On the Geometry of the Conformal Group in Spacetime

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Abstract

The study of the conformal group in $R^{p,q}$ usually involves the conformal compactification of $R^{p,q}$. This allows the transformations to be represented by linear transformations in $R^{p+1,q+1}$. So, for example, the conformal group of Minkowski space, $R^{1,3}$ leads to its isomorphism with $SO(2,4)$. This embedding into a higher dimensional space comes at the expense of the geometric properties of the transformations. This is particularly a problem in $R^{1,3}$ where we might well prefer to keep the geometric nature of the various types of transformations in sight.

In this note, we show that this linearization procedure can be achieved with no loss of geometric insight, if, instead of using this compactification, we let the conformal transformations act on two copies of the associated Clifford algebra. Although we are mostly concerned with the conformal group of Minkowski space (where the geometry is clearest), generalization to the general case is straightforward.

1 Introduction

The conformal group[1] of most interest to physicists is the conformal group on Minkowski space $R^{1,3}$. (We will choose the metric $g = g_{\mu\nu}$ with signature $+, -, -, -$ and $c = 1$, and avoid indices by writing $\langle x, y \rangle = x_0 y_0 - x_1 y_1 - x_2 y_2 -$

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[1]There is not complete unanimity as to what constitutes the conformal group. Most authors restrict it to a connected component. One advantage of a Clifford algebra approach, is that it naturally leads to a description of the covering group and even allows the inclusion of operators such as inversions, which are not normally included in the conformal group, even though they are conformal transformations.

x_3y_3.) Although most of the ideas regarding conformal transformations naturally carry over to general $R^{p,q}$ spaces, the geometric ideas are easier to understand in Minkowski space and we are not burdened with extra notation.

The conformal group can be studied from a number of (equivalent) viewpoints.

We can compactify $R^{1,3}$ to a 4-dimensional submanifold of the projective space $P_5(R)$, in which case the identity connected component of the conformal group is isomorphic to $SO(2,4)$. (See e.g. Schottenloher [1] for a detailed outline of this conventional approach whereby non-linear conformal maps are linearized in a larger space.)

Castro and Pavšič [2] have shown the emergence of the conformal group $SO(2,4)$ from the Clifford algebra of spacetime $Cl_{1,3}$ by pointing out that the conformal group is a subgroup of the Clifford group, but again this leads to an underlying six dimensional space where the extra two components are not easy to identify geometrically.

Hestenes and others [3, 4] developed the idea of the *conformal split* in general $R^{p,q}$ space and used this to highlight the connection between the conformal group on $R^{p,q}$ and spin groups which naturally belong to the Clifford algebra $Cl_{p+1,q+1}$.

Lounesto and Latvamaa [5] extended the Clifford algebra $Cl_{p,q}$ to the larger Clifford algebra $Cl_{p+1,q}$ and found simple commutation relations in $Cl_{p+1,q}$ describing the conformal Lie algebra of the conformal group on $R^{p,q}$. See also Girard [6] for a description of conformal transformations in terms of quaternionic parameters.

In the last two of these approaches, and specializing to Minkowski space, it is recognized that the Clifford algebra $Cl_{1,3}$ is not large enough to accommodate the generators of the conformal Lie algebra on $R^{1,3}$. In a sense, Lounesto and Latvamaa's description is the simplest since only the time index is increased. But in each of these approaches, the geometric nature of the transformations which make up the conformal group tends to become obscured.

The aim of this paper is to show that the conformal group (more properly, the covering group of the conformal group) can be realized by the action of $Cl_{1,3}$ on the *space $Cl_{1,3} \oplus Cl_{1,3}$*. Although this larger space can be viewed as the vector space of the Clifford algebra $Cl_{2,3}$ (which is the approach Lounesto and Latvamaa take), this is unnecessary. Imposing an algebraic structure tends to obscure the more important geometric ideas and also raises problems of interpretation - e.g., what does the extra generator represent physically or geometrically?

