.

The aim of this article is to prove division theorems - step (B) - for $(s;m(r,q))$-summing functions of a given type and order. Next, according to step (C) we indicate that following the arguments of Matos [18], it is possible to get existence and approximation theorems for convolution operators in the spaces of $(s;(r,q))$-quasi-nuclear functions of a given type and order. These results generalize theorems obtained by Gupta [12], Malgrange [14], Martineau [15] and Matos [18, 21].

2 Convolution Operators

To introduce the concept of convolution operators and to prove that it is well-defined we recall the spaces of $(s;(r,q))$-quasi-nuclear functions of a given type and order considered in [11]. Next we show that convolution operators are of the form $\mathcal{O}f = T * f$ as mentioned in step (C).

According to Matos [21], $\mathcal{P}_{(s;m(r,q))}\left({}^{n}E\right)$ is the Banach space of all n-homogeneous polynomials on E which are $(s;m(r,q))$-summing at 0, endowed with the norm $\|\cdot\|_{(s,m(r;q))}$, and $\mathcal{P}_{\tilde{N},(s;(r,q))}\left({}^{n}E\right)$ is the Banach space of all $(s;(r,q))$-quasi-nuclear n-homogeneous polynomials on E, endowed with the norm $\|\cdot\|_{\tilde{N},(s;(r,q))}$, for all $j \in \mathbb{N}$, where $\mathbb{N} = \{0,1,2,...\}$.

In the definitions involving $(s,m(r;q))$-summing polynomials we consider $0 < q \leq r \leq +\infty$ and $s \in [1,+\infty]$ and in the definitions involving $(s;(r,q))$-quasi-nuclear polynomials we consider $s \leq q$, $r \leq q$ and $s,r,q \in [1,+\infty]$.

Definition 2.1. If $\rho > 0$ and $k \geq 1$, we denote by $\mathcal{B}^{k}_{(s,m(r;q)),\rho}(E)$ the complex Banach space of all $f \in \mathcal{H}(E)$ such that $\widehat{d^j} f(0) \in \mathcal{P}_{(s,m(r;q))}\left({}^{j}E\right)$, for all $j \in \mathbb{N}$ and

$$\|f\|_{(s,m(r;q)),k,\rho} = \sum_{j=0}^{\infty} \rho^{-j} \left(\frac{j}{ke}\right)^{\frac{j}{k}} \left\|\frac{1}{j!}\widehat{d^j} f(0)\right\|_{(s,m(r;q))} < +\infty,$$

normed by $\|\cdot\|_{(s,m(r;q)),k,\rho}$. We denote by $\mathcal{B}^{k}_{\tilde{N},(s;(r,q)),\rho}(E)$ the complex Banach space of all $f \in \mathcal{H}(E)$ such that $\widehat{d^j} f(0) \in \mathcal{P}_{\tilde{N},(s;(r,q))}\left({}^{j}E\right)$, for all $j \in \mathbb{N}$ and

$$\|f\|_{\tilde{N},(s;(r,q)),k,\rho} = \sum_{j=0}^{\infty} \rho^{-j} \left(\frac{j}{ke}\right)^{\frac{j}{k}} \left\|\frac{1}{j!}\widehat{d^j} f(0)\right\|_{\tilde{N},(s;(r,q))} < +\infty,$$

normed by $\|\cdot\|_{\tilde{N},(s;(r,q)),k,\rho}$.

Definition 2.2. If $A \in (0,+\infty)$ and $k \geq 1$, we denote by $Exp^{k}_{(s,m(r;q)),A}(E)$ and $Exp^{k}_{\tilde{N},(s;(r,q)),A}(E)$ the complex vector spaces $\bigcup_{\rho<A} \mathcal{B}^{k}_{(s,m(r;q)),\rho}(E)$ and $\bigcup_{\rho<A} \mathcal{B}^{k}_{\tilde{N},(s;(r,q)),\rho}(E)$, respectively, both of them endowed with the corresponding locally convex inductive limit topologies. We consider the complex vector spaces $Exp^{k}_{(s,m(r;q)),0,A}(E) = \bigcap_{\rho>A} \mathcal{B}^{k}_{(s,m(r;q)),\rho}(E)$ and $Exp^{k}_{\tilde{N},(s;(r,q)),0,A}(E) = \bigcap_{\rho>A} \mathcal{B}^{k}_{\tilde{N},(s;(r,q)),\rho}(E)$ both of them endowed with the projective limit topologies.

If $A=+\infty$ and $k\geq 1$, we consider the complex vector spaces $Exp^{k}_{(s,m(r;q)),\infty}(E)=\bigcup_{\rho>0}\mathcal{B}^{k}_{(s,m(r;q)),\rho}(E)$ and $Exp^{k}_{\tilde{N},(s;(r,q)),\infty}(E)=\bigcup_{\rho>0}\mathcal{B}^{k}_{\tilde{N},(s;(r,q)),\rho}(E)$ both of them with the locally convex inductive limit topologies and if $A=0$ and $k\geq 1$, we consider the complex vector spaces $Exp^{k}_{(s,m(r;q)),0}(E)=\bigcap_{\rho>0}\mathcal{B}^{k}_{(s,m(r;q)),\rho}(E)$ and $Exp^{k}_{\tilde{N},(s;(r,q)),0}(E)=\bigcap_{\rho>0}\mathcal{B}^{k}_{\tilde{N},(s;(r,q)),\rho}(E)$ both of them with the projective limit topologies.

Definition 2.3. If $A\in[0,+\infty)$, we denote by $\mathcal{H}_{b(s,m(r;q))}\left(B_{\frac{1}{A}}(0)\right)$ the complex vector space of all $f\in\mathcal{H}\left(B_{\frac{1}{A}}(0)\right)$ such that $\widehat{d}^{j}f(0)\in\mathcal{P}_{(s,m(r;q))}\left({}^{j}E\right)$, for all $j\in\mathbb{N}$ and

$$\limsup_{j\to\infty}\left\|\frac{1}{j!}\widehat{d}^{j}f(0)\right\|_{(s,m(r;q))}^{\frac{1}{j}}\leq A,$$

endowed with the locally convex topology generated by the seminorms $\left(p^{\infty}_{(s,m(r;q)),\rho}\right)_{\rho>A}$, where

$$p^{\infty}_{(s,m(r;q)),\rho}(f)=\sum_{j=0}^{\infty}\rho^{-j}\left\|\frac{1}{j!}\widehat{d}^{j}f(0)\right\|_{(s,m(r;q))}.$$

We denote by $\mathcal{H}_{\tilde{N}b,(s;(r,q))}\left(B_{\frac{1}{A}}(0)\right)$ the complex vector space of all $f\in\mathcal{H}\left(B_{\frac{1}{A}}(0)\right)$ such that $\widehat{d}^{j}f(0)\in\mathcal{P}_{\tilde{N},(s;(r,q))}\left({}^{j}E\right)$, for all $j\in\mathbb{N}$ and

$$\limsup_{j\to\infty}\left\|\frac{1}{j!}\widehat{d}^{j}f(0)\right\|_{\tilde{N},(s;(r,q))}^{\frac{1}{j}}\leq A,$$

endowed with the locally convex topology generated by the seminorms $\left(p^{\infty}_{\tilde{N},(s;(r,q)),\rho}\right)_{\rho>A}$, where

$$p^{\infty}_{\tilde{N},(s;(r,q)),\rho}(f)=\sum_{j=0}^{\infty}\rho^{-j}\left\|\frac{1}{j!}\widehat{d}^{j}f(0)\right\|_{\tilde{N},(s;(r,q))}.$$

We denote $\mathcal{H}_{b(s,m(r;q))}\left(B_{\frac{1}{A}}(0)\right)$ by $Exp^{\infty}_{(s,m(r;q)),0,A}(E)$ and $\mathcal{H}_{\tilde{N}b,(s;(r,q))}\left(B_{\frac{1}{A}}(0)\right)$ by $Exp^{\infty}_{\tilde{N},(s;(r,q)),0,A}(E)$ and we also write $Exp^{\infty}_{(s,m(r;q)),0}(E)=Exp^{\infty}_{(s,m(r;q)),0,0}(E)$ and $Exp^{\infty}_{\tilde{N},(s;(r,q)),0}(E)=Exp^{\infty}_{\tilde{N},(s;(r,q)),0,0}(E)$.

New spaces are now constructed as follows:

Let $L=\bigcup_{\rho<A}\mathcal{H}_{\tilde{N}b,(s;(r,q))}\left(B_{\frac{1}{\rho}}(0)\right)$ and define the following relation:

$$f\sim g\iff \text{there is } \rho\in(0,A) \text{ such that } f|_{B_{\frac{1}{\rho}}(0)}=g|_{B_{\frac{1}{\rho}}(0)}.$$

It is obvious that $\sim$ is an equivalence relation. As usual, $L/\sim$ denotes the quotient set and $[f]$ stands for the equivalence class of f. Now we define the following operations on $L/\sim$:

$$[f]+[g]=\left[f|_{B_{\frac{1}{\rho}}(0)}+g|_{B_{\frac{1}{\rho}}(0)}\right], \text{where } \rho\in(0,A) \text{ is such that}$$

$$f|_{B_{\frac{1}{\rho}}(0)}, g|_{B_{\frac{1}{\rho}}(0)}\in\mathcal{H}_{\tilde{N}b,(s;(r,q))}\left(B_{\frac{1}{\rho}}(0)\right).$$

$$\lambda[f]=[\lambda f],\qquad \lambda\in\mathbb{C},$$

which make $L/\sim$ a vector space. The case $(s,m(r;q))$ is analogous.
For each $\rho\in(0,A)$, let $i_\rho\colon\mathcal{H}_{\tilde{N}b,(s;(r,q))}\left(B_{\frac{1}{\rho}}(0)\right)\longrightarrow L/\sim$ be given by $i_\rho(f)=[f]$.

Definition 2.4. If $A\in(0,+\infty]$, we define $\mathcal{H}_{\tilde{N}b,(s;(r,q))}\left(\overline{B_{\frac{1}{A}}(0)}\right)=L/\sim$ endowed with the locally convex inductive limit topology generated by the family $(i_\rho)_{\rho\in(0,A)}$.
In the same way we construct the space $\mathcal{H}_{b(s,m(r;q))}\left(\overline{B_{\frac{1}{A}}(0)}\right)$.

Now we define the following spaces:

Definition 2.5. If $\rho>0$, we define the complex vector space $\mathcal{H}^{\infty}_{(s,m(r;q))}\left(B_{\frac{1}{\rho}}(0)\right)$ of all $f\in\mathcal{H}\left(B_{\frac{1}{\rho}}(0)\right)$ such that $\widehat{d}^jf(0)\in\mathcal{P}_{(s,m(r;q))}\left({}^jE\right)$, for all $j\in\mathbb{N}$ and

$$\sum_{j=0}^{\infty}\rho^{-j}\left\|\frac{1}{j!}\widehat{d}^jf(0)\right\|_{(s,m(r;q))}<+\infty,$$

which is a Banach space with the norm $p^{\infty}_{(s,m(r;q)),\rho}$. We also define the complex vector space
$\mathcal{H}^{\infty}_{\tilde{N},(s;(r,q))}\left(B_{\frac{1}{\rho}}(0)\right)$ of all $f\in\mathcal{H}\left(B_{\frac{1}{\rho}}(0)\right)$ such that $\widehat{d}^jf(0)\in\mathcal{P}_{\tilde{N},(s;(r,q))}\left({}^jE\right)$, for all $j\in\mathbb{N}$ and

$$\sum_{j=0}^{\infty}\rho^{-j}\left\|\frac{1}{j!}\widehat{d}^jf(0)\right\|_{\tilde{N},(s;(r,q))}<+\infty,$$

which is a Banach space with the norm $p^{\infty}_{\tilde{N},(s;(r,q)),\rho}$.
An equivalence relation $\sim$ is defined on $L=\bigcup_{\rho<A}\mathcal{H}^{\infty}_{(s,m(r;q))}\left(B_{\frac{1}{\rho}}(0)\right)$ as in Definition 2.4. For $A\in(0,+\infty]$, we define

$$Exp^{\infty}_{(s,m(r;q)),A}(E)=L/\sim=\bigcup_{\rho<A}\mathcal{H}^{\infty}_{(s,m(r;q))}\left(B_{\frac{1}{\rho}}(0)\right)/\sim$$

endowed with the locally convex inductive limit topology. We also define

$$Exp^{\infty}_{\tilde{N},(s;(r,q)),A}(E) = \bigcup_{\rho<A} \mathcal{H}^{\infty}_{\tilde{N}b,(s;(r,q))}\left(B_{\frac{1}{\rho}}(0)\right)/\sim$$

endowed with the locally convex inductive limit topology.

The next result, proved in [11], assures that Definitions 2.4 and 2.5 are equivalent:

Proposition 2.6. *The spaces* $\mathcal{H}_{\tilde{N}b,(s;(r,q))}\left(\overline{B_{\frac{1}{A}}(0)}\right)$ *and* $Exp^{\infty}_{\tilde{N},(s;(r,q)),A}(E)$ *coincide algebraically and are topologically isomorphic, and the same holds for the spaces* $\mathcal{H}_{b(s,m(r;q))}\left(\overline{B_{\frac{1}{A}}(0)}\right)$ *and* $Exp^{\infty}_{(s,m(r;q)),A}(E)$.

Now we are in the position to prove a preliminary result that we need to introduce convolution operators.

Proposition 2.7. (a) *If* $a \in E, k \in [1,+\infty], A \in (0,+\infty]$ *and* $f \in Exp^{k}_{\tilde{N},(s;(r,q)),A}(E)$, *then* $\hat{d}^n f(\cdot)a \in Exp^{k}_{\tilde{N},(s;(r,q)),A}(E)$ *and*

$$\hat{d}^n f(\cdot)a = \sum_{j=0}^{\infty}(j!)^{-1}\,\widehat{d^{j+n}f(0)\cdot^{j}}(a),$$

in the sense of the topology of $Exp^{k}_{\tilde{N},(s;(r,q)),A}(E)$.

(b) *If* $a \in E$, $k \in [1,+\infty]$, $A \in [0,+\infty)$ *and* $f \in Exp^{k}_{\tilde{N},(s;(r,q)),0,A}(E)$, *then* $\hat{d}^n f(\cdot)a \in Exp^{k}_{\tilde{N},(s;(r,q)),0,A}(E)$ *and*

$$\hat{d}^n f(\cdot)a = \sum_{j=0}^{\infty}(j!)^{-1}\,\widehat{d^{j+n}f(0)\cdot^{j}}(a),$$

in the sense of the topology of $Exp^{k}_{\tilde{N},(s;(r,q)),0,A}(E)$.

Proof. It is known (see Nachbin [23, p. 29]) that

$$\hat{d}^j f(x)a = \sum_{n=0}^{\infty}(n!)^{-1}\,\widehat{d^{j+n}f(0)x^n}(a) = \sum_{n=0}^{\infty}(n!)^{-1}\,\widehat{d^{j+n}f(0)a^j}(x), \tag{2}$$

for all $x \in E$. By Matos [21, pp. 163-164] we have $\widehat{d^{j+n}f(0)a^j} \in \mathcal{P}_{\tilde{N},(s;(r,q))}(^nE)$ and

$$\left\|\widehat{d^{j+n}f(0)a^j}\right\|_{\tilde{N},(s;(r,q))} \leq \left\|\hat{d}^{n+j}f(0)\right\|_{\tilde{N},(s;(r,q))}\|a\|^j,$$

for all $n \in \mathbb{N}$. If $k \in [1,+\infty)$, let

$$L = \limsup_{n\to\infty}\left(\frac{n+j}{ke}\right)^{\frac{1}{k}}\left\|\frac{\hat{d}^{n+j}f(0)}{(n+j)!}\right\|^{\frac{1}{n+j}}_{\tilde{N},(s;(r,q))}.$$

In both cases (a) and (b), it follows by [11, Proposition 2.5] that $L < +\infty$. Then for every $\varepsilon > 0$, there is $C(\varepsilon) > 0$ such that

$$\left(\frac{n+j}{ke}\right)^{\frac{n+j}{k}} \left\| \frac{\widehat{d}^{n+j} f(0)}{(n+j)!} \right\|_{\tilde{N},(s;(r,q))} \leq C(\varepsilon)\,(L+\varepsilon)^{n+j},$$

for all $n \in \mathbb{N}$. Hence

$$\begin{aligned}
\left(\frac{n}{ke}\right)^{\frac{n}{k}} \frac{1}{n!} \left\| \widehat{d^{j+n} f(0)\, a^{j}} \right\|_{\tilde{N},(s;(r,q))} &\leq \left(\frac{n}{ke}\right)^{\frac{n}{k}} \frac{1}{n!} \left\| \widehat{d}^{n+j} f(0) \right\|_{\tilde{N},(s;(r,q))} \|a\|^{j} \\
&\leq \left(\frac{n}{ke}\right)^{\frac{n}{k}} \frac{(n+j)!}{n!} \left(\frac{ke}{n+j}\right)^{\frac{n+j}{k}} C(\varepsilon)\,(L+\varepsilon)^{n+j} \|a\|^{j} \\
&= \left(\frac{n}{n+j}\right)^{\frac{n}{k}} (n+1)\ldots(n+j) \left(\frac{ke}{n+j}\right)^{\frac{j}{k}} C(\varepsilon)\,(L+\varepsilon)^{n+j} \|a\|^{j}. \qquad (3)
\end{aligned}$$

Since

$$\lim_{n\to\infty} \left(\frac{n}{n+j}\right)^{\frac{1}{k}} \left[(n+1)\ldots(n+j)\right]^{\frac{1}{n}} \left(\frac{ke}{n+j}\right)^{\frac{j}{kn}} = 1,$$

there is $D(\varepsilon) > 0$ such that

$$\left(\frac{n}{n+j}\right)^{\frac{n}{k}} (n+1)\ldots(n+j) \left(\frac{ke}{n+j}\right)^{\frac{j}{k}} \leq D(\varepsilon)\,(1+\varepsilon)^{n}. \qquad (4)$$

From (3) and (4) we obtain

$$\left(\frac{n}{ke}\right)^{\frac{n}{k}} \frac{1}{n!} \left\| \widehat{d^{j+n} f(0)\, a^{j}} \right\|_{\tilde{N},(s;(r,q))} \leq C(\varepsilon)\, D(\varepsilon)\, \|a\|^{j} (L+\varepsilon)^{j} \left[(1+\varepsilon)(L+\varepsilon)\right]^{n},$$

for all $n \in \mathbb{N}$ and $\varepsilon > 0$. Therefore

$$\limsup_{n\to\infty} \left(\frac{n}{ke}\right)^{\frac{1}{k}} \left\| \frac{\widehat{d^{j+n} f(0)\, a^{j}}}{n!} \right\|_{\tilde{N},(s;(r,q))}^{\frac{1}{n}} \leq (1+\varepsilon)(L+\varepsilon),$$

for all $\varepsilon > 0$, which implies

$$\limsup_{n\to\infty} \left(\frac{n}{ke}\right)^{\frac{1}{k}} \left\| \frac{\widehat{d^{j+n} f(0)\, a^{j}}}{n!} \right\|_{\tilde{N},(s;(r,q))}^{\frac{1}{n}} \leq L.$$

By [11, Proposition 2.5], if $f \in Exp^{k}_{\tilde{N},(s;(r,q)),A}(E)$ and $A \in (0,+\infty]$, then $L < A$ and $\widehat{d}^{n} f(\cdot)\, a \in Exp^{k}_{\tilde{N},(s;(r,q)),A}(E)$; and if $f \in Exp^{k}_{\tilde{N},(s;(r,q)),0,A}(E)$ and $A \in [0,+\infty)$, then $L \leq A$ and $\widehat{d}^{n} f(\cdot)\, a \in Exp^{k}_{\tilde{N},(s;(r,q)),0,A}(E)$.
Now we consider $k = +\infty$. The case $\mathcal{H}_{\tilde{N}b,(s;(r,q))}(E) = Exp^{\infty}_{\tilde{N},(s;(r,q)),0}(E)$ (i.e.

