The orthogonal u-invariant of a quaternion algebra

Karim Johannes Becher   Mohammad G. Mahmoudi

Abstract

In quadratic form theory over fields, a much studied field invariant is the u-invariant, defined as the supremum of the dimensions of anisotropic quadratic forms over the field. We investigate the corresponding notions of u-invariant for hermitian and for skew-hermitian forms over a division algebra with involution, with a special focus on skew-hermitian forms over a quaternion algebra with canonical involution. Under certain conditions on the center of the quaternion algebra, we obtain sharp bounds for this invariant.

1 Involutions and hermitian forms

Throughout this article K denotes a field of characteristic different from 2 and $K^\times$ its multiplicative group. We shall employ standard terminology from quadratic form theory, as used in [9]. We say that K is *real* if K admits a field ordering, *nonreal* otherwise. By the Artin-Schreier Theorem, K is real if and only if -1 is not a sum of squares in K.

Let Δ be a division ring whose center is K and with $\dim_K(\Delta) < \infty$; we say that Δ is a *division algebra over K*, for short. We further assume that Δ is endowed with an *involution* σ, that is, a map $\sigma : \Delta \to \Delta$ such that $\sigma(a+b) = \sigma(a) + \sigma(b)$ and $\sigma(ab) = \sigma(b)\sigma(a)$ hold for any $a, b \in \Delta$ and such that $\sigma \circ \sigma = id_\Delta$. Then $\sigma|_K : K \to K$ is an involution of K, and there are two cases to be distinguished.

Received by the editors November 2008.
Communicated by M. Van den Bergh.
2000 *Mathematics Subject Classification :* 11E04, 11E39, 11E81.
Key words and phrases : hermitian form, involution, division algebra, isotropy, system of quadratic forms, discriminant, Tsen-Lang Theory, Kneser's Theorem, local field, Kaplansky field.

If $\sigma|_K = id_K$, then we say that the involution σ is *of the first kind*. In the other case, when $\sigma|_K$ is a nontrivial automorphism of the field K, we say that σ is *of the second kind*. In general, we fix the subfield $k = \{x \in K \mid \sigma(x) = x\}$ and say that σ is a *K/k-involution of* Δ. Note that $\sigma : \Delta \to \Delta$ is k-linear. If σ is of the second kind, then K/k is a quadratic extension. Recall that involutions of the first kind on a division algebra Δ over K exist if and only if Δ is of exponent at most 2, i.e. $\Delta \otimes_K \Delta$ is isomorphic to a full matrix algebra over K. Moreover, an involution σ of the first kind on Δ is either of *orthogonal* or of *symplectic type*, depending on the dimension of the subspace $\{x \in \Delta \mid \sigma(x) = x\}$ (see [9, Chap. 8, (7.6)]).

Let $\varepsilon \in K^\times$ with $\sigma(\varepsilon)\varepsilon = 1$. We are mainly interested in the cases where $\varepsilon = \pm 1$; if σ is of the first kind then these are the only possibilities for ε. An *ε-hermitian form over* (Δ, σ) is a pair (V, h) where V is a finite-dimensional right vector space over Δ and h is a map $h : V \times V \to \Delta$ that is Δ-linear in the second argument and with $\sigma(h(x, y)) = \varepsilon \cdot h(y, x)$ for any $x, y \in V$; it follows that h is 'sesquilinear' in the sense that $h(xa, yb) = \sigma(a)h(x, y)b$ for any $x, y \in V$ and $a, b \in \Delta$. In this situation we may also refer to h as the ε-hermitian form and to V as the *underlying vector space*. We simply say that h is *hermitian* (resp. *skew-hermitian*) if h is 1-hermitian (resp. (-1)-hermitian).

In the simplest case we have $\Delta = K$, $\sigma = id_K$, and $\varepsilon = 1$. A 1-hermitian form over (K, id_K) is a symmetric bilinear form $b : V \times V \to K$ on a finite-dimensional vector space V over K; by the choice of a basis it can be identified with a quadratic form over K in $n = \dim_K(V)$ variables.

An ε-hermitian form h over (Δ, σ) with underlying vector space V is said to be *regular* or *nondegenerate* if, for any $x \in V \setminus \{0\}$, the associated Δ-linear form $V \to \Delta, y \mapsto h(x, y)$ is not the zero map; if this condition fails h is said to be *singular* or *degenerate*. We say that h is *isotropic* if there exists a vector $x \in V \setminus \{0\}$ such that $h(x, x) = 0$, otherwise we say that h is *anisotropic*. Let h_1 and h_2 be two ε-hermitian forms over (Δ, σ) with underlying spaces V_1 and V_2. The *orthogonal sum* of h_1 and h_2 is the ε-hermitian form h on the Δ-vector space $V = V_1 \times V_2$ given by $h(x, y) = h_1(x_1, y_1) + h_2(x_2, y_2)$ for $x = (x_1, x_2), y = (y_1, y_2) \in V$, and it is denoted by $h_1 \perp h_2$. An *isometry* between h_1 and h_2 is an isomorphism of Δ-vector spaces $\tau : V_1 \to V_2$ such that $h_1(x, y) = h_2(\tau(x), \tau(y))$ holds for all $x, y \in V_1$. If an isometry between h_1 and h_2 exists, then we say that h_1 and h_2 are *isometric* and write $h_1 \simeq h_2$. Witt's Cancellation Theorem [2, (6.3.4)] states that, whenever h_1, h_2 and h are ε-hermitian forms on (Δ, σ) such that $h_1 \perp h \simeq h_2 \perp h$, then also $h_1 \simeq h_2$ holds. A regular $2n$-dimensional ε-hermitian form (V, h) is *hyperbolic* if there exits an n-dimensional subspace W of V with $h(x, y) = 0$ for all $x, y \in W$. The (up to isometry) unique regular isotropic 2-dimensional ε-hermitian form is denoted by $\mathbb{H}$.

