Meets of spatial sublocales*

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Abstract

It is well known that the functor $\Omega : \mathbf{Sp} \to \mathbf{Loc}$ does not preserve meets of subspaces. More generally, the meet of a family of spatial sublocales of a locale is in general not spatial. In this paper, we will give a characterization for those topological spaces for which the functor Ω preserves meets of subspaces. As a corollary, we give some characterizations for the meet of some spatial sublocales of a locale to be spatial.

1 Introduction and preliminaries

Recall that a frame A is a complete lattice satisfying the infinite distributive law $a \wedge \bigvee S = \bigvee\{a \wedge s | s \in S\}$ for all $a \in A$ and $S \subseteq A$. Let A, B be frames, $f : A \to B$ is a frame morphism if f preserves arbitrary joins and finite meets. We write **Frm** for the category of frames and frame morphisms and **Loc** for its dual category whose objects are extensionally the same thing, whose morphisms go in the opposite direction, and write $\mathcal{O}(X)$ for the corresponding frame of a locale X.

A sublocale of a locale X is defined to be a regular subobject of X in **Loc**, i.e., the locale corresponding to a regular quotient of $\mathcal{O}(X)$. We write $\mathrm{Sub}(X)$ for the lattice of sublocales of X. We say a sublocale is dense if its closure is the whole of X. Every locale has a smallest dense sublocale X_b, defined by setting $\mathcal{O}(X_b) = (\mathcal{O}(X))_{\neg\neg}$, where $(\mathcal{O}(X))_{\neg\neg}$ denotes the frame of all $\neg\neg$-fixed elements of $\mathcal{O}(X)$. For more details of locales please refer to [1], [2], [3].

We know the functor $\Omega : \mathbf{Sp} \to \mathbf{Loc}$ from the category of topological spaces to the category of locales has a right adjoint $Pt : \mathbf{Loc} \to \mathbf{Sp}$, the spectrum functor. Hence Ω preserves colimits. But in general Ω does not preserve limits, so

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the question that under what conditions the functor Ω preserves some type of limits is very interesting. Many authors have investigated the question. For the preservation of products see J.R. Isbell [4], for the preservation of directed inverse limits see He and Liu [5] and He and Plewe [6]. In categorical sense, the intersection of sublocales (subspaces) can be considered as a special limit. In this paper we investigate the question of preservation of meets of subspaces. For a general background on category theory, we refer to [7], [8].

Let A be a locale. A map $j : A \to A$ is called a nucleus if it satisfies (1) : $j(a \wedge b) = j(a) \wedge j(b)$, (2) : $a \leq j(a)$, (3) : $j(j(a)) \leq j(a)$ for all $a, b \in A$. The partially ordered set $N(A)$ of all nuclei under pointwise partial order is dual to the poset $\mathrm{Sub}(A)$ of all sublocales of A under inclusion order.

We begin with some lemmas, the first of which presents an explicit description of the sublocale induced by a subspace inclusion.

Lemma 1.1. *Let X be a topological space and $Y \rightarrowtail X$ a subspace of X. Then the nucleus j induced by the subspace inclusion satisfies $j(U) = int(U \cup (X \setminus Y))$ for any $U \in \Omega(X)$.*

Proof. Let $i : Y \hookrightarrow X$ be the inclusion map, then $i^{-1} = i^* : \Omega(X) \to \Omega(Y)$, so

$$\begin{aligned} j(U) &= i_* i^*(U) = i_*(U \cap Y) \\ &= \cup\{W \in \Omega(X) | i^*(W) \subseteq U \cap Y\} = \cup\{W \in \Omega(X) | W \cap Y \subseteq U \cap Y\} \\ &= \cup\{W \in \Omega(X) | W \cap Y \subseteq U\} = int(U \cup (X \setminus Y)) = int(U \cup Y^c). \end{aligned}$$ ∎

The following two examples show that the functor Ω does not preserve the intersection of subspaces in general.