$A = 0$) was proved by Matos [21, p. 174]. For $f \in Exp^{\infty}_{\tilde{N},(s;(r,q)),0,A}(E) = \mathcal{H}_{\tilde{N}b,(s;(r,q))}\left(B_{\frac{1}{A}}(0)\right)$, we get

$$\limsup_{n\to\infty} \left\| \frac{\widehat{d}^{n+j} f(0)}{(n+j)!} \right\|_{\tilde{N},(s;(r,q))}^{\frac{1}{n+j}} \leq A$$

and as above we obtain

$$\limsup_{n\to\infty} \left\| \frac{\widehat{d^{j+n} f(0)\, a^j}}{n!} \right\|_{\tilde{N},(s;(r,q))}^{\frac{1}{n}} \leq A.$$

Thus $\widehat{d}^n f(\cdot)\, a \in \mathcal{H}_{\tilde{N}b,(s;(r,q))}\left(B_{\frac{1}{A}}(0)\right) = Exp^{\infty}_{\tilde{N},(s;(r,q)),0,A}(E)$.

For $f \in Exp^{\infty}_{\tilde{N},(s;(r,q)),A}(E)$, we have $f \in \mathcal{H}_{\tilde{N}b,(s;(r,q))}\left(B_{\frac{1}{\rho}}(0)\right)$ for some $\rho < A$. Thus, $\widehat{d}^n f(\cdot)\, a \in \mathcal{H}_{\tilde{N}b,(s;(r,q))}\left(B_{\frac{1}{\rho}}(0)\right)$. This implies $\widehat{d}^n f(\cdot)\, a \in Exp^{\infty}_{\tilde{N},(s;(r,q)),A}(E)$.

We still have to prove the convergence of the series in the respective topologies. If $f \in \mathcal{B}^k_{\tilde{N},(s;(r,q)),\rho}(E)$ for some $\rho > 0$ with $k \in [1,+\infty)$, repeating the argument above with ρ instead of L we get constants $C_1(\varepsilon) > 0$ and $D_1(\varepsilon) > 0$ such that

$$\begin{aligned}
&\left\| \widehat{d}^j f(\cdot)\, a - \sum_{n=0}^{v} (n!)^{-1}\, \widehat{d^{j+n} f(0) \cdot^n}\,(a) \right\|_{\tilde{N},(s;(r,q)),k,\rho_0} \\
&\leq \sum_{n=v+1}^{\infty} \rho_0^{-n} \left(\frac{n}{ke}\right)^{\frac{1}{k}} \left\| (n!)^{-1}\, \widehat{d}^{j+n} f(0) \right\|_{\tilde{N},(s;(r,q))} \|a\|^j \\
&\leq C_1(\varepsilon)\, D_1(\varepsilon)\, \|a\|^j (\rho+\varepsilon)^j \sum_{n=v+1}^{\infty} \left[\rho_0^{-1}(\rho+\varepsilon)(1+\varepsilon)\right]^n,
\end{aligned}$$

and this tends to zero when $v \to \infty$, for $\rho_0 > \rho$ and $\varepsilon > 0$ such that $(\rho+\varepsilon)(1+\varepsilon) < \rho_0$. Now the desired convergence follows from the definiton of the topologies. The case $k = +\infty$, is analogous.

Definition 2.8. For $k \in [1,+\infty]$ and $A \in (0,+\infty]$, a *convolution operator in* $Exp^k_{\tilde{N},(s;(r,q)),A}(E)$ is a continuous linear mapping

$$\mathcal{O}\colon Exp^k_{\tilde{N},(s;(r,q)),A}(E) \longrightarrow Exp^k_{\tilde{N},(s;(r,q)),A}(E)$$

such that $d(\mathcal{O}f)(\cdot)\, a = \mathcal{O}(df(\cdot)\, a)$ for all $a \in E$ and $f \in Exp^k_{\tilde{N},(s;(r,q)),A}(E)$.

For $k \in [1,+\infty]$ and $A \in [0,+\infty)$, a *convolution operator in* $Exp^k_{\tilde{N},(s;(r,q)),0,A}(E)$ is a continuous linear mapping

$$\mathcal{O}\colon Exp^k_{\tilde{N},(s;(r,q)),0,A}(E) \longrightarrow Exp^k_{\tilde{N},(s;(r,q)),0,A}(E)$$

such that $d(\mathcal{O}f)(\cdot)\, a = \mathcal{O}(df(\cdot)\, a)$ for all $a \in E$ and $f \in Exp^k_{\tilde{N},(s;(r,q)),0,A}(E)$.

We denote the set of all convolution operators in $Exp^k_{\tilde{N},(s;(r,q)),A}(E)$ and in

$Exp^k_{\tilde{N},(s;(r,q)),0,A}(E)$ by $\mathcal{A}^k_{\tilde{N},(s;(r,q)),A}$ and $\mathcal{A}^k_{\tilde{N},(s;(r,q)),0,A}$, respectively. We also denote $\mathcal{A}^k_{\tilde{N},(s;(r,q)),\infty} = \mathcal{A}^k_{\tilde{N},(s;(r,q))}$ and $\mathcal{A}^k_{\tilde{N},(s;(r,q)),0,0} = \mathcal{A}^k_{\tilde{N},(s;(r,q)),0}$.

Remark 2.9. From Definition 2.8 it follows that a convolution operator $\mathcal{O}$ commutes with all the directional derivatives of all orders, that is, for all $a \in E$ and $n \in \mathbb{N}$, $\mathcal{O}\left(\widehat{d}^n f(\cdot) a\right) = \widehat{d}^n (\mathcal{O} f)(\cdot) a$. Soon we shall prove that convolution operators could have been defined replacing the condition $d(\mathcal{O}f)(\cdot)a = O(df(\cdot)a)$ by $\tau_{-a}(\mathcal{O}(f)) = \mathcal{O}(\tau_{-a} f)$ for all $a \in E$, where $\tau_{-a} f(x) = f(x+a)$, for all $x \in E$, whenever the translation τ_{-a} is well defined. This means that, in these cases, commutativity with the directional derivatives is equivalent to commutativity with translations.

Proposition 2.10. (a) *For $k \in [1,+\infty)$, if $f \in Exp^k_{\tilde{N},(s;(r,q))}(E)$ and $a \in E$, then $\tau_{-a} f \in Exp^k_{\tilde{N},(s;(r,q))}(E)$ and*

$$\tau_{-a} f = \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{d}^n f(\cdot) a,$$

in the sense of the topology of $Exp^k_{\tilde{N},(s;(r,q))}(E)$.

(b) For $k \in [1,+\infty]$, *if $f \in Exp^k_{\tilde{N},(s;(r,q)),0}(E)$ and $a \in E$, then $\tau_{-a} f \in Exp^k_{\tilde{N},(s;(r,q)),0}(E)$ and*

$$\tau_{-a} f = \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{d}^n f(\cdot) a,$$

in the sense of the topology of $Exp^k_{\tilde{N},(s;(r,q)),0}(E)$.

Proof. The case (b), with $k = +\infty$, was proved by Matos in [21, p. 175]. For $k \in [1,+\infty)$, we suppose that

$$\limsup_{j\to\infty} \left(\frac{j}{ke}\right)^{\frac{1}{k}} \left\| \frac{\widehat{d}^j f(0)}{j!} \right\|_{\tilde{N},(s;(r,q))}^{\frac{1}{j}} = L < +\infty. \tag{5}$$

Then for all $\varepsilon > 0$ there is $C(\varepsilon) > 0$ such that

$$\left(\frac{j}{ke}\right)^{\frac{j}{k}} \left\| \frac{\widehat{d}^j f(0)}{j!} \right\|_{\tilde{N},(s;(r,q))} \le C(\varepsilon)(L+\varepsilon)^j, \tag{6}$$

for all $j \in \mathbb{N}$. Since $\widehat{d}^n (\tau_{-a} f)(0) = \widehat{d}^n f(a)$, we have

$$\left\|\widehat{d}^n (\tau_{-a} f)(0)\right\|_{\tilde{N},(s;(r,q))} = \left\|\widehat{d}^n f(a)\right\|_{\tilde{N},(s;(r,q))} \le \sum_{j=0}^{\infty} \frac{1}{j!} \left\|\widehat{d}^{n+j} f(0)\right\|_{\tilde{N},(s;(r,q))} \|a\|^j$$

and

$$\left(\frac{n}{ke}\right)^{\frac{n}{k}} \frac{1}{n!} \left\|\widehat{d}^n (\tau_{-a} f)(0)\right\|_{\tilde{N},(s;(r,q))} \le \sum_{j=0}^{\infty} \left(\frac{n}{ke}\right)^{\frac{n}{k}} \frac{1}{n!j!} \left\|\widehat{d}^{n+j} f(0)\right\|_{\tilde{N},(s;(r,q))} \|a\|^j$$

$$= \sum_{j=0}^{\infty} \left(\frac{n}{ke}\right)^{\frac{n}{k}} \left(\frac{ke}{n+j}\right)^{\frac{n+j}{k}} \frac{(n+j)!}{n!j!} \left(\frac{n+j}{ke}\right)^{\frac{n+j}{k}} \frac{1}{(n+j)!} \left\|\widehat{d}^{n+j} f(0)\right\|_{\tilde{N},(s;(r,q))} \|a\|^j$$

$$\le \sum_{j=0}^{\infty} \left(\frac{ke}{j}\right)^{\frac{j}{k}} 2^{n+j} \|a\|^j \left(\frac{n+j}{ke}\right)^{\frac{n+j}{k}} \frac{1}{(n+j)!} \left\|\widehat{d}^{n+j} f(0)\right\|_{\tilde{N},(s;(r,q))}.$$

Since $\lim_{j\to\infty}\left(\frac{ke}{j}\right)^{\frac{1}{k}}=0$, for each $\varepsilon>0$ there is $D(\varepsilon)>0$ such that

$$\left(\frac{ke}{j}\right)^{\frac{1}{k}}\leq D(\varepsilon)\,\varepsilon^{j}, \tag{7}$$

for all $j\in\mathbb{N}$. Considering $\varepsilon>0$ such that $2\varepsilon\|a\|(L+\varepsilon)<1$ and using (6), we obtain

$$\begin{aligned}\left(\frac{n}{ke}\right)^{\frac{n}{k}}\frac{1}{n!}\left\|\widehat{d}^{n}(\tau_{-a}f)(0)\right\|_{\tilde{N},(s;(r,q))} &\leq C(\varepsilon)\,D(\varepsilon)\,2^{n}(L+\varepsilon)^{n}\sum_{j=0}^{\infty}2^{j}\varepsilon^{j}\|a\|^{j}(L+\varepsilon)^{j}\\ &= C(\varepsilon)\,D(\varepsilon)\,2^{n}(L+\varepsilon)^{n}\frac{1}{1-2\varepsilon\|a\|(L+\varepsilon)}.\end{aligned}$$

Hence

$$\limsup_{n\to\infty}\left(\frac{n}{ke}\right)^{\frac{1}{k}}\left\|\frac{\widehat{d}^{n}(\tau_{-a}f)(0)}{n!}\right\|_{\tilde{N},(s;(r,q))}^{\frac{1}{n}}\leq 2L<+\infty. \tag{8}$$

Thus if $f\in Exp_{\tilde{N},(s;(r,q))}^{k}(E)$ we have $\tau_{-a}f\in Exp_{\tilde{N},(s;(r,q))}^{k}(E)$ by (8), and if $f\in Exp_{\tilde{N},(s;(r,q)),0}^{k}(E)$ we have (5) with $L=0$ and $\tau_{-a}f\in Exp_{\tilde{N},(s;(r,q)),0}^{k}(E)$ by (8).

In order to prove the convergence, let $f\in Exp_{\tilde{N},(s;(r,q))}^{k}(E)$.

Then $f\in\mathcal{B}_{\tilde{N},(s;(r,q)),L}^{k}(E)$ for some $L>0$. Let $\varepsilon>0$ such that $2\varepsilon\|a\|(L+\varepsilon)<1$. Then for $\rho>2(L+\varepsilon)$ we have

$$\begin{aligned}\left\|\tau_{-a}f-\sum_{n=0}^{v}\frac{1}{n!}\widehat{d}^{n}f(\cdot)\,a\right\|_{\tilde{N},(s;(r,q)),k,\rho} &\leq\sum_{j=0}^{\infty}\rho^{-j}\left(\frac{j}{ke}\right)^{\frac{j}{k}}\sum_{n=v+1}^{\infty}\frac{1}{j!n!}\left\|\widehat{d^{j}}\left(\widehat{d}^{n}f(\cdot)\,a\right)(0)\right\|_{\tilde{N},(s;(r,q))}\\ &\leq\sum_{j=0}^{\infty}\rho^{-j}\left(\frac{j}{ke}\right)^{\frac{j}{k}}\sum_{n=v+1}^{\infty}\frac{1}{j!n!}\left\|\widehat{d}^{n+j}f(0)\right\|_{\tilde{N},(s;(r,q))}\|a\|^{n}\\ &\leq\sum_{j=0}^{\infty}\sum_{n=v+1}^{\infty}\rho^{-j}\left(\frac{ke}{n+j}\right)^{\frac{n}{k}}2^{n+j}\left(\frac{n+j}{ke}\right)^{\frac{n+j}{k}}\left\|\frac{\widehat{d}^{n+j}f(0)}{(n+j)!}\right\|_{\tilde{N},(s;(r,q))}\|a\|^{n}\\ &\leq C(\varepsilon)\,D(\varepsilon)\sum_{j=0}^{\infty}\sum_{n=v+1}^{\infty}\rho^{-j}\varepsilon^{n}(L+\varepsilon)^{n+j}2^{n+j}\|a\|^{n}.\end{aligned}$$

Here we used Proposition 2.7 and the inequalities used in first part of the proof. By our choice of $\varepsilon>0$ we get

$$\begin{aligned}\left\|\tau_{-a}f-\sum_{n=0}^{v}\frac{1}{n!}\widehat{d}^{n}f(\cdot)\,a\right\|_{\tilde{N},(s;(r,q)),\rho} &\leq C(\varepsilon)\,D(\varepsilon)\sum_{j=0}^{\infty}\rho^{-j}(L+\varepsilon)^{j}2^{j}\sum_{n=v+1}^{\infty}\varepsilon^{n}(L+\varepsilon)^{n}2^{n}\|a\|^{n}\\ &= C(\varepsilon)\,D(\varepsilon)\frac{1}{1-2\rho^{-1}(L+\varepsilon)}\cdot\frac{\varepsilon^{v+1}(L+\varepsilon)^{v+1}2^{v+1}\|a\|^{v+1}}{1-2\varepsilon(L+\varepsilon)\|a\|}.\end{aligned}$$

Therefore

$$\lim_{v\to\infty}\left\|\tau_{-a}f-\sum_{n=0}^{v}\frac{1}{n!}\widehat{d}^{n}f(\cdot)\,a\right\|_{\tilde{N},(s;(r,q)),\rho}=0.$$

Now, if $f \in Exp^k_{\tilde{N},(s;(r,q)),0}(E)$, then $f \in \mathcal{B}^k_{\tilde{N},(s;(r,q)),L}(E)$ for all $L > 0$. Hence, if for each $\delta > 0$ we choose $\varepsilon > 0$ and $L > 0$ such that $\frac{\delta-2\varepsilon}{2} > 0$, $L < \frac{\delta-2\varepsilon}{2}$ and $2\varepsilon \|a\| (L+\varepsilon) < 1$, then $\delta > 2(L+\varepsilon)$ and as before we obtain (with δ instead of ρ)

$$\lim_{v\to\infty} \left\| \tau_{-a} f - \sum_{n=0}^{v} \frac{1}{n!} \widehat{d}^n f(\cdot)\, a \right\|_{\tilde{N},(s;(r,q)),\delta} = 0,$$

for all $\delta > 0$. Again the convergence follows by the definition of the topologies.

Using Proposition 2.10 it is not difficult to show the next result.