Given an ε-hermitian form (V, h) over (Δ, σ) we write

$$D(h) = \{h(x, x) \mid x \in V \setminus \{0\}\} \subseteq \Delta .$$

Note that this set contains 0 if and only if h is isotropic. We further put

$$\mathrm{Sym}^\varepsilon(\Delta, \sigma) = \{x \in \Delta \mid \sigma(x) = \varepsilon x\} .$$

For any ε-hermitian form h over (Δ, σ) we have $D(h) \subseteq \mathrm{Sym}^\varepsilon(\Delta, \sigma)$. Given elements $a_1, \ldots, a_n \in \mathrm{Sym}^\varepsilon(\Delta, \sigma)$, an ε-hermitian form h on the Δ-vector space

$V = \Delta^n$ is defined by $h(x,y) = \sigma(x_1)a_1y_1 + \cdots + \sigma(x_n)a_ny_n$ for $x = (x_1,\ldots,x_n)$, $y = (y_1,\ldots,y_n) \in \Delta^n = V$. We denote this form h by $\langle a_1,\ldots,a_n\rangle$ and observe that it is regular if and only if $a_i \neq 0$ for $1 \leqslant i \leqslant n$. As $\mathrm{char}(K) \neq 2$, any ε-hermitian form is isometric to $\langle a_1,\ldots,a_n\rangle$ for some $n \in \mathbb{N}$ and $a_1,\ldots,a_n \in \mathrm{Sym}^\varepsilon(\Delta,\sigma)$ [2, (6.2.4)].

We denote by $\mathrm{Herm}^\varepsilon_n(\Delta,\sigma)$ the set of isometry classes of regular n-dimensional ε-hermitian forms over (Δ,σ). Mapping $a \in \mathrm{Sym}^\varepsilon(\Delta,\sigma)$ to the class of $\langle a\rangle$ yields a surjection

$$\mathrm{Sym}^\varepsilon(\Delta,\sigma) \setminus \{0\} \longrightarrow \mathrm{Herm}^\varepsilon_1(\Delta,\sigma)\,.$$

Two elements a, $b \in \mathrm{Sym}^\varepsilon(\Delta,\sigma)$ are *congruent* if there exists $c \in \Delta$ such that $a = \sigma(c)bc$, which is equivalent to saying that $\langle a\rangle \simeq \langle b\rangle$ over (Δ,σ).

1.1 Remark. In the case where $\Delta = K$ and $\varepsilon = 1$, there is a natural one-to-one correspondence between $\mathrm{Herm}^\varepsilon_1(\Delta,\sigma)$ and $K^\times/K^{\times 2}$. We may thus identify the two sets with one another and endow $\mathrm{Herm}^1_1(\Delta,\sigma)$ with a natural group structure. One can proceed in a similar way in two other cases, first when Δ is a quaternion algebra and σ is its canonical involution, and second when σ is a unitary involution on the field $\Delta = K$.

For an ε-hermitian form h over (Δ,σ) and $a \in k^\times$ where $k = \{x \in K \mid \sigma(x) = x\}$, we define the scaled ε-hermitian form ah in the obvious way. Two ε-hermitian forms h and h' over (Δ,σ) are said to be *similar* if $h' \simeq ah$ holds for some $a \in k^\times$.

2 Hermitian u-invariants

We keep the setting of the previous section. Following [8, Chap. 9, (2.4)] we define

$$u(\Delta,\sigma,\varepsilon) \;=\; \sup\,\{\dim(h) \mid h \text{ anisotropic } \varepsilon\text{-hermitian form over } (\Delta,\sigma)\}$$

in $\mathbb{N} \cup \{\infty\}$ and call this the *u-invariant of* $(\Delta,\sigma,\varepsilon)$. In this context,

$$u(K,id_K,1) \;=\; sup\,\{\dim(\varphi) \mid \varphi \text{ anisotropic quadratic form over } K\}$$

is the u-invariant of the field K, also denoted by $u(K)$. We refer to [8, Chap. 8] for an overview of this invariant for nonreal fields and for a discussion of different versions of this definition that are interesting when dealing with real fields.

To obtain upper bounds on $u(\Delta,\sigma,\varepsilon)$, one can use the theory of systems of quadratic forms. In fact, to every ε-hermitian form h over (Δ,σ) one can associate a system of quadratic forms over k in such a way that the isotropy of h is equivalent to the simultaneous isotropy of this system.

For $r \in \mathbb{N}$, one denotes by $u_r(K)$ the supremum of the $n \in \mathbb{N}$ for which there exists a system of r quadratic forms in n variables over K having no nontrivial common zero. The numbers $u_r(K)$ are called the *system u-invariants of K*. Note that $u_0(K) = 0$ and $u_1(K) = u(K)$. Leep proved that these system u-invariants satisfy the inequalities

$$u_r(K) \;\leqslant\; ru(K) + u_{r-1}(K) \;\leqslant\; \tfrac{r(r+1)}{2}\,u(K)$$

for any integer $r \geqslant 1$. Using systems of quadratic forms, he further showed that $u(L) \leqslant \frac{[L:K]+1}{2} u(K)$ holds for an arbitrary finite field extension L/K. (See [9, Chap. 2, Sect. 16] for these and more facts on systems on quadratic forms.) In the same vein the following result was obtained in [7, (3.6)].

2.1 Proposition. *Let Δ be a division algebra over K, σ an involution on Δ, and $\varepsilon \in K$ with $\varepsilon\sigma(\varepsilon) = 1$. Then*

$$u(\Delta, \sigma, \varepsilon) \leqslant \frac{u_r(k)}{m^2[K:k]} \leqslant \frac{r(r+1)}{2m^2[K:k]} \cdot u(k)$$

where $k = \{x \in K \mid \sigma(x) = x\}$, $m = \deg(\Delta)$ and $r = \dim_k(\mathrm{Sym}^{\varepsilon}(\Delta, \sigma))$. In particular, if $u(k) < \infty$, then $u(\Delta, \sigma, \varepsilon) < \infty$.

