A sharp weighted Wirtinger inequality and some related functional spaces

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Abstract

We consider the generalized Wirtinger inequality

$$\left(\int_0^T a|u|^q\right)^{1/q} \leq C\left(\int_0^T a^{1-p}|u'|^p\right)^{1/p},$$

with $p, q > 1$, $T > 0$, $a \in L^1[0,T]$, $a \geq 0$, $a \not\equiv 0$ and where u is a T-periodic function satisfying the constraint

$$\int_0^T a|u|^{q-2}u = 0.$$

We provide the best constant $C > 0$ as well as all extremals. Furthermore, we characterize the natural functional space where the inequality is defined.

1 Introduction and main results

Wirtinger type inequalities are of interest in various areas of analysis and mathematical physics, including the Wulff theorem [3], quasiconformal mapping theory [8], p-Laplacian systems [11]. In view of their applications, they received a considerable attention in recent years. See, e.g., [1, 2, 4, 9, 14] and the references therein.

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In reference [14], the following weighted Wirtinger inequality is considered:

$$\int_0^{2\pi} au^2 \le C \int_0^{2\pi} a^{-1}u'^2, \tag{1}$$

where $a \ge 0$ and u is a 2π-periodic function satisfying $\int_0^{2\pi} au = 0$. By a technique introduced in [13], the sharp value of the constant $C > 0$ is a key ingredient in [14], which is used in order to obtain a sharp Hölder estimate for two-dimensional, divergence-form, elliptic equations with unit determinant coefficient matrix. Such equations are closely related to quasiconformal mappings and Beltrami equations, see [8, 15]. In this context, the coefficient a, which is related to the coefficient matrix of the elliptic equation, naturally satisfies the assumption $a, a^{-1} \in L^\infty[0,2\pi]$. By the diffeomorphism $y = 2\pi(\int_0^{2\pi} a)^{-1} \int_0^x a(t)\,dt$, inequality (1) reduces to the case $a \equiv 1$, which is the standard Wirtinger inequality. Thus, we obtain $C = \left((2\pi)^{-1}\int_0^{2\pi} a\right)^2$. Moreover, the extremals are given by $u(x) = \alpha \sin\left(\widetilde{a}^{-1}\int_0^x a\,dt + \delta\right)$ for some $\alpha \neq 0$ and $\delta \in \mathbb{R}$, where $\widetilde{a} = (2\pi)^{-1}\int_0^{2\pi} a\,dt$. Such a value of C suggests that inequality (1) should hold true in the more general case $a \in L^1(0,2\pi)$. This fact was established, among other results, in [4]. Furthermore, in [5] the best constant $C > 0$ in the more general inequality

$$\int_0^{2\pi} a|u|^p \le C \int_0^{2\pi} a^{1-p}|u'|^p \tag{2}$$

is given. Here, $a \in L^1(0,2\pi)$, $a \ge 0$, $a \not\equiv 0$ and u is a 2π-periodic function satisfying $\int_0^{2\pi} a|u|^{p-2}u = 0$. We note that the proofs in [4, 5] are based on an approximation argument involving truncations, which does not allow to characterize the extremals.

In this note we consider the following generalized Wirtinger inequality

$$\left(\int_0^T a|u|^q\right)^{1/q} \le C\left(\int_0^T a^{1-p}|u'|^p\right)^{1/p} \tag{3}$$

where $p, q > 1$, $T > 0$, $a \ge 0$, $a \not\equiv 0$, $a \in L^1[0,T]$ and where u is a T-periodic function satisfying $\int_0^T a|u|^{q-2}u = 0$. When $p = q$, (3) reduces to (2). We note that, although the underlying reason for which inequality (3) holds is a rescaling argument, the rigorous analysis is not obvious under our general assumptions on a. Indeed, a may vanish on a set of positive measure. In fact, one of our problems is to identify a natural space $\mathcal{X}$ where inequality (3) is defined. In the space $\mathcal{X}$ we can characterize all extremals in terms of generalized trigonometric functions [7, 9, 10, 11]. Then, we show that $\mathcal{X}$ includes the natural weighted Sobolev spaces defined in the usual way, by approximation by smooth functions or by distributional derivatives. We show that for some particular choices of a, such weighted Sobolev spaces may be strictly included in $\mathcal{X}$.