Example 1.1. *Let $X = \{1, 2, 3\}$, $\Omega(X) = \{\emptyset, X, \{1, 2\}\}$, $Y_1 = \{1\}$, $Y_2 = \{2\}$. Then $\Omega(Y_1) \cong \Omega(Y_2)$ is the two-element frame, but $Y_1 \cap Y_2 = \emptyset$. Thus $\Omega(Y_1) \cap \Omega(Y_2) \neq \Omega(Y_1 \cap Y_2)$.*

Example 1.2. *Let $X = \mathbb{R}$ equipped with the usual topology and $Y_1 = \mathbb{Q}$, $Y_2 = \mathbb{Q}^c$. Then the induced sublocales $\Omega(Y_1) = \{U \in \Omega(X) | U = int(U \cup Y_1^c)\}$, $\Omega(Y_2) = \{U \in \Omega(X) | U = int(U \cup Y_2^c)\}$. Since all the prime elements of $\Omega(X)$ have the form $X - \{x\}$ for some $x \in X$, so $\Omega(Y_1) \cap \Omega(Y_2)$ has no prime element. Thus $\Omega(Y_1) \cap \Omega(Y_2) \neq \Omega(Y_1 \cap Y_2)$.*

2 Conditions for Ω preserving intersections of subspaces

Let X be a space and $Y_1, Y_2 \subseteq X$ be two subspaces. It is clear that $\Omega(Y_1 \cap Y_2) \subseteq \Omega(Y_1) \cap \Omega(Y_2)$. So, by Lemma 1.1, the question that the functor Ω preserves finite intersections of subspaces can be considered to give conditions for a topological space X for which

$$U = int(U \cup A) \text{ and } U = int(U \cup B) \text{ imply } U = int(U \cup A \cup B) \qquad (*)$$

holds for every open set U and any two subspaces A, B of X.

Example 2.1. *Let $X = \{1, 2, 3\}$, $\Omega(X) = \{\emptyset, X, \{1, 2\}, \{2\}, \{2, 3\}\}$. Then for any $U \in \Omega(X)$, and $A, B \subseteq X$, $U = int(U \cup A)$, $U = int(U \cup B) \Rightarrow U = int(U \cup A \cup B)$.*

This example shows that a space satisfying the condition $(*)$ is in general not trivial.

Lemma 2.1. *Let X be a topological space. Then every open set of X is regular open if and only if every open set is closed.*

Lemma 2.2. *Let X be a topological space. If U is an open set and $A \subseteq X \setminus cl(U)$, then $int(U \cup A) = U \cup int(A)$.*

Lemma 2.3. *Let X be a topological space and U an open set of X. If $int(cl(U)) \setminus U \neq \emptyset$, write $T = int(cl(U)) \setminus U$. For every $A \subseteq T$, $U = int(U \cup A)$ if and only if $T \setminus A$ is dense in T.*

Proof.

($\Leftarrow$): Suppose $U \neq int(U \cup A)$. Clearly, $int(U \cup A) \setminus U \neq \emptyset$, so there exists $x \in int(U \cup A), x \notin U$, so $x \in A$. Since $int(U \cup A) \cap T \subseteq (U \cup A) \cap T = A$, then $int(U \cup A) \cap T = int(U \cup A) \cap A$. Thus $int(U \cup A) \cap A$ is a non-empty open set in the subspace T. So $T \setminus A$ cannot be dense in T.