In this article, we are mainly concerned with the u-invariant of an involution of the first kind. Assume that σ is an involution of the first kind on the division algebra Δ over K. In this case $\Delta \otimes_K \Delta$ is isomorphic to a full matrix algebra and $\varepsilon = \pm 1$. In [7] it is explained that $u(\Delta, \sigma, \varepsilon)$ only depends on ε and on the type of σ, i.e., whether it is orthogonal or symplectic. More precisely, given two involutions of the first kind σ and τ on Δ one has $u(\Delta, \sigma, \varepsilon) = u(\Delta, \tau, \varepsilon)$ if σ and τ are of same type and $u(\Delta, \sigma, \varepsilon) = u(\Delta, \tau, -\varepsilon)$ if they are of opposite type. We define

$$u^+(\Delta) = u(\Delta, \sigma, +1) \quad \text{and} \quad u^-(\Delta) = u(\Delta, \sigma, -1)$$

with respect to an arbitrary orthogonal involution σ on Δ, as these numbers do not depend on the choice of σ. We call $u^+(\Delta)$ the *orthogonal* and $u^-(\Delta)$ the *symplectic u-invariant of* Δ. By the previous, for any symplectic involution τ on Δ one has $u(\Delta, \tau, \varepsilon) = u^{-\varepsilon}(\Delta)$.

Let us briefly mention that, in the case of an involution σ of the second kind, $u(\Delta, \sigma, \varepsilon)$ depends only on the field $k = \{x \in K \mid \sigma(x) = x\}$, in particular it does not depend on ε at all.

Let $i \in \mathbb{N}$. Using (2.1) one can obtain estimates for the u-invariants of division algebras with involution over a $\mathcal{C}_i$-field. We recall some facts from Tsen-Lang Theory, following [9, Chap. 2, Sect. 15]. A field K is called a *$\mathcal{C}_i$-field* if every homogeneous polynomial over K of degree d in more than d^i variables has a nontrivial zero. The natural examples of $\mathcal{C}_i$-fields are extensions of transcendence degree i of an arbitrary algebraically closed field and (for $i > 0$) extensions of transcendence degree $i-1$ of a finite field. A result due to Lang and Nagata states that, if K is a $\mathcal{C}_i$-field, then $u_r(K) \leqslant r \cdot 2^i$ for any $r \in \mathbb{N}$ (cf. [9, Chap. 2, (15.8)]). In [8, Chap. 5], variations of the $\mathcal{C}_i$-property and open problems in this context are discussed.

2.2 Corollary. *Let K be a $\mathcal{C}_i$-field and let Δ be a division algebra of exponent 2 and of degree m over K. Then $u^+(\Delta) \leqslant 2^{i-1} \cdot \frac{m+1}{m}$ and $u^-(\Delta) \leqslant 2^{i-1} \cdot \frac{m-1}{m}$.*

Proof: We use (2.1) and the fact that $u_r(k) \leqslant 2^i r$. ■

2.3 Corollary. *Let K be a $\mathcal{C}_i$-field. Let Δ be a quaternion division algebra over K. Then $u^+(\Delta) \leqslant 3 \cdot 2^{i-2}$ and $u^-(\Delta) \leqslant 2^{i-2}$.*

Example (5.4) will show that the first bound in (2.3) is sharp. For the second bound, we leave this as an easy exercise. In fact, determining the symplectic u-invariant of a quaternion algebra is a pure quadratic form theoretic problem in view of Jacobson's Theorem [9, Chap. 10, (1.1)], which relates hermitian forms over a quaternion algebra with canonical involution —the unique symplectic involution on a quaternion algebra— to quadratic forms over the center. This is why our investigation for quaternion algebras concentrates on the orthogonal u-invariant.

3 Kneser's Theorem

In this section, we give an upper bound on the u-invariant of a division algebra with involution in terms of the number of 1-dimensional (skew-)hermitian forms, subject to a condition on the levels of certain subalgebras. This extends an observation due to Kneser [4, Chap. XI, (6.4)] on the commutative case.

From [6] we recall the definition of the level of an involution. Let σ be an involution on a central simple algebra Δ over K. The *level of* σ is defined as

$$s(\Delta,\sigma) \;=\; \sup\{m \in \mathbb{N} \mid m \times \langle 1\rangle \text{ is anisotropic over } (\Delta,\sigma)\}$$

in $\mathbb{N} \cup \{\infty\}$. Whenever $s(\Delta,\sigma)$ is finite, it is equal to the smallest number m for which -1 can be written as a sum of m hermitian squares over (Δ,σ).

3.1 Theorem. *Let Δ be a division algebra over K equipped with an involution σ. Let $\varepsilon \in K$ be such that $\sigma(\varepsilon)\varepsilon = 1$. Let ψ be an ε-hermitian form over (Δ,σ) and let $\alpha \in \Delta^\times$ be such that $\sigma(\alpha) = \varepsilon\alpha$. Let $C_\Delta(\alpha)$ be the centralizer of $K(\alpha)$ in Δ. Suppose that $s(C_\Delta(\alpha),\sigma|_{C_\Delta(\alpha)}) < \infty$. If $\varphi = \psi \perp \langle\alpha\rangle$ is anisotropic then $D(\psi) \subsetneq D(\varphi)$.*

Proof: We write $0 = \sigma(d_0)d_0 + \cdots + \sigma(d_s)d_s$ with $s = s(C_\Delta(\alpha),\sigma|_{C_\Delta(\alpha)})$ and $d_0,\dots,d_s \in C_\Delta(\alpha) \setminus \{0\}$. We suppose that $D(\psi) = D(\varphi)$ and want to conclude that φ is isotropic. We claim that $\alpha \cdot (\sigma(d_0)d_0 + \cdots + \sigma(d_i)d_i) \in D(\varphi)$ for any $0 \leqslant i \leqslant s$. For $i = s$ this yields that φ is isotropic.