More precisely, we define:

$$\mathcal{X} = \left\{u : [0,T] \to \mathbb{R} :\ \exists U \in W^{1,p}_{\mathrm{per}}(\mathbb{R}) \text{ such that } u(x) = U\left(\frac{T}{\int_0^T a}\int_0^x a\right)\right\},$$

where $W^{1,p}_{\mathrm{per}}(\mathbb{R}) = \{u \in W^{1,p}(\mathbb{R}) :\ u \text{ is } T\text{−periodic}\}$. With this notation, we have:

Theorem 1.1. *Let $p,q > 1$ and $T > 0$. Let $a \in L^1[0,T]$, $a \geq 0$, $a \not\equiv 0$. Then, the following Wirtinger inequality holds:*

$$\left(\int_0^T a|u|^q \right)^{1/q} \leq \widetilde{a}^{1/p^*+1/q} C(p,q) \left(\int_0^T a^{1-p}|u'|^p \right)^{1/p} \tag{4}$$

for every $u \in \mathcal{X}$ such that

$$\int_0^T a|u|^{q-2}u = 0,$$

where $\widetilde{a} = T^{-1} \int_0^T a$ and

$$C(p,q) = \left[2\left(\frac{1}{p^*}\right)^{1/q} \left(\frac{1}{q}\right)^{1/p^*} \left(\frac{2}{p^*+q}\right)^{1/p-1/q} B\left(\frac{1}{p^*}, \frac{1}{q}\right) \right]^{-1}. \tag{5}$$

Throughout this note, for every $p \geq 1$ we denote by $p^* = p/(p-1)$ the conjugate exponent of p. Moreover, $B(\alpha, \beta)$ denotes the Beta function

$$B(\alpha,\beta) = \int_0^1 t^{\alpha-1}(1-t)^{\beta-1}\, dt = B(\beta,\alpha)$$

for all $\alpha, \beta > 1$.

In Section 2 we show that $\mathcal{X}$ is a Banach space and we compare it to the weighted Sobolev spaces, as defined in the usual ways. More precisely, setting

$$\mathcal{A} = \left\{ u \in C^1(\mathbb{R}) : \ u \text{ is } T\text{−periodic and } \int_0^T a^{1-p}|u'|^p < +\infty \right\},$$

we show that $\|\cdot\|$ defined by

$$\|u\| = \left(\int_0^T a|u|^q\right)^{1/q} + \left(\int_0^T a^{1-p}|u'|^p\right)^{1/p}$$

is a norm on $\mathcal{A}$ (recall that we allow a to vanish on a set of positive measure). We define $\mathcal{H}$ as the closure of $\mathcal{A}$ with respect to $\|\cdot\|$. We show that $\mathcal{H} \subset \mathcal{X}$. Finally, we compare $\mathcal{X}$ with the space

$$\mathcal{W} = \left\{ u \in L^1[0,T] : \ \int_0^T a^{1-p}|u'|^p < +\infty \text{ and } u(0) = u(T) \right\},$$

which was considered in [4, 5]. We show that $\mathcal{W} = \mathcal{X}$, so that the assumption $u \in L^1[0,T]$, which is needed in order to define the distributional derivative of u, but does not seem natural in (3), is actually not restrictive. We also show that for particular choices of a, we may have $\mathcal{H} \neq \mathcal{W}$, unlike what happens in the usual Sobolev spaces, see [12]. It follows in particular that Theorem 1.1 holds for all functions belonging to the traditional weighted Sobolev spaces.

In order to characterize the extremals for (4), we use generalized trigonometric functions, which we now briefly define. See [7, 9, 10, 11] for more details. Let $p,q > 1$. The function $\arcsin_{pq} : [0,1] \to \mathbb{R}$ is defined by

$$\arcsin_{pq}(\sigma) = \int_0^\sigma \frac{ds}{(1-s^p)^{1/q^*}}.$$

Then, we have

$$\arcsin_{pq}(1) = \frac{1}{p} B\left(\frac{1}{p}, \frac{1}{q}\right) =: \frac{\pi_{pq}}{2}.$$