2   Conformal transformations

The 10-parameter Poincaré group is the semi-direct product of the 6-parameter Lorentz group with the 4-parameter group of space-time translations. The Poincaré group may then be enlarged to the conformal group by adding *dilatations*

$$x \to \rho x \quad (\rho > 0)$$

as well as *special conformal transformations*

$$x \to \frac{x + \langle x, x \rangle a}{\sigma(x)}, \quad \text{where } \sigma(x) = 1 + 2\langle a, x \rangle + \langle a, a \rangle \langle x, x \rangle$$

which correspond to local scale changes.

The special conformal transformations may also be obtained as the product of an inversion

$$I : x \to x^{-1} = \frac{x}{\langle x, x \rangle}$$

followed by a translation and another inversion.

The generators of the (identity component of the) conformal group may be realized as differential operators acting on Minkowski space. The operators corresponding to Lorentz transformations ($M_{\mu\nu}$), translations (P_μ), dilatations (D) and special conformal transformations (K_μ) satisfy the following commutation relations

$$[M_{\mu\nu}, M_{\sigma\rho}] = g_{\mu\rho}M_{\nu\sigma} - g_{\mu\sigma}M_{\nu\rho} + g_{\nu\sigma}M_{\mu\rho} - g_{\nu\rho}M_{\mu\sigma}$$
$$[P_\lambda, M_{\mu\nu}] = g_{\lambda\mu}P_\nu - g_{\lambda\nu}P_\mu$$
$$[D, M_{\mu\nu}] = 0$$
$$[K_\lambda, M_{\mu\nu}] = g_{\lambda\mu}K_\nu - g_{\lambda\nu}K_\mu$$
$$[P_\mu, P_\nu] = 0$$
$$[D, P_\mu] = -P_\mu$$
$$[P_\mu, K_\nu] = 2(M_{\mu\nu} - g_{\mu\nu}D)$$
$$[D, K_\mu] = K_\mu$$
$$[K_\mu, K_\nu] = 0$$

(Note that there is some divergence between authors. Some require that the generators be Hermitian in which case the imaginary number i makes an occasional appearance in these equations. Since we are dealing with Lie, i.e. anti-symmetric, products it is perhaps more logical to define these generators to be skew-Hermitian. This has the added bonus that only real algebras ever have to be used. In this context, we follow the definitions in Barut and Raczka [7] and Lounesto [8].)

3 The Clifford algebra representations

We assume familiarity with the basic definitions and ideas in Clifford algebras. (For these, see Lounesto [8], or Girard [9].) The basis elements for $Cl_{1,3}$ will be e_0, e_1, e_2, e_3 with $e_0^2 = 1$ and $e_1^2 = e_2^2 = e_3^2 = -1$. As usual we write $e_{\mu\nu}$ for $e_\mu e_\nu$ etc. and define the (unit) pseudoscalar $e = e_{0123}$.

We regard $Cl_{1,3}$ as acting on the *vector space $Cl_{1,3} \oplus Cl_{1,3}$* by left multiplication. It is then straightforward to verify that the above commutation relations are

satisfied by the following operators.

$$M_{\mu\nu}(x,y) = \frac{1}{2}(e_{\mu\nu}x, e_{\mu\nu}y)$$
$$P_\mu(x,y) = (e_\mu y, 0)$$
$$K_\mu(x,y) = (0, e_\mu x)$$
$$D(x,y) = \frac{1}{2}(-x, y)$$

where $x, y \in Cl_{1,3}$.

Although the inversion operator is not in the identity component, it too has a very natural representation in this context as

$$I(x,y) = (y, x)$$

It then follows that $K_\mu = IP_\mu I$.

Since these operators are defined through the action of the *associative* algebra $Cl_{1,3}$, they can be expected to have extra algebraic properties.[2] As an example, the translation generators P_μ satisfy the property

$$P_\mu P_\nu = 0$$

(which trivially implies that $[P_\mu, P_\nu] = 0$). As is shown below in this section, this property is important when we want to show that P_μ generates a translation in the direction e_μ.[3]

The transformations that arise from these generators are now easy to describe geometrically.