For $i = 0$, note that α and $\alpha\sigma(d_0)d_0$ are represented by φ. Let now $1 \leqslant i \leqslant s$ and assume that the claim holds for $i-1$. With $\alpha(\sigma(d_0)d_0 + \cdots + \sigma(d_{i-1})d_{i-1}) \in D(\varphi) = D(\psi)$, we obtain readily that $\alpha(\sigma(d_0)d_0 + \cdots + \sigma(d_{i-1})d_{i-1}) + \alpha\sigma(d_i)d_i \in D(\varphi)$, finishing the argument. ■

3.2 Corollary. *Assume that $s(C_\Delta(\alpha),\sigma|_{C_\Delta(\alpha)}) < \infty$ for every $\alpha \in \mathrm{Sym}^\varepsilon(\Delta,\sigma)$. Then $u(\Delta,\sigma,\varepsilon) \leqslant |\mathrm{Herm}_1^\varepsilon(\Delta,\sigma)|$.*

Proof: Let $h \simeq \langle a_1,\dots,a_n\rangle$ be an anisotropic ε-hermitian form of dimension n over (Δ,σ). Set $h_i = \langle a_1,\dots,a_i\rangle$ for $1 \leqslant i \leqslant n$. Using (3.1) we obtain that $D(h_1) \subsetneq D(h_2) \subsetneq \cdots \subsetneq D(h_n) = D(h)$. We conclude that h represents at least n pairwise incongruent elements of $\mathrm{Sym}^\varepsilon(\Delta,\sigma)$, i.e. $|\mathrm{Herm}_1^\varepsilon(\Delta,\sigma)| \geqslant n$. Therefore we have $|\mathrm{Herm}_1^\varepsilon(\Delta,\sigma)| \geqslant u(\Delta,\sigma,\varepsilon)$. ■

3.3 Remark. The hypothesis of (3.2) is trivially satisfied if the subfield of K consisting of the elements fixed by σ is nonreal; this is for example the case when σ is of the first kind and K is a nonreal field.

3.4 Example. Let p be a prime number different from 2 and let Q denote the unique quaternion division algebra over $\mathbb{Q}_p$. Then it follows from [9, Chap. 10, (3.6)] that $u^+(Q) = |\mathrm{Herm}_1^{-1}(Q,\gamma)| = 3$ (see also (4.9), below). Let now m be a positive integer and $K = \mathbb{Q}_p((t_1))\ldots((t_m))$. Then Q_K is a quaternion division algebra over K and $u^+(Q_K) = |\mathrm{Herm}_1^{-1}(Q_K,\gamma)| = 3\cdot 2^m$. This follows from the fact that the u-invariant(s) and the number of 1-dimensional ε-hermitian forms over a division algebra defined over a field K both double when the center is extended from K to $K((t))$.

The upper bound on the u-invariant obtained in (3.2) motivates us to look for criteria for the finiteness of $\mathrm{Herm}_1^{\varepsilon}(\Delta,\sigma)$ where Δ is a division algebra over K, σ an involution on Δ, and $\varepsilon = \pm 1$. We conjecture that $|\mathrm{Herm}_1^{\varepsilon}(\Delta,\sigma)| < \infty$ is equivalent to $|K^\times/K^{\times 2}| < \infty$. In the next section we shall confirm this in the case of skew-hermitian forms over a quaternion division algebra.

4 Congruence of pure quaternions

From this section on we consider a quaternion division algebra Q over K. Let γ denote the canonical involution of Q, π the norm form of Q and π' its pure part, so that $\pi = \langle 1\rangle \perp \pi'$. By a *skew-hermitian form over Q* we always mean a regular skew-hermitian form over (Q,γ). In this section we want to describe $\mathrm{Herm}_1^{-1}(Q,\gamma)$.

Following [10] the *discriminant* of a skew-hermitian form h over Q is defined as the class $\mathrm{disc}(h) = (-1)^n\,\mathrm{Nrd}((h(x_i,x_j))_{ij})K^{\times 2}$ in $K^\times/K^{\times 2}$ where $(x_1,\ldots,x_n)$ is an arbitrary Δ-basis of the underlying vector space and where $\mathrm{Nrd} : M_n(\Delta)\to K$ denotes the reduced norm.

4.1 Remark. For $a\in K^\times$, there exists a skew-hermitian form of dimension 1 and discriminant a over Q if and only if $-a$ is represented by the pure part of the norm form of Q. In particular, any 1-dimensional skew-hermitian form over Q has nontrivial discriminant.

4.2 Proposition. *Skew-hermitian forms of dimension* 1 *over Q are classified up to similarity by their discriminants.*

Proof: More generally, similar skew-hermitian forms over Q have the same discriminant. Assume now that $z_1, z_2 \in Q^\times$ are pure quaternions such that the discriminants of the skew-hermitian forms $\langle z_1\rangle$ and $\langle z_2\rangle$ coincide. Hence there exists $d\in K^\times$ such that $z_2^2 = d^2 z_1^2 = (dz_1)^2$. Therefore the pure quaternions z_2 and dz_1 are congruent in Q, i.e. there exists $\alpha\in Q^\times$ such that $dz_1 = \alpha^{-1}z_2\alpha$. Multiplying this equality by $\mathrm{Nrd}(\alpha) = \gamma(\alpha)\alpha$, it follows that $(\mathrm{Nrd}(\alpha)d)z_1 = \gamma(\alpha)z_2\alpha$. With $c = (\mathrm{Nrd}(\alpha)d)\in K^\times$ we obtain that $\langle cz_1\rangle \simeq \langle z_2\rangle$, so $\langle z_1\rangle$ and $\langle z_2\rangle$ are similar. ■

4.3 Remark. A closer look at the above argument yields the following refinement. Let G be a subgroup of $K^\times$ containing $\mathrm{Nrd}(Q^\times)$. Two 1-dimensional skew-hermitian forms are obtained from one another by scaling with an element of G if and only if their discriminants coincide in $K^\times/G^2$.

4.4 Lemma (Scharlau). *Let $\lambda, \mu \in Q^\times$ be anticommuting elements, so in particular $Q \simeq (a,b)_K$ with $a = \lambda^2, b = \mu^2 \in K^\times$. Let $c \in K^\times$. The skew-hermitian forms $\langle \lambda \rangle$ and $\langle c\lambda \rangle$ over Q are isometric if and only if c is represented by one of the quadratic forms $\langle 1, -a \rangle$ and $\langle b, -ab \rangle$ over K.*

Proof: See [9, Chap. 10, (3.4)]. ■

The following result was obtained in [5], in slightly different terms.