The function $\arcsin_{pq} : [0,1] \to [0, \frac{\pi_{pq}}{2}]$ is strictly increasing and its inverse function is denoted by $\sin_{pq}$. The function $\sin_{pq}$ is extended as an odd function to the interval $[-\pi_{pq}, \pi_{pq}]$ by setting $\sin_{pq}(t) = \sin_{pq}(\pi_{pq} - t)$ in $[\pi_{pq}/2, \pi_{pq}]$, $\sin_{pq}(t) = -\sin_{pq}(-t)$ in $[-\pi_{pq}, 0]$, and to the whole real axis as a $2\pi_{pq}-$periodic function. The function $\xi(t) = \sin_{qp^*}(\pi_{qp^*} t)$ is the unique solution of the initial value problem:

$$(|u'|^{p-2}u')' + \frac{q}{p^*}(|u|^{q-2}u) = 0, \qquad u(0) = 0, \quad u'(0) = 1$$

and it satisfies:

$$\begin{aligned} \frac{\|\xi'\|_p}{\|\xi\|_q} =& \inf\left\{ \frac{\|u'\|_p}{\|u\|_q} : \ u \in W^{1,p}(\mathbb{R}) \setminus \{0\},\ u \text{ is } 2-\text{periodic} \right. \\ & \left. \qquad\qquad \text{and } \int_{-1}^{1} |u|^{q-2}u = 0 \right\} \\ =& C^{-1}(p,q), \end{aligned} \tag{6}$$

where $C(p,q)$ is the constant defined in (5). Moreover, any minimizer for (6) is of the form $u(t) = \alpha\, \xi(t+\delta)$ for some $\alpha \neq 0$ and $\delta \in \mathbb{R}$. It should be mentioned that the existence of the minimum in (6) and the explicit value of $C(p,q)$ were obtained in [2] in a more general setting, and further generalized in [1].

At this point, we can state the sharpness of Theorem 1.1.

Theorem 1.2. *Let $u \in \mathcal{X}$. Then u satisfies inequality* (4) *with the equal sign if and only if u is of the form*

$$u(x) = \alpha \, \sin_{qp^*}\left(\pi_{qp^*} \frac{T}{\int_0^T a} \int_0^x a\, dt + \delta\right), \tag{7}$$

for some $\alpha \in \mathbb{R} \setminus \{0\}$ and for some $\delta \in \mathbb{R}$.

The remaining part of this note is devoted to the proofs of Theorem 1.1 and Theorem 1.2.

2 Some weighted Sobolev spaces

2.1 The space $\mathcal{W}$

The following space was considered in [4, 5]:

$$\mathcal{W} = \left\{ u \in L^1[0,T] : \ \int_0^T a^{1-p}|u'|^p < +\infty \text{ and } u(0) = u(T) \right\},$$

where u' denotes the distributional derivative of u. We first check that $\mathcal{W}$ is well-defined.

Lemma 2.1. *Let $a \in L^1[0,T]$ and let $f : [0,T] \to \mathbb{R}$ be a measurable function. Then,*

$$\int_0^T |f| \leq \left(\int_0^T |f|^p a^{1-p} \right)^{1/p} \left(\int_0^T a \right)^{(p-1)/p}.$$

Proof. The proof follows by the Hölder inequality. ∎

In view of Lemma 2.1, we have that $u' \in L^1[0,T]$ for all $u \in \mathcal{W}$. It follows that u is absolutely continuous. In particular, the periodicity condition $u(0) = u(T)$ is well-defined. When $p = q$, it was shown in [4, 5] that Theorem 1.1 holds for all $u \in \mathcal{W}$ by an approximation argument. A natural question is whether one can define a suitable functional space for inequality (4) which does not require $u \in L^1[0,T]$. Indeed, this condition is only used to define the distributional derivative of u, and does not seem natural for (4).

In what follows, we consider some other natural functional spaces.