Proposition 2.11. (a) *For $k \in [1,+\infty)$, if $f \in Exp^k_{\tilde{N},(s;(r,q))}(E)$ and $a \in E$, then*

$$\lim_{\lambda\to 0} \lambda^{-1} (\tau_{-\lambda a} f - f) = \widehat{d}^1 f(\cdot)\, a,$$

in the sense of the topology of $Exp^k_{\tilde{N},(s;(r,q))}(E)$.
(b) *For $k \in [1,+\infty]$, if $f \in Exp^k_{\tilde{N},(s;(r,q)),0}(E)$ and $a \in E$, then*

$$\lim_{\lambda\to 0} \lambda^{-1} (\tau_{-\lambda a} f - f) = \widehat{d}^1 f(\cdot)\, a,$$

in the sense of the topology of $Exp^k_{\tilde{N},(s;(r,q)),0}(E)$.

Theorem 2.12. (a) *If $k \in [1,+\infty)$ and $\mathcal{O}$ is a continuous linear mapping from $Exp^k_{\tilde{N},(s;(r,q))}(E)$ into itself, then $\mathcal{O}$ is a convolution operator if, and only if, $\mathcal{O}(\tau_a f) = \tau_a(\mathcal{O}f)$ for all $a \in E$ and $f \in Exp^k_{\tilde{N},(s;(r,q))}(E)$.*
(b) *If $k \in [1,+\infty]$ and $\mathcal{O}$ is a continuous linear mapping from $Exp^k_{\tilde{N},(s;(r,q)),0}(E)$ into itself, then $\mathcal{O}$ is a convolution operator if, and only if, $\mathcal{O}(\tau_a f) = \tau_a(\mathcal{O}f)$ for all $a \in E$ and $f \in Exp^k_{\tilde{N},(s;(r,q)),0}(E)$.*

Proof. We saw that $\mathcal{O}\left(\widehat{d}^n f(\cdot)\, a\right) = \widehat{d}^n (\mathcal{O}f)(\cdot)\, a$ for all $n \in \mathbb{N}$ and $a \in E$. By this fact and Proposition 2.10 we have

$$\mathcal{O}(\tau_{-a} f) = \sum_{n=0}^{\infty} \frac{1}{n!} \mathcal{O}\left(\widehat{d}^n f(\cdot)(a)\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \widehat{d}^n (\mathcal{O}f)(\cdot)\, a = \tau_{-a}(\mathcal{O}f),$$

which implies $\mathcal{O}(\tau_a f) = \tau_a(\mathcal{O}f)$. Now, if we suppose that $\mathcal{O}$ is such that $\mathcal{O}(\tau_a f) = \tau_a(\mathcal{O}f)$ for all $a \in E$, it follows from Proposition 2.11 that

$$\widehat{d}^1 (\mathcal{O}f)(\cdot)\, a = \lim_{\lambda\to 0} \lambda^{-1} (\tau_{-\lambda a}(\mathcal{O}f) - \mathcal{O}f) = \lim_{\lambda\to 0} \lambda^{-1} (\mathcal{O}(\tau_{-\lambda a} f) - \mathcal{O}f)$$
$$= \lim_{\lambda\to 0} \mathcal{O}\left(\lambda^{-1}(\tau_{-\lambda a} f - f)\right) = \mathcal{O}\left(\lim_{\lambda\to 0} \lambda^{-1}(\tau_{-\lambda a} f - f)\right) = \mathcal{O}\left(\widehat{d}^1 f(\cdot)\, a\right).$$

Hence $\mathcal{O}$ is a convolution operator.

Definition 2.13. For $k \in [1,+\infty)$, $T \in \left[Exp^k_{\tilde{N},(s;(r,q))}(E)\right]'$ and $f \in Exp^k_{\tilde{N},(s;(r,q))}(E)$, we define *the convolution product of T and f* by $(T * f)(x) = T(\tau_{-x} f)$, for all $x \in E$.
For $k \in [1,+\infty]$, $T \in \left[Exp^k_{\tilde{N},(s;(r,q)),0}(E)\right]'$ and $f \in Exp^k_{\tilde{N},(s;(r,q)),0}(E)$, we define *the convolution product of T and f* by $(T * f)(x) = T(\tau_{-x} f)$, for all $x \in E$.

In order to prove that $T*$ defines a convolution operator on $Exp^k_{\tilde{N},(s;(r,q))}(E)$, for $k \in [1,+\infty)$, and on $Exp^k_{\tilde{N},(s;(r,q)),0}(E)$, for $k \in [1,+\infty]$, we need two preliminary results. Moreover, we are going to show that all the convolution operators on these spaces are of the form $T*$.

Proposition 2.14. *Let $k \in [1,+\infty]$ and $T \in \left[Exp^k_{\tilde{N},(s;(r,q)),0}(E)\right]'$. Then, for every $P \in \mathcal{P}_{\tilde{N},(s;(r,q))}(^nE)$ with $A \in \mathcal{L}_{\tilde{N}s,(s;(r,q))}(^nE)$ such that $P = \hat{A}$, the polynomial*

$$\begin{aligned} T\left(\widehat{A\cdot^m}\right) : E &\longrightarrow \mathbb{C} \\ y &\longmapsto T\left(A\cdot^m y^{n-m}\right) \end{aligned}$$

belongs to $\mathcal{P}_{\tilde{N},(s;(r,q))}(^{n-m}E)$ for every $m \leq n$ and

$$\left\|T\left(\widehat{A\cdot^m}\right)\right\|_{\tilde{N},(s;(r,q))} \leq C\rho^{-m}\left(\frac{m}{ke}\right)^{\frac{m}{k}}\|P\|_{\tilde{N},(s;(r,q))}, \quad \text{if } k \in [1,+\infty),$$

$$\left\|T\left(\widehat{A\cdot^m}\right)\right\|_{\tilde{N},(s;(r,q))} \leq C\rho^{-m}\|P\|_{\tilde{N},(s;(r,q))}, \quad \text{if } k = +\infty,$$

where $C > 0$ and $\rho > 0$ are such that

$$|T(f)| \leq C\|f\|_{\tilde{N},(s;(r,q)),k,\rho}, \quad \text{if } k \in [1,+\infty),$$

$$|T(f)| \leq Cp^{\infty}_{\tilde{N},(s;(r,q)),\rho}(f), \quad \text{if } k = +\infty,$$

for all $f \in Exp^k_{\tilde{N},(s;(r,q)),0}(E)$.

Proof. First we suppose that $P \in \mathcal{P}_{N,(s;(r,q))}(^nE)$. Then

$$P = \sum_{j=0}^{\infty}\lambda_j\varphi_j^n,$$

where $(\lambda_j)_{j=1}^{\infty} \in \ell_s$ ($\in c_0$, if $s = \infty$) and $(\varphi_j)_{j=1}^{\infty} \in \ell_{m(r';q')}(E')$. Furthermore,

$$T\left(\widehat{A\cdot^m}\right)(y) = T\left(A\cdot^m y^{n-m}\right) = T\left(\sum_{j=0}^{\infty}\lambda_j\varphi_j(\cdot)^m\varphi_j(y)^{n-m}\right) = \sum_{j=0}^{\infty}\lambda_j T\left(\varphi_j^m\right)\varphi_j(y)^{n-m},$$

for all $y \in E$, and

$$\begin{aligned} \left\|\left(\lambda_j T\left(\varphi_j^m\right)\right)_{j=1}^{\infty}\right\|_s &\leq \left\|(\lambda_j)_{j=1}^{\infty}\right\|_s \left\|\left(T\left(\varphi_j^m\right)\right)_{j=1}^{\infty}\right\|_{\infty} \\ &\leq \left\|(\lambda_j)_{j=1}^{\infty}\right\|_s C\rho^{-m}\left(\frac{m}{ke}\right)^{\frac{m}{k}}\left\|\left(\|\varphi_j\|\right)_{j=1}^{\infty}\right\|_{\infty}^m \\ &\leq \left\|(\lambda_j)_{j=1}^{\infty}\right\|_s C\rho^{-m}\left(\frac{m}{ke}\right)^{\frac{m}{k}}\left\|(\varphi_j)_{j=1}^{\infty}\right\|_{m(r';q')}^m. \end{aligned}$$

if $k \in [1,+\infty)$, and

$$\left\|\left(\lambda_j T\left(\varphi_j^m\right)\right)_{j=1}^{\infty}\right\|_s \leq \left\|(\lambda_j)_{j=1}^{\infty}\right\|_s C\rho^{-m}\left\|(\varphi_j)_{j=1}^{\infty}\right\|_{m(r';q')}^m$$

if $k=+\infty$. Therefore $T\left(\widehat{A\cdot^m}\right)\in\mathcal{P}_{N,(s;(r,q))}\left({}^{n-m}E\right)$ and

$$\left\|T\left(\widehat{A\cdot^m}\right)\right\|_{N,(s;(r,q))}\leq C\rho^{-m}\left(\frac{m}{ke}\right)^{\frac{m}{k}}\left\|(\lambda_j)_{j=1}^{\infty}\right\|_s\left\|(\varphi_j)_{j=1}^{\infty}\right\|_{m(r';q')}^n,$$

if $k\in[1,+\infty)$, and

$$\left\|T\left(\widehat{A\cdot^m}\right)\right\|_{N,(s;(r,q))}\leq C\rho^{-m}\left\|(\lambda_j)_{j=1}^{\infty}\right\|_s\left\|(\varphi_j)_{j=1}^{\infty}\right\|_{m(r';q')}^n,$$

if $k=+\infty$. Thus,

$$\left\|T\left(\widehat{A\cdot^m}\right)\right\|_{\tilde{N},(s;(r,q))}\leq C\rho^{-m}\left(\frac{m}{ke}\right)^{\frac{m}{k}}\|P\|_{N,(s;(r,q))},$$

if $k\in[1,+\infty)$, and

$$\left\|T\left(\widehat{A\cdot^m}\right)\right\|_{\tilde{N},(s;(r,q))}\leq C\rho^{-m}\|P\|_{N,(s;(r,q))},$$

if $k=+\infty$. Now, if U denotes the closed unit ball of $\left(\mathcal{P}_{N,(s;(r,q))}\left({}^nE\right),\|\cdot\|_{N,(s;(r,q))}\right)$, we can act as Matos [21, Chapter 8] to obtain

$$\left\|T\left(\widehat{A\cdot^m}\right)\right\|_{\tilde{N},(s;(r,q))}\leq C\rho^{-m}\left(\frac{m}{ke}\right)^{\frac{m}{k}},$$

if $k\in[1,+\infty)$, and

$$\left\|T\left(\widehat{A\cdot^m}\right)\right\|_{\tilde{N},(s;(r,q))}\leq C\rho^{-m},$$

if $k=+\infty$, first by considering P in the absolutely convex hull V of U and then P in the weak closure U^{oo} of V. From these inequalities, we get

$$\left\|T\left(\widehat{A\cdot^m}\right)\right\|_{\tilde{N},(s;(r,q))}\leq C\rho^{-m}\left(\frac{m}{ke}\right)^{\frac{m}{k}}\|P\|_{\tilde{N},(s;(r,q))},$$

if $k\in[1,+\infty)$, and

$$\left\|T\left(\widehat{A\cdot^m}\right)\right\|_{\tilde{N},(s;(r,q))}\leq C\rho^{-m}\|P\|_{\tilde{N},(s;(r,q))},$$

if $k=+\infty$, for all $P\in\mathcal{P}_{N,(s;(r,q))}\left({}^nE\right)$. Now the result follows by completion.

Proposition 2.15. *If $k\in[1,+\infty)$ and $T\in\left[Exp_{\tilde{N},(s;(r,q))}^k(E)\right]'$. Then, for every $P\in\mathcal{P}_{\tilde{N},(s;(r,q))}\left({}^nE\right)$ with $A\in\mathcal{L}_{\tilde{N}s,(s;(r,q))}\left({}^nE\right)$ such that $P=\hat{A}$, the polynomial*

$$\begin{aligned}T\left(\widehat{A\cdot^m}\right):E&\longrightarrow\mathbb{C}\\ y&\longmapsto T\left(A\cdot^m y^{n-m}\right)\end{aligned}$$

belongs to $\mathcal{P}_{\tilde{N},(s;(r,q))}\left({}^{n-m}E\right)$, for every $m\leq n$ and

$$\left\|T\left(\widehat{A\cdot^m}\right)\right\|_{\tilde{N},(s;(r,q))}\leq C(\rho)\,\rho^{-m}\left(\frac{m}{ke}\right)^{\frac{m}{k}}\|P\|_{\tilde{N},(s;(r,q))},$$

where the constant $C(\rho)>0$, $\rho>0$, is such that

$$|T(f)|\leq C(\rho)\,\|f\|_{\tilde{N},(s;(r,q)),k,\rho},$$

for all $f\in Exp_{\tilde{N},(s;(r,q))}^k(E)$.

Proof. It is similar to the proof of Proposition 2.14.

Theorem 2.16. (a) *If* $k \in [1, +\infty]$, $T \in \left[Exp^k_{\tilde{N},(s;(r,q)),0}(E)\right]'$ *and* $f \in Exp^k_{\tilde{N},(s;(r,q)),0}(E)$, *then* $T * f \in Exp^k_{\tilde{N},(s;(r,q)),0}(E)$ *and* $T* \in \mathcal{A}^k_{\tilde{N},(s;(r,q)),0}$.
(b) *If* $k \in [1, +\infty)$, $T \in \left[Exp^k_{\tilde{N},(s;(r,q))}(E)\right]'$ *and* $f \in Exp^k_{\tilde{N},(s;(r,q))}(E)$, *then* $T * f \in Exp^k_{\tilde{N},(s;(r,q))}(E)$ *and* $T* \in \mathcal{A}^k_{\tilde{N},(s;(r,q))}$.

Proof. The linearity of $T*$ is clear. By Propositions 2.7 and 2.10 we have in either case that

$$(T * f)(x) = T(\tau_{-x} f) = \sum_{n=0}^{\infty} \frac{1}{n!} T\left(\hat{d}^n f(\cdot)(x)\right) = \sum_{n=0}^{\infty} \frac{1}{n!} \sum_{j=0}^{\infty} \frac{1}{j!} T\left(\widehat{d^{j+n} f(0) \cdot^j (x)}\right).$$

(a) By Proposition 2.14 we have that $T\left(\widehat{d^{j+n} f(0) \cdot^j}\right) \in \mathcal{P}_{\tilde{N},(s;(r,q))}({}^n E)$ and

$$\left\| T\left(\widehat{d^{j+n} f(0) \cdot^j}\right) \right\|_{\tilde{N},(s;(r,q))} \leq C\rho^{-j} \left(\frac{j}{ke}\right)^{\frac{j}{k}} \left\| \hat{d}^{j+n} f(0) \right\|_{\tilde{N},(s;(r,q))},$$

for $k \in [1, +\infty)$, and

$$\left\| T\left(\widehat{d^{j+n} f(0) \cdot^j}\right) \right\|_{\tilde{N},(s;(r,q))} \leq C\rho^{-j} \left\| \hat{d}^{j+n} f(0) \right\|_{\tilde{N},(s;(r,q))},$$

for $k = +\infty$, where C and ρ are as in 2.14. If $k \in [1, +\infty)$ and $0 < \rho' < \rho$, then

$$\begin{aligned}
\sum_{j=0}^{\infty} \frac{1}{j!} \left\| T\left(\widehat{d^{j+n} f(0) \cdot^j}\right) \right\|_{\tilde{N},(s;(r,q))} &\leq C \sum_{j=0}^{\infty} \frac{1}{j!} (\rho')^{-j} \left(\frac{j}{ke}\right)^{\frac{j}{k}} \left\| \hat{d}^{j+n} f(0) \right\|_{\tilde{N},(s;(r,q))} \\
&\leq (\rho')^n Cn! \left(\frac{ke}{n}\right)^{\frac{n}{k}} \sum_{j=0}^{\infty} 2^{j+n} (\rho')^{-(j+n)} \left(\frac{j+n}{ke}\right)^{\frac{j+n}{k}} \left\| \frac{\hat{d}^{j+n} f(0)}{(j+n)!} \right\|_{\tilde{N},(s;(r,q))} \\
&\leq (\rho')^n Cn! \left(\frac{ke}{n}\right)^{\frac{n}{k}} \|f\|_{\tilde{N},(s;(r,q)),k,\frac{\rho'}{2}},
\end{aligned}$$

and this implies

$$P_n = \sum_{j=0}^{\infty} \frac{1}{j!} T\left(\widehat{d^{j+n} f(0) \cdot^j}\right) \in \mathcal{P}_{\tilde{N},(s;(r,q))}({}^n E),$$

for each $n \in \mathbb{N}$ and

$$\|P_n\|_{\tilde{N},(s;(r,q))} \leq (\rho')^n Cn! \left(\frac{ke}{n}\right)^{\frac{n}{k}} \|f\|_{\tilde{N},(s;(r,q)),k,\frac{\rho'}{2}}.$$

Hence

$$\limsup_{n\to\infty} \left(\frac{n}{ke}\right)^{\frac{1}{k}} \left\| \frac{P_n}{n!} \right\|^{\frac{1}{n}}_{\tilde{N},(s;(r,q))} \leq \limsup_{n\to\infty} C^{\frac{1}{n}} \rho' \|f\|^{\frac{1}{n}}_{\tilde{N},(s;(r,q)),k,\frac{\rho'}{2}} = \rho',$$

for all $0 < \rho' < \rho$. Thus $T * f \in Exp^k_{\tilde{N},(s;(r,q)),0}(E)$. If $\rho_1 > 0$, for $0 < \rho' < \rho$ and $\rho' < \rho_1$, we have

$$\|T * f\|_{\tilde{N},(s;(r,q)),k,\rho_1} = \sum_{n=0}^{\infty} \frac{1}{n!} \left(\frac{n}{ke}\right)^{\frac{1}{k}} \rho_1^{-n} \|P_n\|_{\tilde{N},(s;(r,q))}$$
$$\leq C \|f\|_{\tilde{N},(s;(r,q)),k,\frac{\rho'}{2}} \sum_{n=0}^{\infty} \left(\frac{\rho'}{\rho_1}\right)^n = C \left(1 - \frac{\rho'}{\rho_1}\right)^{-1} \|f\|_{\tilde{N},(s;(r,q)),k,\frac{\rho'}{2}}.$$