4.5 Proposition (Lewis). *Let λ be a nonzero pure quaternion in Q. Consider $\mathrm{Herm}_1^{-1}(Q,\gamma)$ as a pointed set with the isometry class of $\langle \lambda \rangle$ as distinguished point. With $L = K(\lambda)$ and $a = \lambda^2 \in K^\times$, one obtains an exact sequence*

$$1 \longrightarrow \mathbb{Z}/2\mathbb{Z} \longrightarrow K^\times/N_{L/K}(L^\times) \xrightarrow{\cdot\lambda} \mathrm{Herm}_1^{-1}(Q,\gamma) \xrightarrow{(-a)\,\mathrm{Nrd}} K^\times/K^{\times 2}.$$

Proof: Let $b \in K^\times$ be such that $Q = (a,b)_K$. By (4.4) the group of elements $x \in K^\times$ such that $\langle x\lambda \rangle \simeq \langle \lambda \rangle$ coincides with $N_{L/K}(L^\times) \cup bN_{L/K}(L^\times)$. This proves the exactness in the first two terms. The exactness at $\mathrm{Herm}_1^{-1}(Q,\gamma)$ follows from (4.2). ■

4.6 Remark. We sketch an alternative, cohomological argument for the exact sequence in (4.5), which was pointed out to us by J.-P. Tignol. Let $\rho = Int(\lambda) \circ \gamma$. Note that $\mathrm{Herm}_1^{-1}(Q,\gamma)$ can be identified with $\mathrm{Herm}_1^{1}(Q,\rho) = H^1(K, O(\rho))$ where $O(\rho) = \{x \in Q \mid \rho(x)x = 1\}$. By [3, Chap. VII, §29], there is an exact sequence $1 \to O^+(\rho) \to O(\rho) \to \mu_2 \to 1$. Moreover, $O^+(\rho) = L^1 = \{x \in L \mid N_{L/K}(x) = 1\}$. This yields the exact sequence $1 \to \mu_2 \to H^1(K, L^1) \to H^1(K, O(\rho)) \to K^\times/K^{\times 2}$. Using that $H^1(K, L^1) \simeq K^\times/N_{L/K}(L^\times)$ we obtain the sequence in (4.5).

4.7 Proposition. *Let $S = \{aK^{\times 2} \mid a \in D(\pi')\} \subseteq K^\times/K^{\times 2}$. For $\alpha \in S$ let $H_\alpha = \{h \in \mathrm{Herm}_1^{-1}(Q,\gamma) \mid \mathrm{disc}(h) = \alpha\}$. Then $\mathrm{Herm}_1^{-1}(Q,\gamma) = \bigcup_{\alpha \in S} H_\alpha$, in particular $|\mathrm{Herm}_1^{-1}(Q,\gamma)| = \sum_{\alpha \in S} |H_\alpha|$. Moreover, for any $\alpha = aK^{\times 2} \in S$ one has $|H_\alpha| \leqslant \frac{1}{2}|K^\times/N_{L/K}(L^\times)|$ with $L = K(\sqrt{-a})$.*

Proof: The first part is clear. For $\alpha \in S$, there is a pure quaternion $\lambda \in Q^\times$ with $\mathrm{disc}(\langle \lambda \rangle) = -\alpha$, and (4.5) applied to $L = K(\lambda)$ yields the last part. ■

4.8 Corollary. *Let $S = \{aK^{\times 2} \mid a \in D(\pi')\}$ and let $\mathcal{L}$ be the set of maximal subfields of Q. Then*

$$\left|\mathrm{Herm}_1^{-1}(Q,\gamma)\right| \leqslant \frac{1}{2} \sup_{L \in \mathcal{L}} \left|K^\times/N_{L/K}(L^\times)\right| \cdot |S|\,.$$

Proof: This is immediate from (4.7). ■

4.9 Remark. We keep the notation of (4.8). Kaplansky showed in [1] that Q is the unique quaternion division algebra over K if and only if

$$\sup_{L \in \mathcal{L}} \left|K^\times/N_{L/K}(L^\times)\right| = 2\,.$$

If this condition holds, then (4.8) yields $|\mathrm{Herm}_1^{-1}(Q,\gamma)| \leqslant |S|$, and as the converse inequality follows from (4.7), we obtain that $|\mathrm{Herm}_1^{-1}(Q,\gamma)| = |S|$. This applies in particular to any local field. Moreover, if K is a non-dyadic local field, then $|K^\times/K^{\times 2}| = 4$ and $|S| = 3$, so that we obtain immediately that $u^+(Q) = |\mathrm{Herm}_1^{-1}(Q,\gamma)| = |S| = 3$.

4.10 Theorem. $\mathrm{Herm}_1^{-1}(Q,\gamma)$ *is finite if and only if* $K^\times/K^{\times 2}$ *is finite.*

Proof: Let $S = \{aK^{\times 2} \mid a \in D(\pi')\}$. We fix a pure quaternion λ in Q and put $L = K(\lambda)$.

Assume that $K^\times/K^{\times 2}$ is finite. Then S is finite. For $\alpha = aK^{\times 2}$, there is a surjection from H_α to the group $K^\times/N_{L/K}(L^\times)$, where $L = K(\sqrt{-a})$, and this group is a quotient of $K^\times/K^{\times 2}$. Therefore H_α is finite for any $\alpha \in S$. Since S is also finite, it follows that $\mathrm{Herm}_1^{-1}(Q,\gamma) = \bigcup_{\alpha\in S} H_\alpha$ is finite.