2.2 The space $\mathcal{H}$

For $k \geq 0$, let $C^k_{\text{per}}(\mathbb{R})$ denote the space of C^k functions defined on $\mathbb{R}$ which are T-periodic. Let $s > 1$, $b \in L^1[0,T]$, $b \geq 0$ and $b \not\equiv 0$. Let $|\cdot|_{b,s}$ be the seminorm defined on $C_{\text{per}}(\mathbb{R})$ by

$$|u|_{b,s} = \left(\int_0^T b|u|^s \right)^{1/s}.$$

Let

$$\mathcal{A} = \left\{ u \in C^1_{\text{per}}(\mathbb{R}) : \int_0^T a^{1-p}|u'|^p < +\infty \right\}.$$

For each $u \in \mathcal{A}$ we define

$$\|u\|_{b,s} = |u|_{b,s} + \left(\int_0^T a^{1-p}|u'|^p \right)^{1/p}.$$

Lemma 2.2. *Let $a, b \in L^1[0,T]$, $a, b \geq 0$, $a, b \not\equiv 0$. Then, $\|\cdot\|_{b,s}$ is a norm on $\mathcal{A}$.*

Proof. We need only prove that $\|u\|_{b,s} = 0$ implies $u = 0$ for every $u \in \mathcal{A}$. To this end, we observe that $\|u\|_{b,s} = 0$ if and only if $\int_0^T b|u|^s = 0$ and $\int_0^T a^{1-p}|u'|^p = 0$. Then, since $a^{1-p} > 0$ a.e. in $[0,T]$, it results $u' = 0$ a.e. and therefore, since u' is continuous, $u' \equiv 0$. It follows that $u \equiv c \equiv$ const. From $\int_0^T b > 0$, we conclude that $\int_0^T b|u|^s = c^s \int_0^T b = 0$ and therefore $c = 0$. ∎

We denote by $\mathcal{H}_{b,s}$ the closure of $\mathcal{A}$ with respect to the norm $\|\cdot\|_{b,s}$.

Proposition 2.3. *Let $s, \widetilde{s} > 1$, $b, \widetilde{b} \in L^1(0,T)$, $b, \widetilde{b} \geq 0$ and $b, \widetilde{b} \not\equiv 0$. Then*

$$\mathcal{H}_{b,s} = \mathcal{H}_{\widetilde{b},\widetilde{s}} \subset L^\infty[0,T]$$

as sets of functions.

Proof. Let $u \in \mathcal{H}_{b,s}$. We note that $\int_0^T |u'| < \infty$. Indeed, by Lemma 2.1, we have:

$$\begin{aligned}\int_0^T |u'| \leq & \left(\int_0^T |u'|^p a^{1-p} \right)^{1/p} \left(\int_0^T a \right)^{(p-1)/p} \\ \leq & \|u\|_{b,s} \left(\int_0^T a \right)^{(p-1)/p} < +\infty.\end{aligned} \tag{8}$$

Let $u_n \in C^1_{\text{per}}(\mathbb{R})$ be a Cauchy sequence in $\mathcal{H}_{b,s}$. Then, $\int_0^T b|u_n - u_m|^s \to 0$ and $\int_0^T a^{1-p}|u_n' - u_m'|^p \to 0$, as $m, n \to \infty$. Furthermore, by (8) with $u = u_n - u_m$, we have

$$\int_0^T |u_n' - u_m'| \leq \left(\int_0^T a^{1-p}|u_n' - u_m'|^p \right)^{1/p} \left(\int_0^T a \right)^{(p-1)/p} \to 0.$$

Let

$$E = \{x \in [0, T] : \ b(x) > 0\}.$$

We observe that the seminorm $|\cdot|_{b,s}$ restricted to the functions defined on E defines an ordinary weighted Lebesgue space $L^s_b(E)$. Then $u_n|_E$ converges in $L^s_b(E)$ and there exists a subsequence u_{n_k} which converges a.e. in E. Let $x_0 \in E$ be such that $u_{n_k}(x_0)$ converges. By the fundamental theorem of calculus, for all $y \in [0, T]$ we have $u_{n_k}(y) = u_{n_k}(x_0) + \int_{x_0}^y u_{n_k}'$, $u_{n_h}(y) = u_{n_h}(x_0) + \int_{x_0}^y u_{n_h}'$ and consequently

$$u_{n_k}(y) - u_{n_h}(y) = u_{n_k}(x_0) - u_{n_h}(x_0) + \int_{x_0}^y (u_{n_k}' - u_{n_h}').$$