Therefore $T*$ is continuous. The case $k = +\infty$ is done in the same way with simpler calculations, since the terms $\left(\frac{n}{ke}\right)^{\frac{n}{k}}$, $n \in \mathbb{N}$, do not appear.
(b) By Proposition 2.15 we obtain that for every $\rho > 0$, there is $C(\rho) > 0$ such that

$$\sum_{j=0}^{\infty} \frac{1}{j!} \left\| T\left(\widehat{d^{j+n} f(0)\, \cdot^j}\right) \right\|_{\tilde{N},(s;(r,q))} \leq C(\rho) \sum_{j=0}^{\infty} \frac{1}{j!} \rho^{-j} \left(\frac{j}{ke}\right)^{\frac{j}{k}} \left\| \widehat{d^{j+n}} f(0) \right\|_{\tilde{N},(s;(r,q))}$$
$$= \rho^n C(\rho)\, n! \sum_{j=0}^{\infty} \frac{(j+n)!}{j!n!} \left(\frac{j}{j+n}\right)^{\frac{j}{k}} \left(\frac{ke}{j+n}\right)^{\frac{n}{k}} \left(\frac{j+n}{ke}\right)^{\frac{j+n}{k}} \rho^{-(j+n)} \left\| \frac{\widehat{d^{j+n}} f(0)}{(j+n)!} \right\|_{\tilde{N},(s;(r,q))}. \tag{9}$$

Since

$$\limsup_{j\to\infty} \binom{j+n}{n}^{\frac{1}{j+n}} = \limsup_{j\to\infty} \left(\frac{1}{n!}\right)^{\frac{1}{j+n}} (j+n)^{\frac{1}{j+n}} \ldots (j+1)^{\frac{1}{j+n}} = 1,$$

it follows that for every $\varepsilon > 0$ there is $D(\varepsilon) > 0$ such that

$$\binom{j+n}{n} \leq D(\varepsilon)(1+\varepsilon)^{j+n},$$

for all $j \in \mathbb{N}$. Hence

$$\sum_{j=0}^{\infty} \frac{1}{j!} \left\| T\left(\widehat{d^{j+n} f(0)\, \cdot^j}\right) \right\|_{\tilde{N},(s;(r,q))} \leq C(\rho) D(\varepsilon) \rho^n n! \left(\frac{ke}{n}\right)^{\frac{n}{k}} \|f\|_{\tilde{N},(s;(r,q)),k,\frac{\rho}{1+\varepsilon}}$$

and

$$\|P_n\|_{\tilde{N},(s;(r,q))} \leq C(\rho) D(\varepsilon) \rho^n n! \left(\frac{ke}{n}\right)^{\frac{n}{k}} \|f\|_{\tilde{N},(s;(r,q)),k,\frac{\rho}{1+\varepsilon}},$$

for all $\rho > 0$ and $\varepsilon > 0$, where

$$P_n = \sum_{j=0}^{\infty} \frac{1}{j!} T\left(\widehat{d^{j+n} f(0)\, \cdot^j}\right).$$

Consequently

$$\limsup_{n\to\infty} \left(\frac{n}{ke}\right)^{\frac{1}{k}} \left\|\frac{P_n}{n!}\right\|_{\tilde{N},(s;(r,q))}^{\frac{1}{n}} \leq \limsup_{n\to\infty} (C(\rho))^{\frac{1}{n}} \rho \|f\|^{\frac{1}{n}}_{\tilde{N},(s;(r,q)),k,\frac{\rho}{1+\varepsilon}} = \rho,$$

if ρ and ε are chosen so that $\|f\|_{\tilde{N},(s;(r,q)),k,\frac{\rho}{1+\varepsilon}} < +\infty$. Hence $T * f \in Exp^k_{\tilde{N},(s;(r,q))}(E)$. For $\rho_1 > 0$, we get

$$\|T * f\|_{\tilde{N},(s;(r,q)),k,\rho_1} = \sum_{n=0}^{\infty} \frac{1}{n!}\left(\frac{n}{ke}\right)^{\frac{1}{k}} \rho_1^{-n} \|P_n\|_{\tilde{N},(s;(r,q))}$$
$$\leq \sum_{n=0}^{\infty} C(\rho) D(\varepsilon) \rho^n n! \left(\frac{ke}{n}\right)^{\frac{n}{k}} \frac{1}{n!}\left(\frac{n}{ke}\right)^{\frac{n}{k}} \rho_1^{-n} \|f\|_{\tilde{N},(s;(r,q)),k,\frac{\rho}{1+\varepsilon}}$$
$$= C(\rho) D(\varepsilon) \sum_{n=0}^{\infty} \left(\frac{\rho}{\rho_1}\right)^n \|f\|_{\tilde{N},(s;(r,q)),k,\frac{\rho}{1+\varepsilon}} = C(\rho) D(\varepsilon) \left(1 - \frac{\rho}{\rho_1}\right)^{-1} \|f\|_{\tilde{N},(s;(r,q)),k,\frac{\rho}{1+\varepsilon}}, \tag{10}$$

if ρ and ε are chosen so that $\|f\|_{\tilde{N},(s;(r,q)),k,\frac{\rho}{1+\varepsilon}} < +\infty$. This gives the continuity of $T*$. The fact that $T*$ commutes with translations is clear.

Definition 2.17. *(a)* For $k \in [1, +\infty]$, we define

$$\gamma^k_{\tilde{N},(s;(r,q)),0} \colon \mathcal{A}^k_{\tilde{N},(s;(r,q)),0} \longrightarrow \left[Exp^k_{\tilde{N},(s;(r,q)),0}(E)\right]'$$

by $\gamma^k_{\tilde{N},(s;(r,q)),0}(\mathcal{O})(f) = (\mathcal{O}f)(0)$, for $f \in Exp^k_{\tilde{N},(s;(r,q)),0}(E)$ and $\mathcal{O} \in \mathcal{A}^k_{\tilde{N},(s;(r,q)),0}$.
(b) For $k \in [1, +\infty)$, we define

$$\gamma^k_{\tilde{N},(s;(r,q))} \colon \mathcal{A}^k_{\tilde{N},(s;(r,q))} \longrightarrow \left[Exp^k_{\tilde{N},(s;(r,q))}(E)\right]'$$

by $\gamma^k_{\tilde{N},(s;(r,q))}(\mathcal{O})(f) = (\mathcal{O}f)(0)$, for $f \in Exp^k_{\tilde{N},(s;(r,q))}(E)$ and $\mathcal{O} \in \mathcal{A}^k_{\tilde{N},(s;(r,q))}$.

Theorem 2.18. *The mappings $\gamma^k_{\tilde{N},(s;(r,q)),0}$ (for $k \in [1, +\infty]$) and $\gamma^k_{\tilde{N},(s;(r,q))}$ (for $k \in [1, +\infty)$) are linear bijections.*

Proof. It is enough to notice that the mappings

$$\Gamma^k_{\tilde{N},(s;(r,q)),0} \colon \left[Exp^k_{\tilde{N},(s;(r,q)),0}(E)\right]' \longrightarrow \mathcal{A}^k_{\tilde{N},(s;(r,q)),0}$$

given by $\Gamma^k_{\tilde{N},(s;(r,q)),0}(T)(f) = T * f$, for $T \in \left[Exp^k_{\tilde{N},(s;(r,q)),0}(E)\right]'$, $f \in Exp^k_{\tilde{N},(s;(r,q)),0}(E)$ and $k \in [1, +\infty]$, and

$$\Gamma^k_{\tilde{N},(s;(r,q))} \colon \left[Exp^k_{\tilde{N},(s;(r,q))}(E)\right]' \longrightarrow \mathcal{A}^k_{\tilde{N},(s;(r,q))}$$

given by $\Gamma^k_{\tilde{N},(s;(r,q))}(T)(f) = T * f$, for $T \in \left[Exp^k_{\tilde{N},(s;(r,q))}(E)\right]'$, $f \in Exp^k_{\tilde{N},(s;(r,q))}(E)$ and $k \in [1, +\infty)$, are the inverse mappings of $\gamma^k_{\tilde{N},(s;(r,q)),0}$ and $\gamma^k_{\tilde{N},(s;(r,q))}$, respectively.

Definition 2.19. For $k \in [1, +\infty]$ and $T_1, T_2 \in \left[Exp^k_{\tilde{N},(s;(r,q)),0}(E)\right]'$ we define *the convolution product* $T_1 * T_2 \in \left[Exp^k_{\tilde{N},(s;(r,q)),0}(E)\right]'$ by

$$T_1 * T_2 = \gamma^k_{\tilde{N},(s;(r,q)),0}(\mathcal{O}_1 \circ \mathcal{O}_2) \in \left[Exp^k_{\tilde{N},(s;(r,q)),0}(E)\right]',$$

where $\mathcal{O}_1 = T_1 *$ and $\mathcal{O}_2 = T_2 *$.
For $k \in [1, +\infty)$ and $T_1, T_2 \in \left[Exp^k_{\tilde{N},(s;(r,q))}(E) \right]'$ we define *the convolution product* $T_1 * T_2 \in \left[Exp^k_{\tilde{N},(s;(r,q))}(E) \right]'$ by

$$T_1 * T_2 = \gamma^k_{\tilde{N},(s;(r,q))} (\mathcal{O}_1 \circ \mathcal{O}_2) \in \left[Exp^k_{\tilde{N},(s;(r,q))}(E) \right]',$$

where $\mathcal{O}_1 = T_1 *$ and $\mathcal{O}_2 = T_2 *$.

It is easy to see that $\gamma^k_{\tilde{N},(s;(r,q)),0}$ and $\gamma^k_{\tilde{N},(s;(r,q))}$ preserve these products, that is, $\gamma^k_{\tilde{N},(s;(r,q)),0} (\mathcal{O}_1 \circ \mathcal{O}_2) = \left(\gamma^k_{\tilde{N},(s;(r,q)),0} \mathcal{O}_1 \right) * \left(\gamma^k_{\tilde{N},(s;(r,q)),0} \mathcal{O}_2 \right)$ and $\gamma^k_{\tilde{N},(s;(r,q))} (\mathcal{O}_1 \circ \mathcal{O}_2) = \left(\gamma^k_{\tilde{N},(s;(r,q))} \mathcal{O}_1 \right) * \left(\gamma^k_{\tilde{N},(s;(r,q))} \mathcal{O}_2 \right)$, and $\delta(f) = f(0)$ is a unit element.

Proposition 2.20. *The spaces* $\left[Exp^k_{\tilde{N},(s;(r,q)),0}(E) \right]'$ *and* $\left[Exp^k_{\tilde{N},(s;(r,q))}(E) \right]'$ *are algebras with unit element* δ.

Proposition 2.21. (a) *For* $k \in [1, +\infty]$, *the Fourier-Borel transform* F *is an algebra isomorphism between* $\left[Exp^k_{\tilde{N},(s;(r,q)),0}(E) \right]'$ *and* $Exp^{k'}_{(s',m(r';q'))}(E')$.
(b) *For* $k \in [1, +\infty)$, *the Fourier-Borel transform* F *is an algebra isomorphism between* $\left[Exp^k_{\tilde{N},(s;(r,q))}(E) \right]'$ *and* $Exp^{k'}_{(s',m(r';q')),0}(E')$.

Proof. It was proved in [11] that F is an algebraic isomorphism between these spaces. Since it is easy to show that $F(T_1 * T_2) = (FT_1)(FT_2)$, in both cases, the result follows.

Remark 2.22. It is not difficult to prove that the following inclusions are continuous for $k \in [1, +\infty]$ and $0 < A < B < +\infty$:

$$\begin{aligned} Exp^k_{(s,m(r;q)),0}(E) &\subset Exp^k_{(s,m(r;q)),A}(E) \subset Exp^k_{(s,m(r;q)),0,A}(E) \\ &\subset Exp^k_{(s,m(r;q)),B}(E) \subset Exp^k_{(s,m(r;q))}(E) \end{aligned}$$

and

$$\begin{aligned} Exp^k_{\tilde{N},(s;(r,q)),0}(E) &\subset Exp^k_{\tilde{N},(s;(r,q)),A}(E) \subset Exp^k_{\tilde{N},(s;(r,q)),0,A}(E) \\ &\subset Exp^k_{\tilde{N},(s;(r,q)),B}(E) \subset Exp^k_{\tilde{N},(s;(r,q))}(E). \end{aligned}$$

Thus if $T \in \left[Exp^k_{\tilde{N},(s;(r,q))}(E) \right]'$, then $T \in \left[Exp^k_{\tilde{N},(s;(r,q)),A}(E) \right]'$ and $T \in \left[Exp^k_{\tilde{N},(s;(r,q)),0,B}(E) \right]'$, for every $A \in (0, +\infty]$ and $B \in [0, +\infty)$ (we are considering the restriction of T to the corresponding space). Hence, if $T \in \left[Exp^\infty_{\tilde{N},(s;(r,q))}(E) \right]'$ we may consider $T * P \in \left[Exp^\infty_{\tilde{N},(s;(r,q)),0}(E) \right]'$ for every $P \in \mathcal{P}_{\tilde{N},(s;(r,q))}(^nE)$ (see Theorem 2.16 (a))

Definition 2.23. The functional $T \in \left[Exp^{k}_{\tilde{N},(s;(r,q)),A}(E)\right]'$, with $A \in (0,+\infty]$ and $k \in [1,+\infty]$, is said to be of *type zero* if it is also in $\left[Exp^{k}_{\tilde{N},(s;(r,q))}(E)\right]'$ or, equivalently, if $FT \in Exp^{k'}_{(s',m(r';q')),0}(E')$.
The functional $T \in \left[Exp^{k}_{\tilde{N},(s;(r,q)),0,B}(E)\right]'$, with $B \in [0,+\infty)$ and $k \in [1,+\infty]$, is said to be of *type zero* if it is also in $\left[Exp^{k}_{\tilde{N},(s;(r,q))}(E)\right]'$ or, equivalently, if $FT \in Exp^{k'}_{(s',m(r';q')),0}(E')$.

Proposition 2.24. *If $P \in \mathcal{P}_{\tilde{N},(s;(r,q))}({}^{n}E)$ and $T \in \left[Exp^{\infty}_{\tilde{N},(s;(r,q))}(E)\right]'$, then for every $\varepsilon > 0$ and $\rho > 0$, with $\rho > \varepsilon$, there is a constant $C(\rho,\varepsilon) \geq 0$, independent of n, such that*

$$p^{\infty}_{\tilde{N},(s;(r,q)),\rho}(T * P) \leq C(\rho,\varepsilon)(\rho-\varepsilon)^{-n}\|P\|_{\tilde{N},(s;(r,q))}.$$

Proof. First we consider $P = \varphi^n$, with $\varphi \in E'$. Thus we have

$$T * P = \sum_{j=0}^{n}\binom{n}{j}T\left(\varphi(\cdot)^{n-j}\right)\varphi^{j}$$

and

$$p^{\infty}_{\tilde{N},(s;(r,q)),\rho}(T * P) = \sum_{j=0}^{n}\binom{n}{j}\left|T\left(\varphi(\cdot)^{n-j}\right)\right|\|\varphi\|^{j}\rho^{-j}. \tag{11}$$

Since $T \in \left[Exp^{\infty}_{\tilde{N},(s;(r,q))}(E)\right]'$, it follows that $FT \in Exp^{1}_{(s',m(r';q')),0}(E')$ and

$$\limsup_{j\to\infty}\left\|\widehat{d^{j}}FT(0)\right\|^{\frac{1}{j}}_{(s',m(r';q'))} = 0.$$

Since

$$\sup_{\phi\neq 0}\frac{|T(\phi^{j})|}{\|\phi\|^{j}} = \left\|\widehat{d^{j}}FT(0)\right\| \leq \left\|\widehat{d^{j}}FT(0)\right\|_{(s',m(r';q'))},$$

for each $\delta > 0$, there is $\alpha(\delta) > 0$ such that

$$\frac{|T(\varphi^{j})|}{\|\varphi\|^{j}} \leq \left\|\widehat{d^{j}}FT(0)\right\|_{(s',m(r';q'))} \leq \alpha(\delta)\delta^{j},$$

for all $j \in \mathbb{N}$, then

$$\left|T\left(\varphi^{j}\right)\right| \leq \alpha(\delta)\delta^{j}\|\varphi\|^{j}, \tag{12}$$

for all $j \in \mathbb{N}$. Now from (11) and (12) (using $\delta = \varepsilon$) we get

$$\begin{aligned} p^{\infty}_{\tilde{N},(s;(r,q)),\rho}(T * P) &\leq \sum_{j=0}^{n}\binom{n}{j}\alpha(\varepsilon)\varepsilon^{n-j}\|\varphi\|^{n-j}\|\varphi\|^{j}\rho^{-j} \\ &= \alpha(\varepsilon)\|\varphi\|^{n}\sum_{j=0}^{n}\binom{n}{j}\varepsilon^{n-j}\rho^{-j} = \alpha(\varepsilon)\|\varphi\|^{n}\left(\rho^{-1}+\varepsilon\right)^{n}. \end{aligned} \tag{13}$$

Let $0 < \varepsilon' < \min\left(\rho, \frac{\varepsilon}{\rho(\rho-\varepsilon)}\right)$. Then $\varepsilon' < \rho$, $\varepsilon' < \frac{\varepsilon}{\rho(\rho-\varepsilon)}$ and $\left(\rho^{-1}+\varepsilon'\right)^n < (\rho-\varepsilon)^{-n}$. Therefore, from (13) (using $\delta = \varepsilon'$) we get

$$p^{\infty}_{\tilde{N},(s;(r,q)),\rho}\left(T * P\right) \leq \alpha\left(\varepsilon'\right)\|\varphi\|^n\left(\rho^{-1}+\varepsilon'\right)^n \leq \alpha\left(\varepsilon'\right)(\rho-\varepsilon)^{-n}\|\varphi\|^n,$$

and since ε' depends only on ρ and ε, we may write $\alpha\left(\varepsilon'\right) = C\left(\rho,\varepsilon\right)$. Therefore the result holds for $P \in \mathcal{P}_f\left({}^nE\right)$. Since $\mathcal{P}_f\left({}^nE\right)$ is dense in $\mathcal{P}_{\tilde{N},(s;(r,q))}\left({}^nE\right)$, the result holds true for $P \in \mathcal{P}_{\tilde{N},(s;(r,q))}\left({}^nE\right)$.