($\Rightarrow$): Suppose $T \setminus A$ is not dense in T when T equipped with the subspace topology, then there exists a non-empty open set $O \in \Omega(T)$, $O \subseteq A$. Hence there exists an open set $G \in \Omega(X)$, such that $O = G \cap T$. Let $G_1 = G \setminus (X \setminus (int(cl(U))))$, then G_1 is open, $G_1 \cap T = O$ and $G_1 \subseteq int(cl(U))$. Furthermore $G_1 \subseteq U \cup G_1 = U \cup O$, hence $U \cup O$ is open. Then $U \subset U \cup O = int(U \cup O) \subseteq int(U \cup A)$. This implies $U \neq int(U \cup A)$. ∎

Lemma 2.4. *Let X be a topological space. Then the following conditions are equivalent:*

(1) $U = int(U \cup A) = int(U \cup B) \Rightarrow U = int(U \cup A \cup B)$ *for every $U \in \Omega(X)$ and $A, B \subseteq X$.*

(2) *The intersection of two dense subspaces of X is dense in X, and for each open set V, either V is regular open or every intersection of two dense subspaces of $int(cl(V)) \setminus V$ is dense in $int(cl(V)) \setminus V$.*

Proof.

($\Rightarrow$): Suppose there exist two dense sets F, G with $F \cap G$ not dense. Clearly, $int(X \setminus F) = X \setminus cl(F) = \emptyset$ and similarly $int(X \setminus G) = \emptyset$,

thus

$int((X \setminus F) \cup (X \setminus G)) = int(X \setminus (F \cap G)) = X \setminus cl(F \cap G) \neq \emptyset$, since $F \cap G$ is not dense.

Let $U = \emptyset$, $A = X \setminus F$ and $B = X \setminus G$, then

$U = \emptyset = int(U \cup A), U = \emptyset = int(U \cup B)$, and $U = \emptyset \neq int(U \cup A \cup B)$,

which leads to a contradiction.

Suppose there exists an open set V which is not regular open, then $int(cl(V)) \setminus V \neq \emptyset$. Let A, B be dense in $T = int(cl(V)) \setminus V$, then $V = int(V \cup (T \setminus A)), V = int(V \cup (T \setminus B))$ by Lemma 2.3. Hence, by the hypothesis (1), $V = int(V \cup ((T \setminus A) \cup (T \setminus B))) = int(V \cup (T \setminus (A \cap B)))$, thus $A \cap B$ is dense in T by Lemma 2.3.

($\Leftarrow$): Suppose $int(U \cup A \cup B) \setminus U \neq \emptyset$, then there exists a point $x \in A \cup B$ and an open set O such that $x \in O \subseteq U \cup A \cup B$ and $x \notin U$. We have the following two cases:

case 1: $x \in (A \cup B) \setminus int(cl(U))$

If $O \cap (X \setminus cl(U)) \neq \emptyset$. Since $U = int(U \cup A)$, then $int(A \setminus U) = \emptyset$, we have $X \setminus (A \setminus U)$ is dense. Similarly we have $X \setminus (B \setminus U)$ is dense. Thus $(X \setminus (A \setminus U)) \cap (X \setminus (B \setminus U)) = X \setminus ((A \cup B) \setminus U)$ is dense. But since $O \cap (X \setminus cl(U))$ is a non-empty open set and satisfies $O \cap (X \setminus cl(U)) \subseteq (A \cup B) \setminus U$, we have a contradiction.

If $O \subseteq cl(U)$, then clearly, $O \subseteq int(cl(U))$, we prove it in case 2.

case 2: $x \in int(cl(U)) \setminus U$. Let $T = int(cl(U)) \setminus U$. Since $U = int(U \cup A) = int(U \cup (A \cap T))$, then $T \setminus (A \cap T)$ is dense in T by Lemma 2.3. Similarly, $T \setminus (B \cap T)$ is dense in T. Thus $(T \setminus (A \cap T)) \cap (T \setminus (B \cap T)) = T \setminus ((A \cup B) \cap T)$ is dense. But $O \cap T$ is a non-empty open set in the subspace T and satisfies $O \cap T \subseteq (A \cup B) \cap T$, which leads to a contradiction.

By the above argument we know that $U = int(U \cup A \cup B)$, which completes the proof. ∎

It should be noted that the first condition in statement (2) is equivalent to saying that (1) holds for $U = \emptyset$, and the second condition in (2) is equivalent to (1) holding for $U \neq \emptyset$.