It follows that

$$\sup_{y\in[0,T]} |u_{n_k}(y) - u_{n_h}(y)| \leq |u_{n_k}(x_0) - u_{n_h}(x_0)| + \int_0^T |u_{n_k}' - u_{n_h}'| \to 0$$

as $h, k \to \infty$. Therefore u_{n_k} is a Cauchy sequence in $L^\infty[0, T]$ and there exists $v \in C_{\text{per}}(\mathbb{R})$ such that $u_{n_k} \to v$ in $L^\infty[0, T]$. We now prove that the whole sequence u_n converges to v in $L^\infty[0, T]$. To this end, let $u_{n_{h,1}}$ and $u_{n_{h,2}}$ be subsequences of u_n such that

$$u_{n_{h,1}} \to v_1 \qquad u_{n_{h,2}} \to v_2,$$

with $v_1, v_2 \in C[0, T]$. We have:

$$\begin{aligned}\left(\int_0^T b|u_{n_{h,1}} - u_{n_{h,2}}|^s\right)^{1/s} \geq & \left(\int_0^T b|v_1 - v_2|^s\right)^{1/s} - \\ & \left(\int_0^T b|u_{n_{h,1}} - v_1|^s\right)^{1/s} - \left(\int_0^T b|u_{n_{h,2}} - v_2|^s\right)^{1/s}.\end{aligned}$$

Since $\int_0^T b|u_{n_{h,1}} - u_{n_{h,2}}|^s \to 0$, $\int_0^T b|u_{n_{h,1}} - v_1|^s \to 0$, $\int_0^T b|u_{n_{h,2}} - v_2|^s \to 0$ we obtain $\int_0^T b|v_1 - v_2|^s \to 0$. That is, $\displaystyle\int_0^T b|v_1 - v_2|^s = 0$ and consequently $v_1 = v_2$ in E. We have proved that any convergent subsequence of u_n converges to v in

$L^\infty(E)$. We conclude that $u_n \to v$ in $L^\infty(E)$. We fix $x_0 \in E$. By the fundamental theorem of calculus, for all $y \in [0,T]$ we have $u_n(y) = u_n(x_0) + \int_{x_0}^y u_n'$, $u_m(y) = u_m(x_0) + \int_{x_0}^y u_m'$ and therefore

$$u_n(y) - u_m(y) = u_n(x_0) - u_m(x_0) + \int_{x_0}^y (u_n' - u_m').$$

It follows that, for all $y \in [0,T]$:

$$|u_n(y) - u_m(y)| \le |u_n(x_0) - u_m(x_0)| + \int_0^T |u_n' - u_m'| \to 0.$$

Equivalently,

$$\|u_n - u_m\|_\infty = \sup_{y\in[0,T]} |u_n(y) - u_m(y)| \to 0$$

as $m, n \to \infty$. We conclude that u_n is a Cauchy sequence in $L^\infty[0,T]$ and $u_n \to v$ in $L^\infty[0,T]$ for some continuous function v. At this point it is readily seen that u_n is a Cauchy sequence in $\mathcal{H}_{\widetilde{b},\widetilde{s}}$. Indeed, we have:

$$\int_0^T \widetilde{b}|u_n - u_m|^{\widetilde{s}} \le \|u_n - u_m\|_\infty^{\widetilde{s}} \int_0^T \widetilde{b} \to 0.$$

We conclude that a Cauchy sequence in $\mathcal{H}_{b,s}$ is also a Cauchy sequence in $\mathcal{H}_{\widetilde{b},\widetilde{s}}$. Therefore, as sets, $\mathcal{H}_{b,s} = \mathcal{H}_{\widetilde{b},\widetilde{s}}$. ∎

In view of Proposition 2.3, we set

$$\mathcal{H} = \mathcal{H}_{b,s}$$

for any $b \in L^1[0,T]$, $b \ge 0$, $b \not\equiv 0$ and for any $s > 1$.

3 Proof of Theorem 1.1 and of Theorem 1.2

Let $a \in L^1[0,T]$, $a \ge 0$, $a \not\equiv 0$ and let $p > 1$. We consider $y : [0,T] \to [0,T]$ defined by

$$y(x) = \widetilde{a}^{-1} \int_0^x a(t)\,dt,$$

where $\widetilde{a} = T^{-1} \int_0^T a$. The function y is well-defined, nondecreasing, absolutely continuous and differentiable a.e. We denote by $W^{1,p}_{\text{per}}$ the set of functions in $W^{1,p}(\mathbb{R})$ which are T-periodic. For every $U \in W^{1,p}_{\text{per}}$ we define $u(x) = (\Psi U)(x) = U(y(x))$. By taking difference quotients, we see that u is differentiable a.e., and

$$u'(x) = U'(y(x))\frac{a(x)}{\widetilde{a}}$$

for almost every $x \in [0,T]$. We recall the following general version of the change of variables formula, see, e.g., [6], Theorem 9.7.5, p.245.