Theorem 2.25. *Let $k \in [1,+\infty]$, $T \in \left[Exp^k_{\tilde{N},(s;(r,q))}\left(E\right)\right]'$ and f in either $Exp^k_{\tilde{N},(s;(r,q)),A}\left(E\right)$ or $Exp^k_{\tilde{N},(s;(r,q)),0,B}\left(E\right)$, with $A \in (0,+\infty]$ and $B \in [0,+\infty)$. If*

$$T * f = \sum_{n=0}^{\infty} T * \left(\frac{1}{n!}\widehat{d}^n f\left(0\right)\right),$$

then we get

$$T * f \in Exp^k_{\tilde{N},(s;(r,q)),A}\left(E\right) \qquad \text{if} \qquad f \in Exp^k_{\tilde{N},(s;(r,q)),A}\left(E\right)$$

and

$$T * f \in Exp^k_{\tilde{N},(s;(r,q)),0,B}\left(E\right) \qquad \text{if} \qquad f \in Exp^k_{\tilde{N},(s;(r,q)),0,B}\left(E\right).$$

Moreover $T*$ defines a convolution operator on $Exp^k_{\tilde{N},(s;(r,q)),A}\left(E\right)$ and $Exp^k_{\tilde{N},(s;(r,q)),0,B}\left(E\right)$, respectively.

Proof. First we suppose that $k \in [1,+\infty)$. If $f \in Exp^k_{\tilde{N},(s;(r,q)),A}\left(E\right)$, then there is $\rho < A$ such that $\|f\|_{\tilde{N},(s;(r,q)),k,\rho} < +\infty$ and by Remark 2.22 we have $f \in Exp^k_{\tilde{N},(s;(r,q))}\left(E\right)$. Let $\varepsilon > 0$ be such that $\rho\left(1+\varepsilon\right) < A$ and $\rho\left(1+\varepsilon\right) < \rho_1 < A$. Then it follows as in (10) (see the proof of Theorem 2.16(b)) that

$$\begin{aligned}\|T * f\|_{\tilde{N},(s;(r,q)),k,\rho_1} &\leq C\left(\rho\left(1+\varepsilon\right)\right)D\left(\varepsilon\right)\left(1-\frac{\rho\left(1+\varepsilon\right)}{\rho_1}\right)^{-1}\|f\|_{\tilde{N},(s;(r,q)),k,\frac{\rho(1+\varepsilon)}{1+\varepsilon}} \\ &= C\left(\rho\left(1+\varepsilon\right)\right)D\left(\varepsilon\right)\left(1-\frac{\rho\left(1+\varepsilon\right)}{\rho_1}\right)^{-1}\|f\|_{\tilde{N},(s;(r,q)),k,\rho} < +\infty. \end{aligned} \tag{14}$$

Then we get $T * f \in Exp^k_{\tilde{N},(s;(r,q)),A}\left(E\right)$.
If $f \in Exp^k_{\tilde{N},(s;(r,q)),0,B}\left(E\right)$, then $\|f\|_{\tilde{N},(s;(r,q)),k,\rho} < +\infty$, for every $\rho > B$. For $\rho > B$, let $\varepsilon > 0$ such that $\frac{\rho}{(1+\varepsilon)^2} > B$, then $\rho > \frac{\rho}{1+\varepsilon} > \frac{\rho}{(1+\varepsilon)^2}$ and we obtain

$$\|T * f\|_{\tilde{N},(s;(r,q)),k,\rho} \leq C\left(\frac{\rho}{1+\varepsilon}\right)D\left(\varepsilon\right)\left(1-\frac{\rho}{\rho\left(1+\varepsilon\right)}\right)^{-1}\|f\|_{\tilde{N},(s;(r,q)),k,\frac{\rho}{(1+\varepsilon)^2}} < +\infty \tag{15}$$

as before. Hence $T * f \in Exp^k_{\tilde{N},(s;(r,q)),0,B}\left(E\right)$.
Now we have to prove that $T*$ is a convolution operator on $Exp^k_{\tilde{N},(s;(r,q)),A}\left(E\right)$

and $Exp^{k}_{\tilde{N},(s;(r,q)),0,B}(E)$. The linearity is clear. For $T*$ on $Exp^{k}_{\tilde{N},(s;(r,q)),A}(E)$, from the properties of the inductive limit topology it follows that $T*$ is continuous if, and only if, $(T*)\circ i\rho$ is continuous for all $\rho \in (0, A)$.
Let $p\colon Exp^{k}_{\tilde{N},(s;(r,q)),A}(E) \longrightarrow \mathbb{R}$ be a continuous seminorm. Then there is $\alpha(\rho_1) > 0$ such that $p(f) \leq \alpha(\rho_1)\|f\|_{\tilde{N},(s;(r,q)),k,\rho_1}$ for all $f \in B^{k}_{\tilde{N},(s;(r,q)),\rho_1}(E)$ (ρ_1 as in (14)). Thus

$$p(T*f) \leq \alpha(\rho_1)\|T*f\|_{\tilde{N},(s;(r,q)),k,\rho_1} \leq \alpha(\rho_1)K(\rho,\rho_1,\varepsilon)\|f\|_{\tilde{N},(s;(r,q)),k,\rho},$$

for all $f \in B^{k}_{\tilde{N},(s;(r,q)),\rho}(E) \subset B^{k}_{\tilde{N},(s;(r,q)),\rho_1}(E)$, where

$$K(\rho,\rho_1,\varepsilon) = C(\rho(1+\varepsilon))D(\varepsilon)\left(1-\frac{\rho(1+\varepsilon)}{\rho_1}\right)^{-1}.$$

Therefore $T*$ is continuous.
On the other hand, let $T*$ on $Exp^{k}_{\tilde{N},(s;(r,q)),0,B}(E)$. Then the continuity of $T*$ follows from (15), since the topology of $Exp^{k}_{\tilde{N},(s;(r,q)),0,B}(E)$ is defined by the family $\|\cdot\|_{\tilde{N},(s;(r,q)),k,\rho}$, $\rho > B$.
Now, in any of the cases above it is possible to show that the mapping $f \longmapsto d^1f(\cdot)x$ is continuous for any $x \in E$, and $d^1(T*P)(\cdot)x = T*\left(d^1P(\cdot)x\right)$ for all $P \in \mathcal{P}_{\tilde{N},(s;(r,q))}({}^nE)$, $n \in \mathbb{N}$. Thus these two facts imply that $d^1(T*f)(\cdot)x = T*\left(d^1f(\cdot)x\right)$ for all $f \in Exp^{k}_{\tilde{N},(s;(r,q)),A}(E)$ in the first case and for all $f \in Exp^{k}_{\tilde{N},(s;(r,q)),0,B}(E)$ in the second case.
Now we suppose that $k = +\infty$. From Proposition 2.24 we get

$$\sum_{n=0}^{\infty} p^{\infty}_{\tilde{N},(s;(r,q)),\rho}\left(T*\left(\frac{1}{n!}\widehat{d}^nf(0)\right)\right) \leq C(\rho,\varepsilon)\sum_{n=0}^{\infty}(\rho-\varepsilon)^{-n}\left\|\frac{1}{n!}\widehat{d}^nf(0)\right\|_{\tilde{N},(s;(r,q))},$$

for each $\varepsilon > 0$ and $\rho > \varepsilon$. If $f \in Exp^{\infty}_{\tilde{N},(s;(r,q)),0,B}(E)$, let $\rho > B$ and $\varepsilon > 0$ be such that $\rho - \varepsilon > B$. Thus,

$$\sum_{n=0}^{\infty} p^{\infty}_{\tilde{N},(s;(r,q)),\rho}\left(T*\left(\frac{1}{n!}\widehat{d}^nf(0)\right)\right) \leq C(\rho,\varepsilon)\,p^{\infty}_{\tilde{N},(s;(r,q)),\rho-\varepsilon}(f) < +\infty,$$

and since for each $\rho > B$ we have $p^{\infty}_{\tilde{N},(s;(r,q)),\rho}(T*f) < +\infty$, it follows that

$$T*f = \sum_{n=0}^{\infty} T*\left(\frac{1}{n!}\widehat{d}^nf(0)\right)$$

converges in the topology of $Exp^{\infty}_{\tilde{N},(s;(r,q)),0,B}(E)$. The continuity of $T*$ follows from

$$p^{\infty}_{\tilde{N},(s;(r,q)),\rho}(T*f) \leq C(\rho,\varepsilon)\,p^{\infty}_{\tilde{N},(s;(r,q)),\rho-\varepsilon}(f)$$

and the linearity is obvious.
If $f \in Exp^{\infty}_{\tilde{N},(s;(r,q)),A}(E)$, then there is $\rho < A$ such that $f \in \mathcal{H}^{\infty}_{\tilde{N},(s;(r,q))}\left(B_{\frac{1}{\rho}}(0)\right)$.

Let $\varepsilon > 0$ be such that $\rho + 2\varepsilon < A$, then $\rho + \varepsilon < A$ and

$$\sum_{n=0}^{\infty} p^{\infty}_{\tilde{N},(s;(r,q)),\rho+\varepsilon}\left(T * \left(\frac{1}{n!}\widehat{d}^n f(0)\right)\right) \leq C'(\rho,\varepsilon)\sum_{n=0}^{\infty}\rho^{-n}\left\|\frac{1}{n!}\widehat{d}^n f(0)\right\|_{\tilde{N},(s;(r,q))}$$
$$= C'(\rho,\varepsilon)\, p^{\infty}_{\tilde{N},(s;(r,q)),\rho}(f) < +\infty.$$

By $C'(\rho,\varepsilon)$ we mean $C(\rho+\varepsilon,\varepsilon)$. Therefore $T * f \in Exp^{\infty}_{\tilde{N},(s;(r,q)),\rho+\varepsilon}(E)$ and this implies that $T * f \in Exp^{\infty}_{\tilde{N},(s;(r,q)),A}(E)$.
Now we have to prove that $T*$ is continuous. Let $q\colon Exp^{\infty}_{\tilde{N},(s;(r,q)),A}(E) \longrightarrow \mathbb{R}$ be a continuous seminorm. Then there is $M > 0$ such that

$$q(g) \leq M p^{\infty}_{\tilde{N},(s;(r,q)),\rho+2\varepsilon}(g) \text{ for all } g \in \mathcal{H}^{\infty}_{\tilde{N},(s;(r,q))}\left(B_{\frac{1}{\rho+2\varepsilon}}(0)\right).$$

Consequently

$$q(T * f) \leq M p^{\infty}_{\tilde{N},(s;(r,q)),\rho+2\varepsilon}(T * f) \leq C'(\rho,\varepsilon)\, M p^{\infty}_{\tilde{N},(s;(r,q)),\rho}(f),$$

for every $f \in \mathcal{H}^{\infty}_{\tilde{N},(s;(r,q))}\left(B_{\frac{1}{\rho}}(0)\right) \subset \mathcal{H}^{\infty}_{\tilde{N},(s;(r,q))}\left(B_{\frac{1}{\rho+2\varepsilon}}(0)\right)$. Thus $T*$ is continuous. That $T*$ commutes with the directional derivatives follows as in the case $k \in [1,+\infty)$.

Remark 2.26. The proofs of Theorems 2.16 and 2.25 correct the proof of the 1-nuclear case of [18, Theorem 3.20] (note that [18, Proposition 3.17], which is used in the proof of [18, Theorem 3.20], is false).

Definition 2.27. For $\mathcal{O} \in \mathcal{A}^{k}_{\tilde{N},(s;(r,q)),A}$, $k \in [1,+\infty]$ and $A \in (0,+\infty]$, we say that $\mathcal{O}$ is of *type zero* if $F\left(\gamma^{k}_{\tilde{N},(s;(r,q)),A}\mathcal{O}\right) \in Exp^{k'}_{(s',m(r';q')),0}(E')$, where $\left(\gamma^{k}_{\tilde{N},(s;(r,q)),A}\mathcal{O}\right)(f) = \mathcal{O}f(0)$ for all $f \in Exp^{k}_{\tilde{N},(s;(r,q)),A}(E)$.
For $\mathcal{O} \in \mathcal{A}^{k}_{\tilde{N},(s;(r,q)),0,B}$, $k \in [1,+\infty]$ and $B \in [0,+\infty)$, we say that $\mathcal{O}$ is of *type zero* if $F\left(\gamma^{k}_{\tilde{N},(s;(r,q)),0,B}\mathcal{O}\right) \in Exp^{k'}_{(s',m(r';q')),0}(E')$, where $\left(\gamma^{k}_{\tilde{N},(s;(r,q)),0,B}\mathcal{O}\right)(f) = \mathcal{O}f(0)$ for all $f \in Exp^{k}_{\tilde{N},(s;(r,q)),0,B}(E)$.

Theorem 2.28. *If $k \in [1,+\infty]$ and $A \in (0,+\infty]$, then $\gamma^{k}_{\tilde{N},(s;(r,q)),A}$ is a linear bijection between the space of convolution operators of type zero on $Exp^{k}_{\tilde{N},(s;(r,q)),A}(E)$ and the space of continuous linear functionals of type zero on $Exp^{k}_{\tilde{N},(s;(r,q)),A}(E)$.*
If $k \in [1,+\infty]$ and $B \in [0,+\infty)$, then $\gamma^{k}_{\tilde{N},(s;(r,q)),0,B}$ is a linear bijection between the space of convolution operators of type zero on $Exp^{k}_{\tilde{N},(s;(r,q)),0,B}(E)$ and the space of continuous linear functionals of type zero on $Exp^{k}_{\tilde{N},(s;(r,q)),0,B}(E)$.

Proof. We define $\left(\Gamma^{k}_{\tilde{N},(s;(r,q)),A}(T)\right)(f) = T * f$ for $T \in \left[Exp^{k}_{\tilde{N},(s;(r,q)),A}(E)\right]'$ of type zero and $f \in Exp^{k}_{\tilde{N},(s;(r,q)),A}(E)$. Then $T \in \left[Exp^{k}_{\tilde{N},(s;(r,q))}(E)\right]'$ and by

Theorem 2.25 we have

$$\gamma^k_{\tilde{N},(s;(r,q)),A}\left(\Gamma^k_{\tilde{N},(s;(r,q)),A}(T)\right)(f)=\left(\Gamma^k_{\tilde{N},(s;(r,q)),A}(T)\right)(f)(0)=(T*f)(0)$$
$$=\sum_{n=0}^{\infty}\left(T*\left(\frac{1}{n!}\widehat{d}^n f(0)\right)\right)(0)=\sum_{n=0}^{\infty}T\left(\frac{1}{n!}\widehat{d}^n f(0)\right)=T(f).$$

Hence $\gamma^k_{\tilde{N},(s;(r,q)),A}\circ\Gamma^k_{\tilde{N},(s;(r,q)),A}$ is the identity mapping on the subspace of $\left[Exp^k_{\tilde{N},(s;(r,q)),A}(E)\right]'$ of all functionals of type zero.

On the other hand, if $\mathcal{O}$ is of type zero we get $\gamma^k_{\tilde{N},(s;(r,q)),A}(\mathcal{O})\in\left[Exp^k_{\tilde{N},(s;(r,q))}(E)\right]'$ and by Theorem 2.25 we have

$$\Gamma^k_{\tilde{N},(s;(r,q)),A}\left(\gamma^k_{\tilde{N},(s;(r,q)),A}(\mathcal{O})\right)(f)(x)=\left(\left(\gamma^k_{\tilde{N},(s;(r,q)),A}(\mathcal{O})\right)*f\right)(x)$$
$$=\sum_{n=0}^{\infty}\left(\left(\gamma^k_{\tilde{N},(s;(r,q)),A}(\mathcal{O})\right)*\left(\frac{1}{n!}\widehat{d}^n f(0)\right)\right)(x)=\sum_{n=0}^{\infty}\mathcal{O}\left(\tau_{-x}\left(\frac{1}{n!}\widehat{d}^n f(0)\right)\right)(0)$$
$$=\sum_{n=0}^{\infty}\tau_{-x}\left(\mathcal{O}\left(\frac{1}{n!}\widehat{d}^n f(0)\right)\right)(0)=\sum_{n=0}^{\infty}\mathcal{O}\left(\frac{1}{n!}\widehat{d}^n f(0)\right)(x)=$$
$$\sum_{n=0}^{\infty}\frac{1}{n!}\widehat{d}^n(\mathcal{O}f)(0)(x)=\mathcal{O}f(x).$$

Hence $\Gamma^k_{\tilde{N},(s;(r,q)),A}\circ\gamma^k_{\tilde{N},(s;(r,q)),A}$ is the identity mapping on the subspace of $\mathcal{A}^k_{\tilde{N},(s;(r,q)),A}$ of all operators of type zero.

Now, if we define $\left(\Gamma^k_{\tilde{N},(s;(r,q)),0,B}(T)\right)(f)=T*f$ for $T\in\left[Exp^k_{\tilde{N},(s;(r,q)),0,B}(E)\right]'$ of type zero and $f\in Exp^k_{\tilde{N},(s;(r,q)),0,B}(E)$, then we prove that $\Gamma^k_{\tilde{N},(s;(r,q)),0,B}(T)$ is the inverse of $\gamma^k_{\tilde{N},(s;(r,q)),0,B}$ by an argument similar to the one used in the first part.