Corollary 2.1. *Let X be a topological space. If X satisfies*

(1) *the intersection of two dense sets is dense and*

(2) *every open set is closed,*

then $U = int(U \cup A) = int(U \cup B) \Rightarrow U = int(U \cup A \cup B)$ for every $U \in \Omega(X)$ and $A, B \subseteq X$.

Proof. By Lemma 2.1 and Lemma 2.4, it is obvious. ∎

The following example shows that a space satisfying the conditions of Corollary 2.1 can be non-trivial.

Example 2.2. *Let $\mathbb{R}$ be the set of real numbers and $\mathcal{B} = \{(a,b) \cap \mathbb{Q} | a, b \in \mathbb{Q}^c\} \cup \{\{x\} | x \in \mathbb{Q}^c\}$. Then $\mathcal{B}$ generates a topology τ on $\mathbb{R}$. It is straightforward to prove that the space $(\mathbb{R}, \tau)$ satisfies the conditions of Corollary 2.1.*

Now we can get our main result.

Theorem 2.1. *Let X be a topological space. Then $\Omega(Y_1 \cap Y_2) \cong \Omega(Y_1) \cap \Omega(Y_2)$ for any two subspaces Y_1, Y_2 of X if and only if X satisfies the equivalent conditions of Lemma 2.4. Therefore if X satisfies the equivalent conditions of Lemma 2.4, then $\Omega(Y_1) \cap \Omega(Y_2)$ is spatial for any two subspaces Y_1, Y_2 of X.*

Proof.

($\Rightarrow$): Suppose $\Omega(Y_1) \cap \Omega(Y_2) \cong \Omega(Y_1 \cap Y_2)$ is satisfied for any two subspaces Y_1, Y_2 of X, then $U = int(U \cup Y_1^c), U = int(U \cup Y_2^c) \Rightarrow U = int(U \cup Y_1^c \cup Y_2^c)$ is satisfied for every $U \in \Omega(X)$ and any two subspaces Y_1, Y_2 of X.

($\Leftarrow$): Let $U \in \Omega(Y_1) \cap \Omega(Y_2)$, then $U = int(U \cup Y_1^c) = int(U \cup Y_2^c)$. By Lemma 2.4, we then have $U = int(U \cup Y_1^c \cup Y_2^c)$, hence $U \in \Omega(Y_1 \cap Y_2)$, so $\Omega(Y_1) \cap \Omega(Y_2) \subseteq \Omega(Y_1 \cap Y_2)$. ∎

Corollary 2.2. *Let X be a topological space. If X satisfies the condition of Corollary 2.1, then $\Omega(Y_1 \cap Y_2) \cong \Omega(Y_1) \cap \Omega(Y_2)$ for any two subspaces Y_1, Y_2 of X .*

It is well known that a complemented sublocale of a spatial locale is spatial (see [2]). However, not every spatial sublocale of a spatial locale is complemented.

Example 2.3. *Let $X = \mathbb{R}, S = \mathbb{Q}$, then S and $X \setminus S$ are dense in X, so any sublocale T of X satisfying $S \cup T = X$ must contain all the points of $X \setminus S$, then both S and T are dense and thus $X_b \subseteq S \cap T$. Hence S does not have a complement in $Sub(X)$.*

Now, we give a sufficient condition for a spatial sublocale of a spatial locale to be complemented.

Proposition 2.1. *Let X be a spatial locale and Y a sublocale of X. If X satisfies the equivalent conditions of Lemma* 2.4*, then Y is complemented if and only if Y is spatial.*

Proof. We only need to show the sufficiency.

($\Leftarrow$): Since X is a spatial locale, then $X \cong \Omega Pt(X)$. Denote the complement subspace of $Pt(Y)$ as Y', so

$Pt(Y) \cup Y' = Pt(X)$, $Pt(Y) \cap Y' = \varnothing$

In categorical sense, the unions of subspaces can be considered as coproducts, and the left adjoint Ω preserves colimits, so

$\Omega(Pt(Y) \cup Y') = \Omega(Pt(Y)) \cup \Omega(Y') = Y \cup \Omega(Y') = \Omega(Pt(X)) = X$.