Lemma 3.1 ([6]). *Let g be a nondecreasing absolutely continuous function on $[\alpha,\beta]$ and let f be integrable on $[g(\alpha),g(\beta)]$. Then, $(f\circ g)g'\in L^1[\alpha,\beta]$ and*

$$\int_\alpha^\beta (f\circ g)g' = \int_{g(\alpha)}^{g(\beta)} f.$$

We set

$$\mathcal{X} = \Psi(W^{1,p}_{\mathrm{per}}).$$

Lemma 3.2. *The mapping $\Psi : W^{1,p}_{\mathrm{per}} \to \mathcal{X}$ is an isomorphism of Banach spaces.*

Proof. In view of Lemma 3.1, we have:

$$\int_0^T a|u|^q\,dx = \widetilde{a}\int_0^T |U|^q\,dy \tag{9}$$

$$\int_0^T a^{1-p}|u'|^p\,dx = \widetilde{a}^{1-p}\int_0^T |U'|^p\,dy. \tag{10}$$

We note that $\mathcal{X}$ is complete with respect to the norm $\|\cdot\|_{a,q}$ defined in Section 2. Indeed, in view of (9)–(10), if $u_n = \Psi(U_n)\in\mathcal{X}$ is a Cauchy sequence, then U_n is a Cauchy sequence in $W^{1,p}_{\mathrm{per}}$. Then $U_n\to U$ in $W^{1,p}_{\mathrm{per}}$ and $u_n\to u=\Psi(U)$. Now the claim follows by the Open Mapping Theorem. ∎

In the next lemma we clarify the relations between $\mathcal{H},\mathcal{W}$ and $\mathcal{X}$.

Lemma 3.3. *There holds:*

$$\mathcal{H}\subset\mathcal{W}=\mathcal{X}.$$

Proof. We first show that $\mathcal{H}\subset\mathcal{X}$. Let $u\in\mathcal{H}$. We may assume that $u\in C^1$. In the degenerate case where $\inf a=0$, the function $y(x)$ may have some "flat regions". Namely, there may be $x',x''\in I$, $x'<x''$ such that $y(x')=y(x'')=y(x)$ for all $x\in[x',x'']$. In this case, $a=0$ a.e. in $[x',x'']$. Since $\int_0^T a^{1-p}u'<+\infty$, we derive that u =const in $[x',x'']$ and in particular $u(x')=u(x'')$. It follows that $U(y)=u(x(y))$ is well-defined and continuous and moreover $u(x)=U(y(x))$. Furthermore, since $x(y)$ is monotone, it is differentiable a.e. It follows that U is differentiable a.e. By change of variables, as in Lemma 3.1, $U\in W^{1,p}_{\mathrm{per}}$ and $u=\Psi U$. Hence, $u\in\mathcal{X}$.

Now we show that $\mathcal{W}\subset\mathcal{X}$. Let $u\in\mathcal{W}$. Let $x_1\in[0,T]$ be a jump discontinuity point for $x(y)$. Then, there exists $x_2>x_1$ such that $y(x)=y(x_1)=y(x_2)$ for all $x\in[x_1,x_2]$. In particular, $a=0$ for almost every $x\in[x_1,x_2]$ and consequently $u'=0$ for almost every $x\in[x_1,x_2]$. It follows that $u(x)=u(x_1)=u(x_2)$ for every $x\in[x_1,x_2]$. Therefore, the function $U(y)=u(x(y))$ is well-defined and continuous. Moreover, consideration of difference quotients yields $U'(y)=u'(x(y))x'(y)$ for almost every $y\in[0,T]$. We claim that the almost everywhere derivative U' is the distributional derivative of U. To this end, let $\varphi\in C^1_{\mathrm{per}}(\mathbb{R})$. Since u' is the distributional derivative of u, we have

$$\begin{aligned}\int_0^T U'(y)\varphi(y)\,dy &= \int_0^T u'(x)\varphi(y(x))\,dx = -\int_0^T u(x)\varphi'(y(x))y'(x)\,dx\\ &= -\int_0^T U(y)\varphi'(y)\,dy.\end{aligned}$$

Hence, $u = \Psi U$ with $U \in W^{1,p}_{\text{per}}$ and therefore $u \in \mathcal{X}$.