Remark 2.29. (1) Since the elements of $\left[Exp^\infty_{\tilde{N},(s;(r,q))}(E)\right]'$ are of type zero, then $\gamma^\infty_{\tilde{N},(s;(r,q))}$ is a linear bijection between $\left[Exp^\infty_{\tilde{N},(s;(r,q))}(E)\right]'$ and $\mathcal{A}^\infty_{\tilde{N},(s;(r,q))}$.

(2) For $k\in[1,+\infty]$, $B\in[0,+\infty)$ and $T_1,T_2\in\left[Exp^k_{\tilde{N},(s;(r,q)),0,B}(E)\right]'$, with T_2 of type zero, we may define $T_1*T_2\in\left[Exp^k_{\tilde{N},(s;(r,q)),0,B}(E)\right]'$ in the following way: If $f\in Exp^k_{\tilde{N},(s;(r,q)),0,B}(E)$, let

$$P_n=\sum_{j=0}^{n}\frac{1}{j!}\widehat{d}^j f(0),$$

for each $n\in\mathbb{N}$. By Remark 2.22 we have $T_1,T_2\in\left[Exp^k_{\tilde{N},(s;(r,q)),0}(E)\right]'$ and from Definition 2.19 we have $T_1*T_2\in\left[Exp^k_{\tilde{N},(s;(r,q)),0}(E)\right]'$. Thus we set

$$(T_1*T_2)(f)=\lim_{n\to\infty}(T_1*T_2)(P_n)=\lim_{n\to\infty}\gamma^k_{\tilde{N},(s;(r,q)),0}((T_1*)\circ(T_2*))(P_n)=$$
$$=\lim_{n\to\infty}(T_1*(T_2*P_n))(0)=\lim_{n\to\infty}T_1(T_2*P_n)=T_1\left(\lim_{n\to\infty}T_2*P_n\right)=T_1(T_2*f)$$

and the last equality is valid since P_n converges to f in $Exp^k_{\tilde{N},(s;(r,q))}(E)$ and $T_2 \in \left[Exp^k_{\tilde{N},(s;(r,q))}(E)\right]'$.

Moreover, from Theorem 2.25 we get $T_2 * f \in Exp^k_{\tilde{N},(s;(r,q)),0,B}(E)$ and $T_2 *$ is a convolution operator on $Exp^k_{\tilde{N},(s;(r,q)),0,B}(E)$.

Since $T_1 \in \left[Exp^k_{\tilde{N},(s;(r,q)),0,B}(E)\right]'$, we have that $T_1 \circ (T_2 *)$ is continuous on $Exp^k_{\tilde{N},(s;(r,q)),0,B}(E)$. Hence $T_1 * T_2$ is continuous on $Exp^k_{\tilde{N},(s;(r,q)),0,B}(E)$ and $F(T_1 * T_2) = F(T_1) F(T_2)$ as in Proposition 2.21.

(3) As in (2), if $k \in [1, +\infty]$, $A \in (0, +\infty]$ and $T_1, T_2 \in \left[Exp^k_{\tilde{N},(s;(r,q)),A}(E)\right]'$, with T_2 of type zero, we define $T_1 * T_2(f) = T_1(T_2 * f)$ for all $f \in Exp^k_{\tilde{N},(s;(r,q)),A}(E)$ and we obtain $T_1 * T_2 \in \left[Exp^k_{\tilde{N},(s;(r,q)),A}(E)\right]'$ satisfying $F(T_1 * T_2) = F(T_1) F(T_2)$.

Definition 2.30. The product $*$ defined in (2) and (3) is called *the convolution product of T_1 and T_2 on* $\left[Exp^k_{\tilde{N},(s;(r,q)),0,B}(E)\right]'$ and $\left[Exp^k_{\tilde{N},(s;(r,q)),A}(E)\right]'$, respectively.

3 Division Theorems

In order to prove division theorems we need the following two division results obtained by Matos [18]:

Proposition 3.1. *For $k \in [1, +\infty)$, $f \in Exp^k_{0,A}(E)$ and $g \in Exp^k_{0,B}(E)$, with $A, B \in [0, +\infty)$ and $g \neq 0$, if f/g is entire on E, then $f/g \in Exp^k_{0,L}(E)$, where*

$$L = \inf_{\lambda > 0} \left((A(1+\lambda))^k + (B(1+\lambda))^k \left(\left(\frac{1+\lambda}{\lambda} \right)^2 - 1 \right) \right)^{k^{-1}}.$$

Proposition 3.2. *Let $f \in Exp^\infty_{0,A}(E)$ and $g \in Exp^\infty_{0,B}(E)$, with $A \geq B \geq 0$ and $g \neq 0$. If f/g is holomorphic on $B_{A^{-1}}(0) \subset E$, then $f/g \in Exp^\infty_{0,A}(E)$.*

Remark 3.3. The spaces $Exp^k_{0,A}(E)$ are the analogues of the spaces $Exp^k_{(s,m(r;q)),0,A}(E)$ with the usual norm of polynomials being replaced by the norm $(s, m(r;q))$. For further details, see [17, 18].

A technical result is also needed to prove a division theorem for $k \in [1, +\infty)$.

Lemma 3.4. *For each $\varepsilon > 0$ there is a constant $D(\varepsilon) > 0$ such that*

$$\frac{j}{j-l} \left(\frac{j}{j-l} \right)^{j-l} \left(\frac{j}{l} \right)^l \frac{l!\,(j-l)!}{j!} \leq D(\varepsilon)(1+\varepsilon)^l,$$

for all $j, l \in \mathbb{N}$, with $1 \leq l \leq j-1$.

Proof. First of all, for each $l \in \mathbb{N}$ it is not difficult to see that the sequence $(a_j)_{j>l}$ is increasing, where

$$a_j = \left(\frac{j}{j-l}\right)^{j-l} \left(\frac{j}{l}\right)^{l} \frac{l!\,(j-l)!}{j!}.$$

Since

$$a_j = \frac{1}{\left(1-\frac{l}{j}\right)^{j}} \cdot \frac{l!}{l^l} \cdot \frac{(j-l)^l}{j\,(j-1)\cdots(j-(l-1))},$$

$$\lim_{j\to\infty} \frac{(j-l)^l}{j\,(j-1)\cdots(j-(l-1))} = \lim_{j\to\infty} \frac{\left(1-\frac{l}{j}\right)^{l}}{1\left(1-\frac{1}{j}\right)\cdots\left(1-\frac{(l-1)}{j}\right)} = 1$$

and

$$\lim_{j\to\infty} \left(1-\frac{l}{j}\right)^{j} = e^{-l},$$

we get

$$\lim_{j\to\infty} \left(\frac{j}{j-l}\right)^{j-l} \left(\frac{j}{l}\right)^{l} \frac{l!\,(j-l)!}{j!} = \frac{l!}{l^l} \lim_{j\to\infty} \frac{1}{\left(1-\frac{l}{j}\right)^{j}} \cdot \frac{(j-l)^l}{j\,(j-1)\cdots(j-(l-1))} = \frac{l!}{l^l}e^l.$$

Hence

$$\left(\frac{j}{j-l}\right)^{j-l} \left(\frac{j}{l}\right)^{l} \frac{l!\,(j-l)!}{j!} \le \frac{l!}{l^l}e^l,$$

for all $j > l$, $j \in \mathbb{N}$. Multiplying both sides by $\frac{j}{j-l}$ we get

$$\frac{j}{j-l}\left(\frac{j}{j-l}\right)^{j-l} \left(\frac{j}{l}\right)^{l} \frac{l!\,(j-l)!}{j!} \le \frac{j}{j-l} \cdot \frac{l!}{l^l}e^l.$$

Furthermore

$$\frac{j}{j-l} \le l+1 \iff j \ge l+1 \iff j > l,$$

then we obtain

$$\frac{j}{j-l}\left(\frac{j}{j-l}\right)^{j-l} \left(\frac{j}{l}\right)^{l} \frac{l!\,(j-l)!}{j!} \le (l+1)\,\frac{l!}{l^l}e^l. \tag{16}$$

Now the two limits

$$\lim_{l\to\infty} \left(\frac{l!}{l^l}e^l\right)^{\frac{1}{l}} = \lim_{l\to\infty} \frac{e\,(l!)^{\frac{1}{l}}}{l} = 1$$

and

$$\lim_{l\to\infty} (l+1)^{\frac{1}{l}} = 1$$

assure that

$$\lim_{l\to\infty} \left((l+1)\,\frac{l!}{l^l}e^l\right)^{\frac{1}{l}} = 1.$$

So there is $D(\varepsilon) > 0$ such that

$$(l+1)\,\frac{l!}{l^l}e^l \le D(\varepsilon)\,(1+\varepsilon)^l, \tag{17}$$

for all $l \in \mathbb{N}$. Now the result follows from (16) and (17).

Theorem 3.5. *Let $k \in [1,+\infty)$, $A,B \in [0,+\infty)$, $f \in Exp^k_{(s,m(r;q)),0,A}(E)$ and $g \in Exp^k_{(s,m(r;q)),0,B}(E)$, with $g \neq 0$.*
If f/g is entire on E, then $f/g \in Exp^k_{(s,m(r;q)),0,(L+B)}(E)$, where

$$L = \inf_{\lambda>0}\left(\left(A\left(1+\lambda\right)\right)^k + \left(B\left(1+\lambda\right)\right)^k\left(\left(\frac{1+\lambda}{\lambda}\right)^2 - 1\right)\right)^{k^{-1}}.$$

Proof. Since $\|\cdot\| \leq \|\cdot\|_{(s,m(r;q))}$ and $\mathcal{P}_{(s,m(r;q))}\left({}^jE\right) \subset \mathcal{P}\left({}^jE\right)$, we get $Exp^k_{(s,m(r;q)),0,A}(E) \subset Exp^k_{0,A}(E)$ and $Exp^k_{(s,m(r;q)),0,B}(E) \subset Exp^k_{0,B}(E)$. By Proposition 3.1 we get $h = f/g \in Exp^k_{0,L}(E)$. Thus for each $\varepsilon > 0$, there is $C(\varepsilon) > 0$ such that

$$\left\|\widehat{d}^j h(0)\right\| \leq C(\varepsilon)\left(\frac{ke}{j}\right)^{\frac{j}{k}} j!\left(L+\varepsilon\right)^j,$$

for all $j \in \mathbb{N}$. Let $(x_m)_{m=1}^\infty \in \ell_{m(r;q)}(E)$ with $\left\|(x_m)_{m=1}^\infty\right\|_{m(r;q)} \leq 1$. Then $\|x_m\| \leq 1$ for all $m \in \mathbb{N}$. So we have

$$\left|\widehat{d}^j h(0)(x_m)\right| \leq C(\varepsilon)\left(\frac{ke}{j}\right)^{\frac{j}{k}} j!\left(L+\varepsilon\right)^j,$$

for all $j,m \in \mathbb{N}$. Suppose first that $g(0) \neq 0$. Since $f = g \cdot h$, it follows by the uniqueness of the power series of a holomorphic function around a point of its domain that

$$\frac{\widehat{d}^j f(0)(x)}{j!} = g(0)\frac{\widehat{d}^j h(0)(x)}{j!} + \sum_{l=1}^{j}\frac{\widehat{d}^l g(0)(x)}{l!}\frac{\widehat{d}^{j-l}h(0)(x)}{(j-l)!},$$

for all $x \in E$. Then

$$\widehat{d}^j h(0)(x_m) = \frac{1}{g(0)}\widehat{d}^j f(0)(x_m) - \frac{j!}{g(0)}\sum_{l=1}^{j}\frac{\widehat{d}^l g(0)(x_m)}{l!}\frac{\widehat{d}^{j-l}h(0)(x_m)}{(j-l)!}$$

and

$$\begin{aligned}\left|\widehat{d}^j h(0)(x_m)\right| \leq{}& \frac{1}{|g(0)|}\left|\widehat{d}^j f(0)(x_m)\right| + \frac{|h(0)|}{|g(0)|}\left|\widehat{d}^j g(0)(x_m)\right| \\ &+\frac{C(\varepsilon)}{|g(0)|}\sum_{l=1}^{j-1}\binom{j}{l}\left(\frac{ke}{j-l}\right)^{\frac{j-l}{k}}(j-l)!\left(L+\varepsilon\right)^{j-l}\left|\widehat{d}^l g(0)(x_m)\right|.\end{aligned}$$

Thus

$$\begin{aligned}\left\|\left(\widehat{d}^j h(0)(x_m)\right)_{m=1}^\infty\right\|_s \leq{}& \frac{1}{|g(0)|}\left\|\left(\widehat{d}^j f(0)(x_m)\right)_{m=1}^\infty\right\|_s + \frac{|h(0)|}{|g(0)|}\left\|\left(\widehat{d}^j g(0)(x_m)\right)_{m=1}^\infty\right\|_s \\ &+\frac{C(\varepsilon)}{|g(0)|}\sum_{l=1}^{j-1}\binom{j}{l}\left(\frac{ke}{j-l}\right)^{\frac{j-l}{k}}(j-l)!\left(L+\varepsilon\right)^{j-l}\left\|\left(\widehat{d}^l g(0)(x_m)\right)_{m=1}^\infty\right\|_s.\end{aligned}$$

By the definition of $\|\cdot\|_{(s,m(r;q))}$(see Matos [21, pp. 97-98]), we have

$$\left\|\left(\widehat{d}^n f(0)(x_m)\right)_{m=1}^\infty\right\|_s \leq \left\|\widehat{d}^n f(0)\right\|_{(s,m(r;q))}\left(\left\|(x_m)_{m=1}^\infty\right\|_{m(r;q)}\right)^n$$

and
$$\left\| \left(\widehat{d}^n g(0)(x_m) \right)_{m=1}^{\infty} \right\|_s \le \left\| \widehat{d}^n g(0) \right\|_{(s,m(r;q))} \left(\left\| (x_m)_{m=1}^{\infty} \right\|_{m(r;q)} \right)^n,$$
for all $n \in \mathbb{N}$. Since $\left\| (x_m)_{m=1}^{\infty} \right\|_{m(r;q)} \le 1$, we get
$$\left\| \left(\widehat{d}^j h(0)(x_m) \right)_{m=1}^{\infty} \right\|_s \le \frac{1}{|g(0)|} \left\| \widehat{d}^j f(0) \right\|_{(s,m(r;q))} + \frac{|h(0)|}{|g(0)|} \left\| \widehat{d}^j g(0) \right\|_{(s,m(r;q))}$$
$$+ \frac{C(\varepsilon)}{|g(0)|} \sum_{l=1}^{j-1} \binom{j}{l} \left(\frac{ke}{j-l} \right)^{\frac{j-l}{k}} (j-l)! (L+\varepsilon)^{j-l} \left\| \widehat{d}^l g(0) \right\|_{(s,m(r;q))}.$$
and consequently we obtain
$$\left\| \widehat{d}^j h(0) \right\|_{(s,m(r;q))} \le \frac{1}{|g(0)|} \left\| \widehat{d}^j f(0) \right\|_{(s,m(r;q))} + \frac{|h(0)|}{|g(0)|} \left\| \widehat{d}^j g(0) \right\|_{(s,m(r;q))}$$
$$+ \frac{C(\varepsilon)}{|g(0)|} \sum_{l=1}^{j-1} \binom{j}{l} \left(\frac{ke}{j-l} \right)^{\frac{j-l}{k}} (j-l)! (L+\varepsilon)^{j-l} \left\| \widehat{d}^l g(0) \right\|_{(s,m(r;q))}.$$
Now since $f \in Exp^k_{(s,m(r;q)),0,A}(E)$ and $g \in Exp^k_{(s,m(r;q)),0,B}(E)$ we have that for each $\varepsilon > 0$, there are $\alpha(\varepsilon) > 0$ and $\beta(\varepsilon) > 0$ such that
$$\left\| \widehat{d}^n f(0) \right\|_{(s,m(r;q))} \le \alpha(\varepsilon) n! \left(\frac{ke}{n} \right)^{\frac{n}{k}} (A+\varepsilon)^n$$
and
$$\left\| \widehat{d}^n g(0) \right\|_{(s,m(r;q))} \le \beta(\varepsilon) n! \left(\frac{ke}{n} \right)^{\frac{n}{k}} (B+\varepsilon)^n,$$
for all $n \in \mathbb{N}$. Therefore
$$\left\| \widehat{d}^j h(0) \right\|_{(s,m(r;q))} \le \frac{\alpha(\varepsilon)}{|g(0)|} j! \left(\frac{ke}{j} \right)^{\frac{j}{k}} (A+\varepsilon)^j + \frac{|h(0)| \beta(\varepsilon)}{|g(0)|} j! \left(\frac{ke}{j} \right)^{\frac{j}{k}} (B+\varepsilon)^j$$
$$+ \frac{C(\varepsilon) \beta(\varepsilon)}{|g(0)|} \sum_{l=1}^{j-1} \binom{j}{l} \left(\frac{ke}{j-l} \right)^{\frac{j-l}{k}} \left(\frac{ke}{l} \right)^{\frac{l}{k}} l! (j-l)! (L+\varepsilon)^{j-l} (B+\varepsilon)^l.$$
Note that
$$\left(\frac{j}{ke} \right)^{\frac{j}{k}} \frac{1}{j!} \binom{j}{l} \left(\frac{ke}{j-l} \right)^{\frac{j-l}{k}} \left(\frac{ke}{l} \right)^{\frac{l}{k}} l! (j-l)! = \binom{j-1}{l} \frac{j}{j-l} \left(\frac{j}{j-l} \right)^{\frac{j-l}{k}} \left(\frac{j}{l} \right)^{\frac{l}{k}} \frac{l! (j-l)!}{j!}.$$
Then
$$\sum_{l=1}^{j-1} \binom{j}{l} \left(\frac{ke}{j-l} \right)^{\frac{j-l}{k}} \left(\frac{ke}{l} \right)^{\frac{l}{k}} l! (j-l)! (L+\varepsilon)^{j-l} (B+\varepsilon)^l =$$
$$= \left(\frac{ke}{j} \right)^{\frac{j}{k}} j! \sum_{l=1}^{j-1} \binom{j-1}{l} \frac{j}{j-l} \left(\frac{j}{j-l} \right)^{\frac{j-l}{k}} \left(\frac{j}{l} \right)^{\frac{l}{k}} \frac{l! (j-l)!}{j!} (L+\varepsilon)^{j-l} (B+\varepsilon)^l$$
$$\le \left(\frac{ke}{j} \right)^{\frac{j}{k}} j! \sum_{l=1}^{j-1} \binom{j-1}{l} \frac{j}{j-l} \left(\frac{j}{j-l} \right)^{j-l} \left(\frac{j}{l} \right)^{l} \frac{l! (j-l)!}{j!} (L+\varepsilon)^{j-l} (B+\varepsilon)^l,$$

and by Lemma 3.4 we obtain

$$\sum_{l=1}^{j-1}\binom{j}{l}\left(\frac{ke}{j-l}\right)^{\frac{j-l}{k}}\left(\frac{ke}{l}\right)^{\frac{l}{k}} l!\,(j-l)!\,(L+\varepsilon)^{j-l}\,(B+\varepsilon)^{l}$$
$$\leq\left(\frac{ke}{j}\right)^{\frac{j}{k}} j!D\,(\varepsilon)\sum_{l=1}^{j-1}\binom{j-1}{l}(L+\varepsilon)^{j-l}\left[(1+\varepsilon)\,(B+\varepsilon)\right]^{l}.$$