By Theorem 2.1, $\Omega(Pt(Y) \cap Y') = \Omega(Pt(Y)) \cap \Omega(Y') = Y \cap \Omega(Y') = \Omega(\varnothing)$ is the least sublocale of X.

Thus Y is a complemented sublocale of X, with complement $\Omega(Y')$. ∎

The binary case can be generalized to the case of an arbitrary family of subspaces of X.

Lemma 2.5. *Let X be a topological space. Then the following conditions are equivalent:*

(1) *$U = int(U \cup A_i)$, $i \in I$ implies $U = int(U \cup \bigcup_{i\in I} A_i)$ for any $U \in \Omega(X)$ and any family $\{A_i | i \in I\}$ of subspaces of X.*

(2) *The intersection of any family of dense subspaces of X is dense in X, and for each open set V, either V is regular open or every intersection of any family of dense subspaces of $int(cl(V)) \setminus V$ is dense in $int(cl(V)) \setminus V$.*

Corollary 2.3. *Let X be a topological space. If X satisfies*

(1) *the intersection of any family of dense subspaces of X is dense in X and*

(2) *every open set is closed,*

then $U = int(U \cup A_i)$, $i \in I$ implies $U = int(U \cup \bigcup_{i\in I} A_i)$ for any $U \in \Omega(X)$ and any family $\{A_i | i \in I\}$ of subspaces of X.

Theorem 2.2. *Let X be a topological space. Then $\Omega(\bigcap_{i\in I} Y_i) \cong \bigcap_{i\in I} \Omega(Y_i)$ for any family $\{Y_i | i \in I\}$ of subspaces of X if and only if X satisfies the equivalent conditions of Lemma* 2.5*. Therefore if X satisfies the conditions of Lemma* 2.5*, then $\bigcap_{i\in I} \Omega(Y_i)$ is spatial for any family $\{Y_i | i \in I\}$ of subspaces of X.*

Corollary 2.4. *Let X be a topological space. Then $\Omega(\bigcap_{i\in I} Y_i) \cong \bigcap_{i\in I} \Omega(Y_i)$ for any family $\{Y_i | i \in I\}$ of subspaces of X if X satisfies the conditions of Corollary* 2.3.

3 Some special cases

In this section, we consider the case whether for some special given subspaces Y_i, $i \in I$, of X, $\Omega(\bigcap_{i\in I} Y_i) = \bigcap_{i\in I}\Omega(Y_i)$ holds.

We first consider the case for Y_i to be open subspaces.

Lemma 3.1. *Let X be a topological space. If $A, B \subseteq X$ are two closed subsets, then $U = int(U \cup A)$ and $U = int(U \cup B)$ imply $U = int(U \cup A \cup B)$ for any open set U.*

Proof. Let $A_1 = A \cap int(cl(U))$, $A_2 = A \setminus int(cl(U))$, $A_3 = A \cap (int(cl(U)) \setminus U)$. and $B_1 = B \cap int(cl(U))$, $B_2 = B \setminus int(cl(U))$, $B_3 = B \cap (int(cl(U)) \setminus U)$. Clearly, A_2, B_2 are closed and $int(U \cup A_1) = int(U \cup A_3) = U$, $int(U \cup B_1) = int(U \cup B_3) = U$, $int(A_2) = int(B_2) = \emptyset$.

We first prove that $int(A_2 \cup B_2) = \emptyset$. Suppose $int(A_2 \cup B_2) \neq \emptyset$. Then there exists a point x and an open set V such that $x \in V \subseteq int(A_2 \cup B_2)$. We then have the following:

If $V \subseteq A_2$, then $int(A_2) \neq \emptyset$, which is a contradiction.