Lorentz transformations. It is well known that the elements $M_{\mu\nu}$ generate (the proper orthochronous) Lorentz transformations on $R^{1,3}$. In the Clifford algebra setting, where we could define $M_{\mu\nu} = \dfrac{e_{\mu\nu}}{2}$, this is particularly simple. For example, a boost in the direction $\mathbf{n} = (n_1, n_2, n_3)$, with velocity $v = \tanh\phi$ (remember that we are using units with $c = 1$), may be represented as

$$x \to x' = axa^{-1}$$

where $a = \exp(\phi n)$ and $n = n_1 e_{01} + n_2 e_{02} + n_3 e_{03}$. Similarly spacial rotations through an angle θ, are of the same form but now $a = \exp(\theta ne)$ with $\mathbf{n}$ describing the axis of rotation and $e = e_{0123}$.

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[2]As a general rule, if a Lie algebra structure is imposed on an associative algebra via $[A, B] = AB - BA$, some properties of AB may be lost

[3]The authors are indebted to the anonymous referee for pointing out that an earlier draft of this paper was too vague and that this point could be clarified.

Translations. Consider now the generator P_μ defined by

$$P_\mu(x,y) = (e_\mu y, 0)$$

Clearly $P_\mu^2 = 0$ so that

$$\exp(tP_\mu)\,(x,y) = (x + te_\mu y, y)$$

More generally, if $a = (a^\mu e_\mu)$ is a vector in $R^{1,3}$, then

$$\exp(ta^\mu P_\mu)\,(x,y) = (x + ta^\mu e_\mu y, y)$$

and in particular

$$\exp(ta^\mu P_\mu)\,(x,1) = (x + ta, 1)$$

so that $P_a = a^\mu P_\mu$ generates the translation operation $T_a : x \to x + a$ in $R^{1,3}$ when this space is identified with the hyperplane $R^{1,3} \oplus (1)$ in $Cl_{1,3} \oplus Cl_{1,3}$, (see figure 1).

It might also be worth pointing out, that there is no *algebraic* reason why we should only consider translations in the direction of a 1-vector. If u is *any* element of $Cl_{1,3}$, we can define an operator P_u by

$$P_u(x,y) = (uy, 0)$$

and this generates a translation in $Cl_{1,3} \oplus Cl_{1,3}$. This then leads to a generalization of the conformal group and it would be interesting to characterize this extended group further.

Special conformal transformations. The generator K_μ behaves much like P_μ, but on the second component space of $Cl_{1,3} \oplus Cl_{1,3}$. It too generates a translation $U_b : y \to y + b$ in $R^{1,3}$ when the space in identified with the hyperplane $(1) \oplus R^{1,3}$, this time of the form

$$\exp(ta^\mu K_\mu)\,(1,y) = (1, y + ta).$$

This illustrates an advantage of our approach. The special conformal transformations act in an entirely similar way to translations, but on the second component subspace rather than the first. In that sense, they are no more non-linear than translations.

Again we can generalize special conformal transformations to operators K_u defined by

$$K_u(x,y) = (0, ux).$$

Dilatations. The operator D defined by $D(x,y) = \frac{1}{2}(-x,y)$ generates the transformations

$$\exp(tD)(x,y) = \left(e^{-\frac{t}{2}}x, e^{\frac{t}{2}}y\right).$$

which, in the special cases where either x or y is 0, can represent dilations of 1-vectors in $R^{1,3}$. Again, there is no algebraic reason why dilatations cannot be considered on all of $Cl_{1,3}$ or in fact, on all of $Cl_{1,3} \oplus Cl_{1,3}$.