Hence

$$\left\|\widehat{d^j}h\,(0)\right\|_{(s,m(r;q))}\leq\frac{\alpha\,(\varepsilon)}{|g\,(0)|}j!\left(\frac{ke}{j}\right)^{\frac{j}{k}}(A+\varepsilon)^{j}+\frac{|h\,(0)|\,\beta\,(\varepsilon)}{|g\,(0)|}j!\left(\frac{ke}{j}\right)^{\frac{j}{k}}(B+\varepsilon)^{j}$$
$$+\frac{C\,(\varepsilon)\,\beta\,(\varepsilon)\,D\,(\varepsilon)}{|g\,(0)|}\left(\frac{ke}{j}\right)^{\frac{j}{k}} j!\left[L+\varepsilon+(1+\varepsilon)\,(B+\varepsilon)\right]^{j}.$$

Since $A\leq L$, it follows that

$$\left\|\widehat{d^j}h\,(0)\right\|_{(s,m(r;q))}\leq\frac{\alpha\,(\varepsilon)}{|g\,(0)|}j!\left(\frac{ke}{j}\right)^{\frac{j}{k}}(L+\varepsilon)^{j}+\frac{|h\,(0)|\,\beta\,(\varepsilon)}{|g\,(0)|}j!\left(\frac{ke}{j}\right)^{\frac{j}{k}}(B+\varepsilon)^{j}$$
$$+\frac{C\,(\varepsilon)\,\beta\,(\varepsilon)\,D\,(\varepsilon)}{|g\,(0)|}\left(\frac{ke}{j}\right)^{\frac{j}{k}} j!\left[L+\varepsilon+B+\varepsilon+\varepsilon B+\varepsilon^{2}\right]^{j}$$
$$\leq\left(\frac{\alpha\,(\varepsilon)}{|g\,(0)|}+\frac{|h\,(0)|\,\beta\,(\varepsilon)}{|g\,(0)|}+\frac{C\,(\varepsilon)\,\beta\,(\varepsilon)\,D\,(\varepsilon)}{|g\,(0)|}\right)j!\left(\frac{ke}{j}\right)^{\frac{j}{k}}(L+B+\varepsilon(2+B+\varepsilon))^{j}.$$

Since $\varepsilon>0$ was chosen arbitrarily we have

$$\limsup_{j\to\infty}\left(\frac{j}{ke}\right)^{\frac{1}{k}}\left\|\frac{1}{j!}\widehat{d^j}h\,(0)\right\|_{(s,m(r;q))}^{\frac{1}{j}}\leq L+B.$$

Hence $h\in Exp^{k}_{(s,m(r;q)),0,(L+B)}\,(E)$.
If $g\,(0)=0$ we consider $f_0\,(x)=f\,(x)+\psi\,(x)\,h\,(x)$ and $g_0\,(x)=g\,(x)+\psi\,(x)$ for all $x\in E$, where $\psi\in Exp^{k}_{(s,m(r;q)),0}\,(E)$, $\psi\,(0)\neq0$ and ψ is non constant. Then $f_0=g_0h$ and $g_0\,(0)\neq0$. By Remark 2.22 we get $\psi\in Exp^{k}_{(s,m(r;q)),0,B}\,(E)$ which implies $g_0\in Exp^{k}_{(s,m(r;q)),0,B}\,(E)$. If $f_0\in Exp^{k}_{(s,m(r;q)),0,L}\,(E)$, we apply the result we just proved and obtain $h\in Exp^{k}_{(s,m(r;q)),0,(L+B)}\,(E)$.
In order to prove that $f_0\in Exp^{k}_{(s,m(r;q)),0,L}\,(E)$ it is enough to show that $\psi h\in Exp^{k}_{(s,m(r;q)),0,L}\,(E)$. Since

$$\widehat{d^j}\,(\psi h)\,(0)\,(x_m)=j!\sum_{l=0}^{j}\frac{\widehat{d^l}\psi\,(0)\,(x_m)}{l!}\frac{\widehat{d^{j-l}}h\,(0)\,(x_m)}{(j-l)!}$$
$$=\sum_{l=0}^{j-1}\binom{j}{l}\widehat{d^l}\psi\,(0)\,(x_m)\,\widehat{d^{j-l}}h\,(0)\,(x_m)+\widehat{d^j}\psi\,(0)\,(x_m)\,h\,(0),$$

and $h\in Exp^{k}_{0,L}\,(E)$, we get

$$\left|\widehat{d^{j-l}}h\,(0)\,(x_m)\right|\leq C\,(\varepsilon)\left(\frac{ke}{j-l}\right)^{\frac{j-l}{k}}(j-l)!\,(L+\varepsilon)^{j-l},$$

if $\left\|(x_m)_{m=1}^{\infty}\right\|_{m(r;q)} \leq 1$. Thus

$$\left|\widehat{d^j}\,(\psi h)\,(0)\,(x_m)\right| \leq |h\,(0)|\left|\widehat{d^j}\psi\,(0)\,(x_m)\right|$$
$$+C\,(\varepsilon)\sum_{l=0}^{j-1}\binom{j}{l}\left|\widehat{d^l}\psi\,(0)\,(x_m)\right|\left(\frac{ke}{j-l}\right)^{\frac{j-l}{k}}(j-l)!\,(L+\varepsilon)^{j-l}.$$

Since $\psi \in Exp^k_{(s,m(r;q)),0}\,(E)$, there is $N\,(\varepsilon) > 0$ such that

$$\left\|\widehat{d^j}\psi\,(0)\right\|_{(s,m(r;q))} \leq N\,(\varepsilon)\left(\frac{ke}{j}\right)^{\frac{j}{k}} j!\varepsilon^j,$$

for all $j \in \mathbb{N}$. Using the same arguments of the result we just proved, we get

$$\left\|\widehat{d^j}\,(\psi h)\,(0)\right\|_{(s,m(r;q))} \leq$$
$$|h\,(0)|\left\|\widehat{d^j}\psi\,(0)\right\|_{(s,m(r;q))} + C\,(\varepsilon)\sum_{l=0}^{j-1}\binom{j}{l}\left\|\widehat{d^l}\psi\,(0)\right\|_{(s,m(r;q))}\left(\frac{ke}{j-l}\right)^{\frac{j-l}{k}}(j-l)!\,(L+\varepsilon)^{j-l}.$$

Since $\mathcal{P}_{(s,m(r;q))}\left({}^0E\right) = \mathbb{C}$, we have that $\left\|\widehat{d^0}\psi\,(0)\right\|_{(s,m(r;q))} = |\psi\,(0)|$ and

$$\left\|\widehat{d^j}\,(\psi h)\,(0)\right\|_{(s,m(r;q))} \leq |h\,(0)|\,N\,(\varepsilon)\left(\frac{ke}{j}\right)^{\frac{j}{k}} j!\varepsilon^j + C\,(\varepsilon)\,|\psi\,(0)|\left(\frac{ke}{j}\right)^{\frac{j}{k}} j!\,(L+\varepsilon)^j +$$
$$+C\,(\varepsilon)\,N\,(\varepsilon)\sum_{l=1}^{j-1}\binom{j}{l}\left(\frac{ke}{l}\right)^{\frac{l}{k}} l!\left(\frac{ke}{j-l}\right)^{\frac{j-l}{k}}(j-l)!\varepsilon^l\,(L+\varepsilon)^{j-l}.$$

From Lemma 3.4 and the previous arguments, we get

$$\sum_{l=1}^{j-1}\binom{j}{l}\left(\frac{ke}{j-l}\right)^{\frac{j-l}{k}}\left(\frac{ke}{l}\right)^{\frac{l}{k}} l!\,(j-l)!\,(L+\varepsilon)^{j-l}\,\varepsilon^l$$
$$\leq \left(\frac{ke}{j}\right)^{\frac{j}{k}} j!D\,(\varepsilon)\sum_{l=1}^{j-1}\binom{j-1}{l}(L+\varepsilon)^{j-l}\left[(1+\varepsilon)\,\varepsilon\right]^l.$$

Therefore

$$\left\|\widehat{d^j}\,(\psi h)\,(0)\right\|_{(s,m(r;q))} \leq C\,(\varepsilon)\,|\psi\,(0)|\left(\frac{ke}{j}\right)^{\frac{j}{k}} j!\,(L+\varepsilon)^j + |h\,(0)|\,N\,(\varepsilon)\left(\frac{ke}{j}\right)^{\frac{j}{k}} j!\varepsilon^j$$
$$+C\,(\varepsilon)\,N\,(\varepsilon)\,D\,(\varepsilon)\left(\frac{ke}{j}\right)^{\frac{j}{k}} j!\sum_{l=1}^{j-1}\binom{j-1}{l}(L+\varepsilon)^{j-l}\left[(1+\varepsilon)\,\varepsilon\right]^l$$
$$\leq C\,(\varepsilon)\,|\psi\,(0)|\left(\frac{ke}{j}\right)^{\frac{j}{k}} j!\,(L+\varepsilon)^j + |h\,(0)|\,N\,(\varepsilon)\left(\frac{ke}{j}\right)^{\frac{j}{k}} j!\varepsilon^j$$
$$+C\,(\varepsilon)\,N\,(\varepsilon)\,D\,(\varepsilon)\left(\frac{ke}{j}\right)^{\frac{j}{k}} j!\,(L+\varepsilon+(1+\varepsilon)\,\varepsilon)^{j-1}$$
$$\leq \left(C\,(\varepsilon)\,|\psi\,(0)| + |h\,(0)|\,N\,(\varepsilon) + \frac{C\,(\varepsilon)\,N\,(\varepsilon)\,D\,(\varepsilon)}{(L+\varepsilon+(1+\varepsilon)\,\varepsilon)}\right)\left(\frac{ke}{j}\right)^{\frac{j}{k}} j!\,(L+\varepsilon+(1+\varepsilon)\,\varepsilon)^j.$$

Since $\varepsilon > 0$ was chosen arbitrarily we have

$$\limsup_{j\to\infty}\left(\frac{j}{ke}\right)^{\frac{1}{k}}\left\|\frac{1}{j!}\widehat{d}^j(\psi h)(0)\right\|_{(s,m(r;q))}^{\frac{1}{j}}\leq L.$$

Hence $\psi h \in Exp^k_{(s,m(r;q)),0,L}(E)$.

Remark 3.6. Note that if $B = 0$, then $L = A$ and $f/g \in Exp^k_{(s,m(r;q)),0,A}(E)$.

Example 3.7. Now we give some examples of $\psi \in Exp^k_{(s,m(r;q)),0}(E)$, with $\psi(0) \neq 0$ and ψ non constant.
If $k \neq 1$, then $\psi = e^{\varphi}$ with $0 \neq \varphi \in E'$, is such that $e^{\varphi} \in Exp^k_{(s,m(r;q)),0}(E)$, $e^{\varphi(0)} = 1$ (see [11, Proposition 2.15]).
For $k \in [1,+\infty)$, $\psi(x) = 1 + P(x)$ with $P \in \mathcal{P}_{(s,m(r;q))}({}^nE)$ for some $0 \neq n \in \mathbb{N}$, is such that $\psi \in Exp^k_{(s,m(r;q)),0}(E)$, $\psi(0) = 1$ and ψ is non constant.

Theorem 3.8. *Let $f \in Exp^\infty_{(s,m(r;q)),0,A}(E)$ and $g \in Exp^\infty_{(s,m(r;q)),0,B}(E)$, with $A \geq B \geq 0$ and $g \neq 0$. If f/g is holomorphic on $B_{A^{-1}}(0) \subset E$, then*

$$f/g \in Exp^\infty_{(s,m(r;q)),0,(A+B)}(E).$$

Proof. Since $\|\cdot\| \leq \|\cdot\|_{(s,m(r;q))}$ and $\mathcal{P}_{(s,m(r;q))}({}^jE) \subset \mathcal{P}({}^jE)$ for all $j \in \mathbb{N}$, we get $f \in Exp^\infty_{0,A}(E)$ and $g \in Exp^\infty_{0,B}(E)$. By Proposition 3.2 we have $h = f/g \in Exp^\infty_{0,A}(E)$, and for each $\varepsilon > 0$, there is $C(\varepsilon) > 0$ such that

$$\left\|\widehat{d}^j h(0)\right\| \leq C(\varepsilon)\, j!\,(A+\varepsilon)^j,$$

for all $j \in \mathbb{N}$. Let $(x_m)_{m=1}^\infty \in \ell_{m(r;q)}(E)$ with $\|(x_m)_{m=1}^\infty\|_{m(r;q)} \leq \min\{1, A^{-1}\}$. Then $\|x_m\| \leq \min\{1, A^{-1}\}$ for all $m \in \mathbb{N}$. Then

$$\left|\widehat{d}^j h(0)(x_m)\right| \leq C(\varepsilon)\, j!\,(A+\varepsilon)^j,$$

for all $j \in \mathbb{N}$. First we suppose that $g(0) \neq 0$. Thus

$$\widehat{d}^j f(0)(x) = g(0)\,\widehat{d}^j h(0)(x) + j!\sum_{l=1}^{j}\frac{\widehat{d}^l g(0)(x)}{l!}\frac{\widehat{d}^{j-l} h(0)(x)}{(j-l)!},$$

for all $x \in B_{A^{-1}}(0)$ and

$$\begin{aligned}\left|\widehat{d}^j h(0)(x_m)\right| &\leq \frac{1}{|g(0)|}\left|\widehat{d}^j f(0)(x_m)\right| + \frac{1}{|g(0)|}\sum_{l=1}^{j}\binom{j}{l}\left|\widehat{d}^l g(0)(x_m)\right|\left|\widehat{d}^{j-l} h(0)(x_m)\right| \\ &\leq \frac{1}{|g(0)|}\left|\widehat{d}^j f(0)(x_m)\right| + \frac{C(\varepsilon)}{|g(0)|}\sum_{l=1}^{j}\binom{j}{l}(j-l)!\,(A+\varepsilon)^{j-l}\left|\widehat{d}^l g(0)(x_m)\right|.\end{aligned}$$

Therefore

$$\begin{aligned}\left\|\widehat{d}^j h(0)\right\|_{(s,m(r;q))} &\leq \frac{1}{|g(0)|}\left\|\widehat{d}^j f(0)\right\|_{(s,m(r;q))} \\ &+ \frac{C(\varepsilon)}{|g(0)|}\sum_{l=1}^{j}\binom{j}{l}(j-l)!\,(A+\varepsilon)^{j-l}\left\|\widehat{d}^l g(0)\right\|_{(s,m(r;q))},\end{aligned}$$

and since for each $\varepsilon > 0$ there is $\alpha(\varepsilon) > 0$ and $\beta(\varepsilon) > 0$ such that

$$\left\|\widehat{d}^j f(0)\right\|_{(s,m(r;q))} \leq \alpha(\varepsilon)\, j!\,(A+\varepsilon)^j$$

and

$$\left\|\widehat{d}^l g(0)\right\|_{(s,m(r;q))} \leq \beta(\varepsilon)\, l!\,(B+\varepsilon)^l,$$

we get

$$\begin{aligned}
\left\|\widehat{d}^j h(0)\right\|_{(s,m(r;q))} &\leq \frac{\alpha(\varepsilon)}{|g(0)|} j!\,(A+\varepsilon)^j + \frac{C(\varepsilon)\,\beta(\varepsilon)}{|g(0)|}\sum_{l=1}^{j}\binom{j}{l} l!\,(j-l)!\,(A+\varepsilon)^{j-l}(B+\varepsilon)^l \\
&\leq \frac{\alpha(\varepsilon)}{|g(0)|} j!\,(A+\varepsilon)^j + \frac{C(\varepsilon)\,\beta(\varepsilon)}{|g(0)|} j!\sum_{l=1}^{j}\binom{j}{l}(A+\varepsilon)^{j-l}(B+\varepsilon)^l \\
&\leq \frac{\alpha(\varepsilon)}{|g(0)|} j!\,(A+\varepsilon)^j + \frac{C(\varepsilon)\,\beta(\varepsilon)}{|g(0)|} j!\,(A+B+2\varepsilon)^j \\
&\leq \left(\frac{\alpha(\varepsilon)}{|g(0)|} + \frac{C(\varepsilon)\,\beta(\varepsilon)}{|g(0)|}\right) j!\,(A+B+2\varepsilon)^j.
\end{aligned}$$

Since $\varepsilon > 0$ was chosen arbitrarily we have

$$\limsup_{j\to\infty}\left\|\frac{1}{j!}\widehat{d}^j h(0)\right\|_{(s,m(r;q))}^{\frac{1}{j}} \leq A+B,$$

and since $B_{(A+B)^{-1}}(0) \subset B_{A^{-1}}(0)$ it follows that h is holomorphic on $B_{(A+B)^{-1}}(0)$. Hence $h \in Exp^{\infty}_{(s,m(r;q)),0,(A+B)}(E)$.