So, $(X \setminus A_2) \cap V \neq \emptyset$, and$(X \setminus A_2) \cap V \subseteq int(B_2)$, then $int(B_2) \neq \emptyset$, which is a contradiction also.

Now we show $int(U \cup A_1 \cup B_1) = U$.

Let $T = int(cl(U)) \setminus U$. Suppose $int(U \cup A_1 \cup B_1) \setminus U \neq \emptyset$. Then $int(U \cup A_1 \cup B_1) \cap T$ is a non-empty open subset in the subspace T. Clearly, $A_1 \cap T = A \cap T$ and $B_1 \cap T = B \cap T$. So, $A_1 \cap T$ and $B_1 \cap T$ are two closed subsets in the subspace T.

Note that $int(U \cup A_1 \cup B_1) \cap T$ is a non-empty open set in the subspace T.

If $int(U \cup A_1 \cup B_1) \cap T \subseteq A_1 \cap T$, then $int(A_1) \cap T$ is a non-empty open set in the subspace T and $int(A_1) \cap T \subseteq A_3$, so $T \setminus A_3$ is not dense in T. Then we have $int(U \cup A_3) \neq U$ by Lemma 2.3, which leads to a contradiction.

So, $(int(U \cup A_1 \cup B_1) \cap T) \cap (X \setminus (A_1 \cap T)) = (int(U \cup A_1 \cup B_1) \cap T) \cap (T \setminus A_1) \neq \emptyset$, then $(int(U \cup A_1 \cup B_1) \cap T) \cap (T \setminus A_1)$ is a non-empty open subset in the subspace T, which also leads to a contradiction by Lemma 2.3.

It suffices to prove $U = int(U \cup A \cup B)$. Suppose $int(U \cup A_1 \cup B_1 \cup A_2 \cup B_2) \setminus U \neq \emptyset$, then there exists a point $x \in A_2 \cup B_2$ and $x \in int(U \cup A_1 \cup B_1 \cup A_2 \cup B_2)$, thus there exists an open set O such that $x \in O \subseteq U \cup A_1 \cup B_1 \cup A_2 \cup B_2$, then $O \cap (A_3 \cup B_3)$ is a non-empty open set in the subspace T. Similar to the proof of the necessity part of Lemma 2.3, we have $U \cup (O \cap (A_3 \cup B_3))$ is open in X. So $U \subset U \cup (O \cap (A_3 \cup B_3)) = int(U \cup (O \cap (A_3 \cup B_3))) \subseteq int(U \cup A_1 \cup B_1)$, this implies $U \neq int(U \cup A_1 \cup B_1)$, and we have a contradiction. ∎

Proposition 3.1. *Let X be a topological space and Y_1, Y_2 be two open subspaces of X. Then $\Omega(Y_1 \cap Y_2) \cong \Omega(Y_1) \cap \Omega(Y_2)$.*

Corollary 3.1. *Let X be a locale and Y_1, Y_2 be two open spatial sublocales of X. Then $Y_1 \cap Y_2$ is spatial.*

Lemma 3.1 cannot be generalized to an arbitrary family of closed sets as the following example shows.

Example 3.1. *Let $\mathbb{R}$ be the set of real numbers endowed with the usual topology. Then all single point sets $\{x\}$, are closed. Let $U = \emptyset$. Then $U = int(U \cup \{x\})$, $x \in \mathbb{R}$. But $\emptyset = U \neq int(U \cup \bigcup_{x\in\mathbb{R}} \{x\}) = \mathbb{R}$.*

But for a family of locally finite closed sets, we have the following:

Lemma 3.2. *Let X be a topological space. If $\{A_i | i \in I\}$ is a family of locally finite closed sets, then $U = int(U \cup A_i), i \in I$ implies $U = int(U \cup \bigcup_{i\in I} A_i)$ for any open set U.*

Proof. Let $T = int(cl(U)) \setminus U$, $A_i^1 = A_i \cap (X \setminus int(cl(U))), A_i^2 = A_i \cap T, i \in I$. Then $\{A_i^1 | i \in I\}$ is a family of locally finite closed sets. Since $U = int(U \cup A_i), i \in I$, then $int(A_i^1) = \emptyset$ for each $i \in I$.