Suppose now that $\mathrm{Herm}_1^{-1}(Q,\gamma)$ is finite. Then $K^\times/N_{L/K}(L^\times)$ is finite by (4.5). As $K^\times/\mathrm{Nrd}(Q^\times)$ is a quotient of this group, it is also finite. Moreover, the image of $\mathrm{disc} : \mathrm{Herm}_1^{-1}(Q,\gamma) \longrightarrow K^\times/K^{\times 2}$ is finite, which means that S is finite. Since the group of reduced norms $\mathrm{Nrd}(Q^\times)$ is generated by the elements of $D(\pi')$, it follows that $\mathrm{Nrd}(Q^\times)/K^{\times 2}$ is finite. Hence, $K^\times/K^{\times 2}$ is finite. ■

5 Anisotropic forms of dimension three

We keep the setting of the previous section. In this section we show that 3-dimensional anisotropic skew-hermitian forms over Q do exist in all but a few exceptional cases.

5.1 Lemma. *Let* $x, y, z \in Q^\times$ *be pure quaternions. If* $\mathrm{Nrd}(xyz) \notin D(\pi')$, *then the skew-hermitian form* $\langle x, y, z\rangle$ *over* Q *is anisotropic.*

Proof: If $\langle x, y, z\rangle$ is isotropic, then $\langle x, y, z\rangle \simeq \mathbb{H} \perp \langle w\rangle$ for some pure quaternion $w \in Q^\times$ and it follows that $\mathrm{Nrd}(xyz) = \mathrm{Nrd}(w) \in D(\pi')$. ■

Recall that a *preordering* of a field K is a subset $T \subseteq K$ that is closed under addition and under multiplication and contains all squares in K.

5.2 Theorem. *The following are equivalent:*

(1) $D(\pi') \cup \{0\}$ *is a preordering of* K.

(2) $D(\pi')$ *is closed under multiplication.*

(3) $D(\pi') = D(\pi)$.

(4) *For any* $a, b, c \in D(\pi')$ *one has* $abc \in D(\pi')$.

If any of these conditions holds, then K *is a real field and* $Q_{K(\sqrt{-1})}$ *is split.*

Proof: By the definition of a preordering, (1) implies (2). Since any element of Q is a product of two pure quaternions, the group of nonzero norms $D(\pi)$ is generated by the elements of $D(\pi')$. Therefore (2) implies (3). Since $D(\pi)$ is always a group, it is clear that (3) implies (4).

Assume now that (4) holds. Take a diagonalization $\pi' \simeq \langle a, b, c\rangle$. Then $a, b, c \in D(\pi')$, so (4) yields that $abc \in D(\pi')$. Since π' has determinant 1, we have $abc \in K^{\times 2}$ and conclude that $1 \in D(\pi')$. Fixing $c = 1 \in D(\pi')$ we conclude from (4) that $D(\pi')$ is closed under multiplication. Hence (2) and (3) are satisfied. For $a, b \in D(\pi')$, we have $a^{-1}b \in D(\pi')$, whence $1 + a^{-1}b \in D(\pi) = D(\pi')$ by (3) and $a + b = a(1 + a^{-1}b) \in D(\pi')$ by (2). Hence $D(\pi')$ is closed under addition. Therefore $D(\pi') \cup \{0\}$ is a preordering, showing (1). Since $\pi = \langle 1 \rangle \perp \pi'$ is anisotropic, this preordering does not contain -1, so K is real. Moreover, $Q_{K(\sqrt{-1})}$ is split because $1 \in D(\pi')$. ∎

5.3 Corollary. *If $D(\pi') \neq D(\pi)$ or if K is nonreal or if $Q_{K(\sqrt{-1})}$ is a division algebra, then $u^+(Q) \geqslant 3$.*

Proof: By (5.2), in each case there are $a, b, c \in D(\pi')$ with $abc \notin D(\pi')$. With pure quaternions $x, y, z \in Q$ such that $\mathrm{Nrd}(x) = a$, $\mathrm{Nrd}(y) = b$, and $\mathrm{Nrd}(z) = c$, the skew-hermitian form $\langle x, y, z \rangle$ is anisotropic by (5.1). ∎

5.4 Example. Let $k = \mathbb{C}(X_1, X_2)$, $Q = (X_1, X_2)$, and $K = \mathbb{C}(X_1, \ldots, X_n)$ for some $n \geqslant 2$. Then Q_K is a division algebra and $u^+(Q_K) \leqslant 3 \cdot 2^{n-2}$ by (2.3), because K is a $\mathcal{C}_n$-field. By (5.3), there is an anisotropic 3-dimensional skew-hermitian form h over Q. Multiplying this form h by the quadratic form $\langle 1, X_3 \rangle \otimes \cdots \otimes \langle 1, X_n \rangle$ over K, we obtain a skew-hermitian form of dimension $3 \cdot 2^{n-2}$ over Q_K. Therefore $u^+(Q_K) = 3 \cdot 2^{n-2}$.

6 Kaplansky fields

Kaplansky [1] noticed that most statements about quadratic over local fields remain valid over what he called 'generalized Hilbert fields', which are called 'pre-Hilbert fields' in [4, Chap. XII, Sect. 6]. As the relation to Hilbert's work is vague (based on the notion of the 'Hilbert symbol' for a local field), we use the term 'Kaplansky field' instead. To be precise, K is called a *Kaplansky field* if there is a unique quaternion division algebra over K (up to isomorphism). Natural examples of such fields are local fields and real closed fields. For the construction of other examples we refer to [4, Chap. XII, Sect. 7].

Tsukamoto [10] obtained a classification for skew-hermitian forms over the unique quaternion division algebra over a field K that is either real closed or a local number field. As observed in [10], the same result holds more generally under the condition that the field K satisfies 'local class field theory'. In this section we show that Tsukamoto's classification for skew-hermitian forms over a quaternion division algebra Q over K is valid whenever K is a Kaplansky field, which is a strictly weaker condition. The proof is adapted from [10] and [9, Chap. 10, (3.6)].