Inversions. Although inversions are not part of the connected component of the conformal group (and hence do not appear in the conformal Lie algebra), they are conformal transformations which in our picture, interchange the two component subspaces and thus provide a link between special conformal transformations and translations.
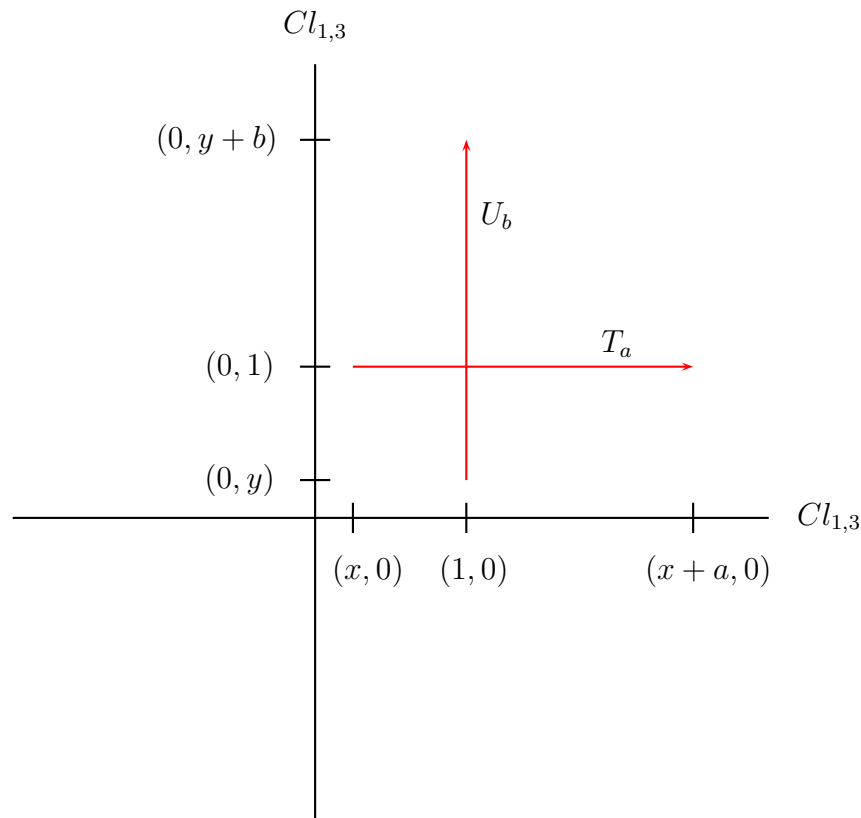

Figure 1: Geometric representation of the translations and special conformal transformations on the space $Cl_{1,3} \oplus Cl_{1,3}$. The generators T_a and U_b act on the hyperspaces $R^{1,3} \oplus (1)$ and $(1) \oplus R^{1,3}$ respectively.

4   Concluding remarks

In this note we have put forth an alternative linearization procedure for the conformal group of $R^{p,q}$. To achieve linearization, we let the conformal transformations act on two copies of the associated Clifford algebra instead of the standard procedure which involves compactifying $R^{p,q}$,(so that the conformal transformations may be represented by linear transformations in $R^{p+1,q+1}$). In particular we have considered the conformal transformations of Minkowski space $R^{1,3}$ to highlight the geometrical advantages provided by this Clifford algebra approach.

Representing the conformal algebra in $R^{1,3}$ in terms of $Cl_{1,3}$ rather than some larger Clifford algebra, preserves and in fact emphasizes the geometric nature of
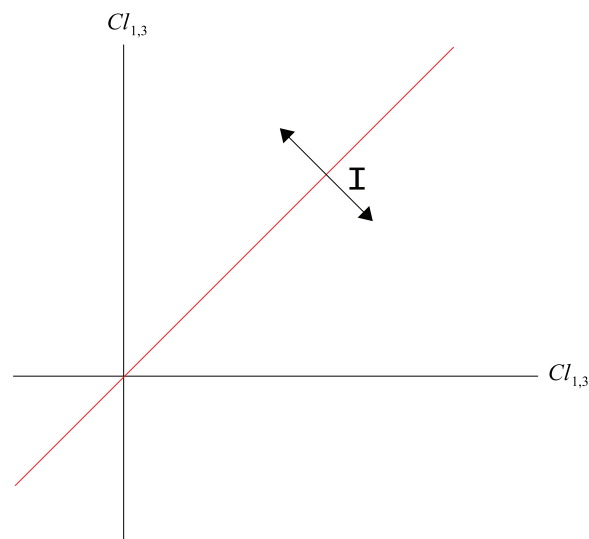
Figure 2: Geometric representation of an inversion. An inversion corresponds to a reflection about the 'line' (x, x).

conformal transformations. It also makes it easier to treat algebras such as the stabilized Poincaré-Heisenberg algebra (SPHA). This is regarded as a stabilized version of the direct sum of the Heisenberg and Poincaré algebras (see Chryssomalakos and Okon [10]), but may also be considered in terms of simple commutation relations in $Cl_{1,3}$ (See Gresnigt et al. [11]).

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