We first prove $int(\bigcup_{i\in I} A_i^1) = \emptyset$. Suppose $int(\bigcup_{i\in I} A_i^1) \neq \emptyset$, then there exists a point $x \in int(\bigcup_{i\in I} A_i^1)$. Since $\{A_i^1 | i \in I\}$ is a family of locally finite closed sets, then exists an open set U_x, such that $x \in U_x$ and U_x intersects with finitely many members of $\{A_i^1 | i \in I\}$. We denote these members by $A_j, j = 1, 2, \ldots, k$. Let $U_x^{'} = U_x \cap int(\bigcup_{i\in I} A_i^1)$, then, clearly, $x \in U_x^{'}$ and $U_x^{'} \subseteq \bigcup_{j=1}^{k} A_j$. Thus $int(\bigcup_{j=1}^{k} A_j) \neq \emptyset$. But from the proof of Lemma 3.1 and using induction, we have $int(\bigcup_{j=1}^{k} A_j) = \emptyset$, which leads to a contradiction. So, we have $int(\bigcup_{i\in I} A_i^1) = \emptyset$.

Since locally finiteness is a hereditary property, similar to the above proof, we then have $int_T(\bigcup_{i\in I} A_i^2) = \emptyset$, where int_T denotes the interior operation in the subspace T.

The rest is similar to the proof of the sufficiency part of Lemma 2.4. ■

Proposition 3.2. *Let X be a topological space and $\{Y_i | i \in I\}$ be a family of open subspaces. If $\{Y_i^c | i \in I\}$ is locally finite, then $\Omega(\bigcap_{i\in I} Y_i) \cong \bigcap_{i\in I} \Omega(Y_i)$.*

Corollary 3.2. *Let X be a locale and $\{Y_i | i \in I\}$ be a family of open spatial sublocales such that $\{Y_i^c | i \in I\}$ is locally finite. Then $\bigcap_{i\in I} Y_i$ is spatial.*

Now we consider the case for closed subspaces.

Proposition 3.3. *Let X be a topological space. Then $\Omega(\bigcap_{i\in I} Y_i) \cong \bigcap_{i\in I} \Omega(Y_i)$ for any family $\{Y_i | i \in I\}$ of closed subspaces of X.*

Proof. Let $\{Y_i | i \in I\}$ be any family of closed subspaces of X. If $U \in \bigcap_{i\in I} \Omega(Y_i)$, then $U = int(U \cup Y_i^c)$, $i \in I$, we have $Y_i^c \subseteq U$, $i \in I$ since Y_i^c is open for each $i \in I$. So $\bigcup_{i\in I} Y_i^c \subseteq U$, thus $U = int(U \cup \bigcup_{i\in I} Y_i^c)$, then we have $U \in \Omega(\bigcap_{i\in I} Y_i)$, hence $\bigcap_{i\in I} \Omega(Y_i) \subseteq \Omega(\bigcap_{i\in I} Y_i)$. The rest is obvious. ■

Corollary 3.3. *Let X be a locale and $\{Y_i | i \in I\}$ be any family of closed spatial sublocales. Then $\bigcap_{i \in I} Y_i$ is spatial.*

Proposition 3.4. *Let X be a topological space. Then $\Omega(Y \cap \bigcap_{i \in I} Y_i) \cong \Omega(Y) \cap \bigcap_{i \in I} \Omega(Y_i)$ for any family $\{Y_i | i \in I\}$ of closed subspaces of X and any open subspace Y of X.*

Corollary 3.4. *Let X be a locale. Then $Y \cap \bigcap_{i \in I} Y_i$ is spatial for any family $\{Y_i | i \in I\}$ of closed spatial sublocales of X and any open spatial sublocale Y of X.*

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