If $g(0) = 0$ we consider $f_0(x) = f(x) + e^{\varphi(x)}h(x)$ and $g_0(x) = g(x) + e^{\varphi(x)}$ for all $x \in E$, where $\varphi \in E'$, $\varphi \neq 0$. Then $f_0 = g_0 h$, $g_0(0) \neq 0$ and since $e^{\varphi} \in Exp^{\infty}_{(s,m(r;q)),0,B}(E)$ we get $g_0 \in Exp^{\infty}_{(s,m(r;q)),0,B}(E)$. If $f_0 \in Exp^{\infty}_{(s,m(r;q)),0,A}(E)$ we apply the result we just proved to obtain $h \in Exp^{\infty}_{(s,m(r;q)),0,(A+B)}(E)$.

In order to prove that $f_0 \in Exp^{\infty}_{(s,m(r;q)),0,A}(E)$ it is enough to show that $e^{\varphi}h \in Exp^{\infty}_{(s,m(r;q)),0,A}(E)$, but this follows analogously.

The proofs of the following three division theorems involving the Fourier-Borel transform are similar to the proofs of Theorems 4.9, 4.10 and 4.11 obtained by Matos [18]. Just use our results 3.5 and 3.8 where Matos uses his results 4.5 and 4.7.

Theorem 3.9. *If $k \in [1, +\infty]$ and $T_1, T_2 \in \left[Exp^{k}_{\tilde{N},(s;(r,q)),0}(E)\right]'$ are such that $T_2 \neq 0$ and $T_1(P\exp\varphi) = 0$ whenever $T_2 * P\exp\varphi = 0$ with $\varphi \in E'$ and $P \in \mathcal{P}_{\tilde{N},(s;(r,q))}(^nE)$, $n \in \mathbb{N}$, then FT_1 is divisible by FT_2 with the quotient being an element of $Exp^{k'}_{(s',m(r';q'))}(E')$.*

Theorem 3.10. *If $k \in [1, +\infty]$ and $T_1, T_2 \in \left[Exp^{k}_{\tilde{N},(s;(r,q))}(E)\right]'$ are such that $T_2 \neq 0$ and $T_1(P\exp\varphi) = 0$ whenever $T_2 * P\exp\varphi = 0$ with $\varphi \in E'$ and $P \in \mathcal{P}_{\tilde{N},(s;(r,q))}(^nE)$, $n \in \mathbb{N}$, then FT_1 is divisible by FT_2 with the quotient being an element of $Exp^{k'}_{(s',m(r';q')),0}(E')$.*

Theorem 3.11. (a) *For $k \in [1, +\infty]$ and $A \in (0, +\infty)$, if $T_1, T_2 \in \left[Exp^k_{\tilde{N},(s;(r,q)),A}(E)\right]'$ are such that T_2 is of type zero, $T_2 \neq 0$ and $T_1(P \exp \varphi) = 0$ whenever $T_2 * P \exp \varphi = 0$, with $\varphi \in E'$ and $P \in \mathcal{P}_{\tilde{N},(s;(r,q))}({}^nE)$, $n \in \mathbb{N}$, then FT_1 is divisible by FT_2 with the quotient being an element of $Exp^{k'}_{(s',m(r';q')),0,(\theta(k)A)^{-1}}(E')$.*

(b) *For $k \in [1, +\infty]$ and $B \in (0, +\infty)$, if $T_1, T_2 \in \left[Exp^k_{\tilde{N},(s;(r,q)),0,B}(E)\right]'$ are such that T_2 is of type zero, $T_2 \neq 0$ and $T_1(P \exp \varphi) = 0$ whenever $T_2 * P \exp \varphi = 0$, with $\varphi \in E'$ and $P \in \mathcal{P}_{\tilde{N},(s;(r,q))}({}^nE)$, $n \in \mathbb{N}$, then FT_1 is divisible by FT_2 with the quotient being an element of $Exp^{k'}_{(s',m(r';q')),(\theta(k)B)^{-1}}(E')$.*

4 Existence and Approximation Theorems for Convolution Equations

This section is devoted to results concerning approximation and existence of solutions of convolution equations. The three next results are consequences of Theorems 3.9, 3.10 and 3.11. It is enough to follow the arguments of Matos [18, 5.1, 5.2, 5.3 and 5.4].

Theorem 4.1. (a) *If $k \in [1, +\infty]$ and $\mathcal{O} \in \mathcal{A}^k_{\tilde{N},(s;(r,q)),0}$, then the vector subspace of $Exp^k_{\tilde{N},(s;(r,q)),0}(E)$ generated by the exponential polynomial solutions of the homogeneous equation $\mathcal{O}=0$ is dense in the closed subspace of all solutions of the homogeneous equation. That is, the vector subspace of $Exp^k_{\tilde{N},(s;(r,q)),0}(E)$ generated by*

$$\mathcal{L} = \left\{P \exp \varphi; P \in \mathcal{P}_{\tilde{N},(s;(r,q))}({}^nE), n \in \mathbb{N}, \varphi \in E', \mathcal{O}(P \exp \varphi) = 0\right\}$$

is dense in

$$\ker \mathcal{O} = \left\{f \in Exp^k_{\tilde{N},(s;(r,q)),0}(E); \mathcal{O}f = 0\right\}.$$

(b) *If $k \in [1, +\infty]$ and $\mathcal{O} \in \mathcal{A}^k_{\tilde{N},(s;(r,q))}$, then the vector subspace of $Exp^k_{\tilde{N},(s;(r,q))}(E)$ generated by*

$$\mathcal{L} = \left\{P \exp \varphi; P \in \mathcal{P}_{\tilde{N},(s;(r,q))}({}^nE), n \in \mathbb{N}, \varphi \in E', \mathcal{O}(P \exp \varphi) = 0\right\}$$

is dense in

$$\ker \mathcal{O} = \left\{f \in Exp^k_{\tilde{N},(s;(r,q))}(E); \mathcal{O}f = 0\right\}.$$

Theorem 4.2. (a) *If $k \in [1, +\infty]$, $A \in (0, +\infty)$ and $\mathcal{O} \in \mathcal{A}^k_{\tilde{N},(s;(r,q)),0,A}$ is of type zero, then the vector subspace of $Exp^k_{\tilde{N},(s;(r,q)),0,A}(E)$ generated by*

$$\mathcal{L} = \left\{P \exp \varphi; P \in \mathcal{P}_{\tilde{N},(s;(r,q))}({}^nE), n \in \mathbb{N}, \varphi \in E', \mathcal{O}(P \exp \varphi) = 0\right\}$$

is dense in

$$\ker \mathcal{O} = \left\{f \in Exp^k_{\tilde{N},(s;(r,q)),0,A}(E); \mathcal{O}f = 0\right\}.$$

(b) *If* $k \in [1, +\infty]$, $B \in (0, +\infty)$ *and* $\mathcal{O} \in \mathcal{A}^k_{\tilde{N},(s;(r,q)),B}$ *is of type zero, then the vector subspace of* $Exp^k_{\tilde{N},(s;(r,q)),B}(E)$ *generated by*

$$\mathcal{L} = \left\{ P \exp \varphi; P \in \mathcal{P}_{\tilde{N},(s;(r,q))}\left({}^nE\right), n \in \mathbb{N}, \varphi \in E', \mathcal{O}\left(P \exp \varphi\right) = 0 \right\}$$

is dense in

$$\ker \mathcal{O} = \left\{ f \in Exp^k_{\tilde{N},(s;(r,q)),B}(E); \mathcal{O}f = 0 \right\}.$$

Theorem 4.3. (a) *For* $k \in [1, +\infty]$, *if* $\mathcal{O} \in \mathcal{A}^k_{\tilde{N},(s;(r,q)),0}$, $\mathcal{O} \neq 0$, *then its transpose* ${}^t\mathcal{O} \colon \left[Exp^k_{\tilde{N},(s;(r,q)),0}(E)\right]' \longrightarrow \left[Exp^k_{\tilde{N},(s;(r,q)),0}(E)\right]'$ *is such that*
(a.1) ${}^t\mathcal{O}\left(\left[Exp^k_{\tilde{N},(s;(r,q)),0}(E)\right]'\right)$ *is the orthogonal of* $\ker \mathcal{O}$ *in* $\left[Exp^k_{\tilde{N},(s;(r,q)),0}(E)\right]'$.
(a.2) ${}^t\mathcal{O}\left(\left[Exp^k_{\tilde{N},(s;(r,q)),0}(E)\right]'\right)$ *is closed for the weak-star topology in* $\left[Exp^k_{\tilde{N},(s;(r,q)),0}(E)\right]'$ *defined by* $Exp^k_{\tilde{N},(s;(r,q)),0}(E)$.

(b) *For* $k \in [1, +\infty]$ *and* $A \in (0, +\infty)$, *if* $\mathcal{O} \in \mathcal{A}^k_{\tilde{N},(s;(r,q)),0,A}$ *is of type zero and* $\mathcal{O} \neq 0$, *then its transpose* ${}^t\mathcal{O} \colon \left[Exp^k_{\tilde{N},(s;(r,q)),0,A}(E)\right]' \longrightarrow \left[Exp^k_{\tilde{N},(s;(r,q)),0,A}(E)\right]'$ *is such that*
(b.1) ${}^t\mathcal{O}\left(\left[Exp^k_{\tilde{N},(s;(r,q)),0,A}(E)\right]'\right)$ *is the orthogonal of* $\ker \mathcal{O}$ *in* $\left[Exp^k_{\tilde{N},(s;(r,q)),0,A}(E)\right]'$.
(b.2) ${}^t\mathcal{O}\left(\left[Exp^k_{\tilde{N},(s;(r,q)),0,A}(E)\right]'\right)$ *is closed for the weak-star topology in* $\left[Exp^k_{\tilde{N},(s;(r,q)),0,A}(E)\right]'$ *defined by* $Exp^k_{\tilde{N},(s;(r,q)),0,A}(E)$.

An analogous result is valid for $\mathcal{O} \in \mathcal{A}^k_{\tilde{N},(s;(r,q))}$, $\mathcal{O} \neq 0$ and for $\mathcal{O} \in \mathcal{A}^k_{\tilde{N},(s;(r,q)),B}$, $\mathcal{O} \neq 0$ of type zero, with $B \in (0, +\infty)$.

The last result of this article is a theorem about existence of solution of convolution equations. In order to prove this result we need the following Dieudonné-Schwartz result (see [13, p. 308]).

Lemma 4.4. *If* E *and* F *are Fréchet spaces and* $u \colon E \longrightarrow F$ *is a linear continuous mapping, then the following conditions are equivalent:*
(a) $u(E) = F$;
(b) ${}^tu \colon F' \longrightarrow E'$ *is injective and* ${}^tu(F')$ *is closed for the weak-star topology of* E' *defined by* E.

Theorem 4.5. (a) *For* $k \in [1, +\infty]$, *if* $\mathcal{O} \in \mathcal{A}^k_{\tilde{N},(s;(r,q)),0}$, $\mathcal{O} \neq 0$, *then*
$\mathcal{O}\left(Exp^k_{\tilde{N},(s;(r,q)),0}(E)\right) = Exp^k_{\tilde{N},(s;(r,q)),0}(E)$.
(b) *For* $k \in [1, +\infty]$ *and* $A \in (0, +\infty)$, *if* $\mathcal{O} \in \mathcal{A}^k_{\tilde{N},(s;(r,q)),0,A}$ *is of type zero and* $\mathcal{O} \neq 0$, *then* $\mathcal{O}\left(Exp^k_{\tilde{N},(s;(r,q)),0,A}(E)\right) = Exp^k_{\tilde{N},(s;(r,q)),0,A}(E)$.

Proof. By [11, Propositions 2.6 and 2.12], $Exp^k_{\tilde{N},(s;(r,q)),0}(E)$ and $Exp^k_{\tilde{N},(s;(r,q)),0,A}(E)$ are Fréchet spaces. By Lemma 4.4(b) and by Theorem 4.3

items (a.2) and (c.2), it is enough to show that ${}^t\mathcal{O}$ is injective. Since $\mathcal{O} = T*$ for some T in the domain of ${}^t\mathcal{O}$, then for all S in the domain of ${}^t\mathcal{O}$ and f in the domain of $\mathcal{O}$ we have $\left({}^t\mathcal{O}\mathcal{S}\right)(f) = S(\mathcal{O}f) = S(T*f) = (S*T)(f)$. Thus ${}^t\mathcal{O}\mathcal{S} = S*T$ and if ${}^t\mathcal{O}\mathcal{S} = 0$, then $S*T = 0$ and $F(S*T) = 0$. Since $\mathcal{O} \neq 0$, it follows that $T \neq 0$ and $FT \neq 0$ and since $F(S*T) = FS.FT$, we get $FS = 0$. Hence $S = 0$ and ${}^t\mathcal{O}$ is injective.

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References

[1] P. J. Boland, *Malgrange theorem for entire functions on nuclear spaces*. Proceedings on Infinite Dimensional Holomorphy. Lecture Notes in Math. **364**, Springer Verlag (1974), 135-144.

[2] G. Botelho, *Weakly compact and absolutely summing polynomials*. J. Math. Anal. Appl. **265** (2002), 458-462.

[3] G. Botelho, D. Pellegrino, *Scalar-valued dominated polynomials on Banach spaces*. Proc. Amer. Math. Soc. **134** (2006), no.6, 1743-1751 (electronic).

[4] R. Cilia, J. M. Gutiérrez. *Dominated, diagonal polynomials on ℓ_p spaces*. Arch. Math. (Basel) **84** (2005), no. 5, 421-431.

[5] J. F. Colombeau, M. C. Matos. *Convolution equations in spaces of infinite dimensional entire functions*. Indag. Math., **42** (1980), 375-389.

[6] J.F Colombeau, B. Perrot, *Convolution equations in spaces of infinite dimensional entire functions of exponencial and related types*. Trans. Amer. Math. Soc. **258** (1980), 191-198.

[7] S. Dineen, *Holomorphy types on a Banach space*. Studia Math. **39** (1971), 241-288.

[8] J. Diestel, H. Jarchow, A. Tonge, *Absolutely Summing Operators*. Cambridge Univ. Press, Cambridge, 1995.

[9] T.A.W. Dwyer III, *Convolution equations for vector-valued entire functions of nuclear bounded type*. Trans. Amer. Math. Soc. **217** (1976), 105-119.

[10] T.A.W. Dwyer III, *Partial differential equations in Fischer-Fock spaces for the Hilbert-Schmidt holomorphy type*. Bull. Amer. Math. Soc. **77** (1971), 725-730.

[11] V.V. Fávaro, *The Fourier-Borel transform between spaces of entire functions of a given type and order*. Port. Math. **65** (2008), 285-309.

[12] C. Gupta, *Convolution Operators and Holomorphic Mappings on a Banach Space*. Séminaire d'Analyse Moderne, 2. Université de Sherbrooke. Sherbrooke, 1969.

[13] J. Horváth, *Topological Vector Spaces and Distribuitions*. Addison-Wesley, Reading, MA, 1966.

[14] B. Malgrange, *Existence et approximation des équations aux dérivées partielles et des equations des convolutions*. Ann. Inst. Fourier (Grenoble)**6** (1955/56), 271-355.

[15] A. Martineau, *Équations différentielles d'ordre infini*. Bull. Soc. Math. France **95** (1967), 109-154.

[16] M. C. Matos, *On Malgrange Theorem for nuclear holomorphic functions in open balls of a Banach space*. Math. Z. **171** (1978), 171-290.

[17] M. C. Matos, *On the Fourier-Borel transformation and spaces of entire functions in a normed space*, in: Functional Analysis, Holomorphy and Approximation Theory II (G. I. Zapata, ed.), North-Holland Math. Studies, North-Holland, Amsterdam, 1984, pp. 139-170.

[18] M. C. Matos, *On convolution operators in spaces of entire functions of a given type and order*, in: Complex Analysis, Functional Analysis and Approximation Theory (J. Mujica, ed.), North-Holland Math. Studies **125**, North-Holland, Amsterdam, 1986, pp. 129-171.

[19] M. C. Matos, *Absolutely summing holomorphic mappings*. An. Acad. Brasil. Ciênc. **68** (1996), 1-13.

[20] M.C. Matos, *Nonlinear absolutely summing mappings*. Math. Nachr. **258** (2003), 71-89.

[21] M. C. Matos, *Absolutely Summing Mappings, Nuclear Mappings and Convolution Equations*. IMECC-UNICAMP, 2007. Web: http://www.ime.unicamp.br/~matos

[22] M. C. Matos, L. Nachbin, *Entire functions on locally convex spaces and convolution operators*. Comp. Math. **44** (1981), 145-181.

[23] L. Nachbin, *Topology on Spaces of Holomorphic Mappings*. Springer, New York, 1969.

[24] D. Pellegrino, *Cotype and absolutely summing homogeneous polynomials in $\mathcal{L}_p$-spaces*. Studia. Math. **157** (2003), no. 2, 121-131.

[25] D. Pellegrino, *On scalar-valued nonlinear absolutely summing mappings*. Ann. Polon. Math. **83** (2004), 281-288.

[26] D. Pérez-García, I. Villanueva, *Multiple summing operators on Banach spaces*. J. Math. Anal. Appl. **285** (2003), 86-96.

[27] A. Pietsch, *Operator Ideals*. Deutscher Verlag, Berlin, 1979 and North-Holland Publishing, North-Holland, 1980.

[28] A. Pietsch, *Ideals of multilinear functionals*. In: Proceedings of the Second International Conference on Operator Algebras, Ideals and Their Applications in Theoretical Physics, pp.185-199, Teubner, 1983.

Faculdade de Matemática, Universidade Federal de Uberlândia,
38.400-902 - Uberlândia, Brazil,
e-mail: `vvfavaro@gmail.com`.