Finally, we show that $\mathcal{X} \subset \mathcal{W}$. To this end, let $U \in W^{1,p}_{\text{per}}(\mathbb{R})$ and let $u(x) = U(y(x))$. Then, u is continuous and T-periodic. Moreover, by taking difference quotients, we have that u is differentiable a.e. in $[0,T]$ and the pointwise derivative is given by

$$u'(x) = U'(y(x))y'(x)$$

for a.e. $x \in [0,T]$. We have to show that u' is the distributional derivative of u. Since U' is the distributional derivative of U, for any $y_1 \in [0,T]$ we have $U(y_1) = U(0) + \int_0^{y_1} U'(y)\,dy$. Let $x_1 = \inf\{x \in [0,T] : y(x) = y_1\}$. By the change of variables formula, Lemma 3.1, we obtain $u(x_1) = u(0) + \int_0^{x_1} u'(x)\,dx$. Let $x_2 = \sup\{x \in [0,T] : y(x) = y_1\}$. Then, $y'(x) = 0$ for all $x \in (x_1, x_2)$ and in view of (3), we have $u(x) = u(0) + \int_0^x u'(t)\,dt$ for all $x \in [x_1, x_2]$. Since $x(y)$ only admits jump discontinuities, we conclude that $u(x) = u(0) + \int_0^x u'(t)\,dt$ for all $x \in [0,T]$. It follows that u' is indeed the distributional derivative of u. In view of (10), we conclude that $u \in \mathcal{W}$. ∎

By the following example we see that for some particular choices of a, the space $\mathcal{H}$ may degenerate to the space of constant functions, and in particular $\mathcal{H} \neq \mathcal{W}$, unlike what happens in the usual Sobolev spaces, see [12].

EXAMPLE 3.1. There exists $a \in L^1[0,T]$, $a \geq 0$, $a \not\equiv 0$ such that

$$\mathcal{H} = \{c\}_{c \in \mathbb{R}} \neq \mathcal{W}.$$

Indeed, let $\mathcal{C} \subset [0,T]$ be a Cantor set such that $|\mathcal{C}| = T/2 = |[0,T] \setminus \mathcal{C}|$. Let $a = \chi_{\mathcal{C}}$, the characteristic function of $\mathcal{C}$. We claim that $\mathcal{H} = \{c\}_{c \in \mathbb{R}}$. To see this, recall that $\mathcal{C}$ and $[0,T] \setminus \mathcal{C}$ are dense in $[0,T]$. Let $u \in \mathcal{A}$, where $\mathcal{A}$ is defined in Section 2. Since $a^{1-p} = +\infty$ on $[0,T] \setminus \mathcal{C}$, we have $u' = 0$ on $[0,T] \setminus \mathcal{C}$. By continuity, $u' = 0$ on $[0,T]$. It follows that u is constant. Now let $u \in \mathcal{H}$. Then, there exists $u_n = c_n \in \mathcal{A}$ such that $c_n \to u$ with the respect to the norm $\|\cdot\|_{a,q}$. It follows that c_n is bounded, $c_n \to c \in \mathbb{R}$ and $u = c$. We conclude that for this choice of a, $\mathcal{H}$ is the space of constant functions.

Finally, we provide the proofs of our main results.

Proof of Theorem 1.1. Let $u \in \mathcal{X}$. Then $u(x) = U(y(x))$ for some $U \in W^{1,p}_{\text{per}}(\mathbb{R})$. The functions u, U satisfy the identities (9)–(10) and moreover:

$$\int_0^T a|u|^{q-2}u = \widetilde{a} \int_0^T |U|^{q-2}U = 0.$$

Therefore, using (6) we conclude the proof. ∎

Proof of Theorem 1.2. Uniqueness of the extremals in $W^{1,p}_{\text{per}}(\mathbb{R})$ implies uniqueness of the extremals of the form (7) in $\mathcal{X}$. ∎

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