6.1 Lemma. *Let K be a Kaplansky field and let Q be the unique quaternion division algebra over K. For any pure quaternion $\lambda \in Q^\times$ and any $d \in K^\times$ we have $\langle \lambda \rangle \simeq \langle d\lambda \rangle$ as skew-hermitian forms over Q.*

Proof: Let $\mu \in Q^\times$ be such that $\mu\lambda = -\lambda\mu$. Then $Q \simeq (a, b)_K$ for $a = \lambda^2$ and $b = \mu^2$. Assume that there exists $d \in K^\times$ with $\langle \lambda \rangle \not\simeq \langle d\lambda \rangle$. By (4.4), none of the forms $\langle 1, -a \rangle$ and $\langle b, -ab \rangle$ represents d. Then $(a, d)_K$ is a quaternion division algebra and not isomorphic to Q, contradicting the hypothesis. ∎

6.2 Theorem (Tsukamoto)**.** *Let K be a Kaplansky field and let Q be the unique quaternion division algebra over K.*

(a) *Any skew-hermitian form of dimension at least* 4 *over* Q *is isotropic.*

(b) *Skew-hermitian forms over* Q *are classified by their dimension and discriminant.*

(c) *A* 2*-dimensional skew-hermitian form over* Q *is isotropic if and only if it has trivial discriminant.*

(d) *Any* 3*-dimensional skew-hermitian form over* Q *with trivial discriminant is anisotropic.*

Proof: Let γ denote the canonical involution on Q. We first show that 1-dimensional skew-hermitian forms over Q are classified by the discriminant. Suppose that $z_1, z_2 \in \mathrm{Sym}^-(Q,\gamma)$ are such that the skew-hermitian forms $\langle z_1\rangle$ and $\langle z_2\rangle$ over Q have the same discriminant. According to (4.2), then $\langle z_1\rangle \simeq \langle cz_2\rangle$ for some $c \in K$. Since also $\langle z_2\rangle \simeq \langle cz_2\rangle$ by (6.1), we obtain that $\langle z_1\rangle \simeq \langle z_2\rangle$.

(a) Let $z_1, z_2 \in \mathrm{Sym}^-(Q,\gamma)$ be such that the skew-hermitian form $\langle z_1, z_2\rangle$ over Q has trivial discriminant. Then $\mathrm{Nrd}(z_1)$ and $\mathrm{Nrd}(z_2)$ represent the same class in $K^\times/K^{\times 2}$. This means that the 1-dimensional forms $\langle z_1\rangle$ and $\langle -z_2\rangle$ have the same discriminant, whence $\langle z_1\rangle \simeq \langle -z_2\rangle$ by what we showed above.

(b) Let φ be a 3-dimensional skew-hermitian form over Q. If φ is isotropic, then $\varphi \simeq \mathbb{H}\perp\langle a\rangle$ where $a \in \mathrm{Sym}^-(Q,\gamma)$, and it follows that φ has the same discriminant as $\langle a\rangle$, which cannot be trivial by part (a).

(c) Let φ be a 4-dimensional skew-hermitian form over Q. Choose $a_1,\dots,a_4 \in \mathrm{Sym}^-(Q,\gamma)$ such that $\varphi \simeq \langle a_1,a_2,a_3,a_4\rangle$. As $\dim_K(\mathrm{Sym}^-(Q,\gamma)) = 3$, there exist $c_1,\dots,c_4 \in K$, not all zero, such that $c_1a_1 + c_2a_2 + c_3a_3 + c_4a_4 = 0$. By the first paragraph of the proof, for $1 \leqslant i \leqslant 4$ there is some $d_i \in Q$ with $c_ia_i = \gamma(d_i)a_id_i$. Then $\sum_{i=1}^4 \gamma(d_i)a_id_i = 0$ and thus φ is isotropic.

(d) Let φ and ψ be two n-dimensional skew-hermitian forms over Q for some $n \geqslant 1$, and assume that both forms have the same discriminant. By (b), the $2n$-dimensional form $\varphi \perp -\psi$ then splits off $n-1$ hyperbolic planes. The remaining 2-dimensional form has trivial discriminant and thus is hyperbolic by (a). Therefore $\varphi \perp -\psi$ is hyperbolic, which means that $\varphi \simeq \psi$. ∎

6.3 Corollary. *Let* Q *be a quaternion division algebra over* K*. Skew-hermitian forms over* Q *are classified by dimension and discriminant if and only if* K *is a Kaplansky field.*

Proof: By (6.2) the condition is sufficient. To show its necessity, suppose that Q is not the unique quaternion division algebra over K. By (4.9), there exists $\lambda \in Q \setminus K$ such that, for the field $L = K(\lambda) \subseteq Q$, the index of $N_{L/K}(L^\times)$ in $K^\times$ is at least 4. Let $a, b \in K^\times$ be such that $\lambda^2 = a$ and $Q \simeq (a,b)_K$. Now, there exists $c \in K^\times$ such that neither c nor bc is a norm of L/K. Then the two 1-dimensional skew-hermitian forms $\langle\lambda\rangle$ and $\langle c\lambda\rangle$ over Q have the same discriminant, but they are not isometric by (4.4). ∎

6.4 Corollary. *Let* K *be a nonreal Kaplansky field and let* Q *be the unique quaternion division algebra over* K*. Then* $u^+(Q) = 3$*.*

Proof: We have $u^+(Q) \leqslant 3$ by (6.2) and $u^+(Q) \geqslant 3$ by (5.3). ∎

The field K is said to be *euclidean* if $K^{\times 2} \cup \{0\}$ is an ordering of K, or equivalently, if K is real and $K^\times = K^{\times 2} \cup -K^{\times 2}$ (cf. [4, Chap. VIII, (4.2)]). If K is euclidean, then $(-1,-1)_K$ is the unique quaternion division algebra over K, in particular K is a Kaplansky field.

6.5 Proposition. *Let Q be a quaternion division algebra over K and γ its canonical involution. The following are equivalent:*

(1) $u^+(Q) = 1$.

(2) $|\mathrm{Herm}_1^{-1}(Q,\gamma)| = 1$.

(3) *K is euclidean and $Q \simeq (-1,-1)_K$.*

Proof: The equivalence of (1) and (2) is clear. If (3) holds, then K is a Kaplansky field and any 1-dimensional skew-hermitian form over Q has trivial discriminant, and by (6.2) this implies (2).

Suppose that (1) and (2) hold. From (2) it follows that $D(\pi') = K^{\times 2}$, whence $\pi' \simeq \langle 1,1,1\rangle$ and $\sum K^{\times 2} = K^{\times 2}$. Therefore we have $Q \simeq (-1,-1)_K$ and furthermore $-1 \notin K^{\times 2} = \sum K^{\times 2}$, as Q is not split. So K is real. To prove (3), it remains to show that $K^\times = K^{\times 2} \cup -K^{\times 2}$. We fix $i \in Q$ with $i^2 = -1$ and $L = K(i)$. For any $a \in K^\times$, the skew-hermitian form $\langle i, -ai\rangle$ over Q is isotropic by (1), whence $a \in N_{L/K}(L^\times) \cup -N_{L/K}(L^\times) = K^{\times 2} \cup -K^{\times 2}$ by (4.4). ■

6.6 Proposition. *Let K be a real Kaplansky field and let $Q = (-1,-1)_K$. Then $u^+(Q) \leqslant 2$.*

Proof: Let i be a pure quaternion in Q with $i^2 = -1$. By (6.2), the skew-hermitian form $\langle i, i\rangle$ over Q is isotropic. We claim that every 2-dimensional skew-hermitian form over Q is isometric to $\langle i, z\rangle$ for some pure quaternion $z \in Q^\times$. Once this is shown, it follows that every 3-dimensional skew-hermitian form over Q contains $\langle i, i\rangle$ and therefore is isotropic.

Let h be a 2-dimensional skew-hermitian form over Q. We write $\mathrm{disc}(h) = aK^{\times 2}$ with $a \in K^\times$. Then $a \in \mathrm{Nrd}(Q^\times)$ and a is a sum of four squares in K. Since K is a real Kaplansky field, the quaternion algebra $(-1,a)_K$ is split, because it is not isomorphic to $(-1,-1)_K$. Therefore a is a sum of two squares in K. It follows that there is a pure quaternion z in Q with $\mathrm{Nrd}(z) = a$. Then the skew-hermitian form $\langle i, z\rangle$ over Q has discriminant a and is therefore isometric to h, by (6.2). ■

6.7 Example. Let K be a maximal subfield of $\mathbb{R}$ with $2 \notin K^{\times 2}$. Then K is a real field with four square classes represented by $\pm 1, \pm 2$, and $Q = (-1,-1)_K$ is the unique quaternion division algebra over K. Since $Q \simeq (-1,-2)_K$, there are anticommuting pure quaternions $\alpha, \beta \in Q$ with $\alpha^2 = 1$ and $\beta^2 = 2$. Then the skew-hermitian form $\langle \alpha, \beta\rangle$ over Q has nontrivial discriminant $2K^{\times 2}$, so it is anisotropic. This together with (6.6) shows that $u^+(Q) = 2$.

6.8 Theorem. *Let K be a Kaplansky field and let Q be the unique quaternion division algebra over K. Then*

$$u^+(Q) = \begin{cases} 1 & \text{if } K \text{ is real euclidean,} \\ 2 & \text{if } K \text{ is real non-euclidean,} \\ 3 & \text{if } K \text{ is nonreal.} \end{cases}$$

Proof: This follows from (6.2), (6.5), (6.6), and (5.3). ■

Acknowledgements

We wish to express our gratitude to David Lewis and to Jean-Pierre Tignol for inspiring discussions on the subject of this work. We further wish to acknowledge financial support provided by the European RTN Network 'Algebraic *K*-Theory, Linear Algebraic Groups and Related Structures' (HPRN CT-2002-00287), the Swiss National Science Foundation (Grant No. 200020-100229/1), the Irish Research Council for Science, Engineering, and Technology (Basic Research Grant SC/02/265), the Research Council of Sharif University of Technology, the Deutsche Forschungsgemeinschaft (project Quadratic Forms and Invariants, BE 2614/3-1), and by the Zukunftskolleg, Universität Konstanz.

References

[1] I. Kaplansky. Fröhlich's local quadratic forms. *J. Reine Angew. Math.* **239/240** (1969): 74–77.

[2] M.-A. Knus. *Quadratic and Hermitian forms over rings.* Grundlehren der Mathematischen Wissenschaften, **294**. Springer-Verlag, Berlin, 1991.

[3] M.-A. Knus, S. A. Merkurjev, M. Rost, J.-P. Tignol. *The Book of Involutions.* American Mathematical Society Colloquium Publications, **44**. Providence, RI, 1998.

[4] T. Y. Lam, *Introduction to Quadratic Forms over Fields.* Graduate Studies in Mathematics, **67**. American Mathematical Society, Providence, RI, 2005.

[5] D. W. Lewis. *Quadratic forms, quaternion algebras and Hilbert's theorem 90.* Preprint (1979).

[6] D. W. Lewis. Sums of Hermitian Squares. *J. Algebra* **115** (1988): 466–480.

[7] M. G. Mahmoudi. Hermitian forms and the u-invariant. *Manuscripta Math.* **116** (2005): 493–516.

[8] A. Pfister. *Quadratic forms with applications to algebraic geometry and topology.* London Mathematical Society Lecture Note Series, **217**. Cambridge University Press, Cambridge, 1995.

[9] W. Scharlau. *Quadratic and Hermitian forms.* Grundlehren der Mathematischen Wissenschaften, **270**. Springer-Verlag, Berlin, 1985.

[10] T. Tsukamoto. On the local theory of quaternionic anti-hermitian forms. *J. Math. Soc. Japan* **13** (1961): 387–400.

Fachbereich Mathematik und Statistik, D203, Universität Konstanz,
78457 Konstanz, Germany.
Email: `becher@maths.ucd.ie`

Department of Mathematical Sciences, Sharif University of Technology,
P.O. Box: 11155-9415, Tehran, Iran.
Email: `mmahmoudi@sharif.ir`