On a property of PLS-spaces inherited by their tensor products

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Abstract

The class of PLS-spaces covers most of the natural spaces of analysis, e. g. the space of real analytic functions, spaces of distributions. We investigate the property of PLS-spaces called *dual interpolation estimate* and show that in many important and classical cases this property is inherited by tensor products of two PLS-spaces. We establish the inheritance if at least one of the spaces is a nuclear Fréchet space or a PLN-space. The latter includes the important, classical case when one of the spaces is the dual of a nuclear Fréchet space.

1 Introduction

The aim of this paper is to investigate when the dual interpolation estimate is inherited by tensor products of PLS-spaces. Basic definitions (e.g. PLS-space, dual interpolation estimate) and properties are collected in Section 2 while Section 3 contains all the main results. We prove the inheritance if e.g. at least one of the spaces is nuclear Fréchet – Th. 6 or at least one of them is a PLN-space – Th. 9. According to [14, 21.8.4] not every nuclear Fréchet space can be given a PLN-space structure therefore one has to distinguish these two cases.

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The property we are going to consider has its origin in the so called $(DN) - (\Omega)$ type conditions for Fréchet spaces. They were extensively explored by Vogt starting with the paper [30]. These conditions have proved to be very useful in several contexts. They appear in the characterization of subspaces and quotients of power series spaces (see [29], [32] and [25]). If X is a stable power series space of finite type then the conditions $(DN) - (\Omega)$ characterize all those Fréchet spaces Y for which $L(X, Y) = LB(X, Y)$ and $L(Y, X) = LB(Y, X)$, i.e. every continuous and linear operator is bounded in the sense that it maps some zero neighbourhood into a bounded set (see [28, Ths. 2.1, 4.2]). If X belongs to the same class of Fréchet spaces then the properties of type (DN) and (Ω) characterize those Fréchet spaces Y for which the pairs (X, Y) and (Y, X) are tame (see [20]). Tameness is (see [12, Th. 2.3]) related to the problem of Pełczyński (see [19]) whether every complemented subspace of a nuclear Fréchet space with basis has a basis itself. $(DN) - (\Omega)$ type conditions have also much to do with the splitting of short exact sequences of Fréchet spaces (see e.g. [21], [17, Th. 30.1]). The condition we are going to investigate found an interesting application in the proof of the non-existence of basis in the space of real analytic functions $\mathcal{A}(\Omega)$ (see [10]). The dual interpolation estimate also plays an important role in the theory of splitting of short exact sequences of PLS-spaces (see [24], [3]). This property appears also in the context of surjectivity of operators on spaces of vector valued distributions and real analytic parameter dependence of solutions of differential equations (see [4], [2] and [6] for a comprehensive survey of this topic).

2 Preliminaries

Let us recall that *PLS-spaces* are (see [10], [11]) the projective limits of a sequence of duals of Fréchet-Schwartz spaces. This means that every PLS-space X can be viewed as

$$X = \mathrm{proj}_{N\in\mathbb{N}}\mathrm{ind}_{n\in\mathbb{N}}X_{N,n},$$

where all the $X_{N,n}$ are Banach spaces and all the linking maps $\iota_{N,n}^{N,n+1}\colon X_{N,n} \to X_{N,n+1}$ are compact. If, in addition, all these linking maps are nuclear then X is called a *PLN-space*. We also define $X_N := \mathrm{ind}_{n\in\mathbb{N}}X_{N,n}$ and by $\iota_N\colon X \to X_N$ we denote the canonical projection. Moreover $B_{N,n}$ will be the closed unit ball of $X_{N,n}$. Let us also recall (see [3]) that a PLS-space X is said to have *the dual interpolation estimate for small* θ if

$$\begin{aligned}\forall\, N\ \exists\, M\ \forall\, K\ \exists\, n\ \forall\, m\ \exists\, \theta_0 \in (0,1)\ \forall \theta \le \theta_0\ \exists\, k, C > 0\ \forall\, x' \in X_N' :\\ ||x' \circ \iota_N^M||_{M,m}^* \le C(||x' \circ \iota_N^K||_{K,k}^*)^{1-\theta}(||x'||_{N,n}^*)^{\theta}.\end{aligned} \tag{1}$$

If we take $\theta \ge \theta_0$ then X has the *dual interpolation estimate for big* θ and if we take $\theta \in (0,1)$ then X has the *dual interpolation estimate for all* θ. The examples of PLS-spaces with this property are collected in the following result.

Proposition 1 ([2], [3], [4]).

(i) A Fréchet-Schwartz space has the dual interpolation estimate for big θ *iff it has* $(\underline{\Omega})$. *It has the dual interpolation estimate for small (equivalently, all)* θ *iff it has* $(\overline{\overline{\Omega}})$.

(ii) *An LS-space has the dual interpolation estimate for small θ iff its strong dual has $(\underline{DN})$. It has the dual interpolation estimate for big (equivalently, all) θ iff its strong dual has (DN).*

(iii) *The space of distributions $\mathcal{D}'(\Omega)$ or the space of Beurling ultradistributions $\mathcal{D}'_\omega(\Omega)$ has the the dual interpolation estimate for all θ.*

(iv) *The space of real analytic functions $\mathcal{A}(\Omega)$, $\Omega \subset \mathbb{R}^n$ has the the dual interpolation estimate for small θ.*

(v) *The PLS-type power series space $\Lambda_{r,s}(\alpha, \beta)$ has the dual interpolation estimate for big θ iff $s = \infty$ or it is a Fréchet space. It has the dual interpolation estimate for small θ iff it is an LS-space.*

This property unifies four other conditions previously defined: $(P\Omega)$, $(P\overline{\overline{\Omega}})$ – see [4] and (PA), $(P\underline{A})$ – see [2]. The relation between these conditions is expressed in the result below.

Proposition 2 ([3]). *A PLS-space X has the dual interpolation: for small θ iff it has both $(P\underline{A})$ and $(P\overline{\overline{\Omega}})$; for big θ iff it has both (PA) and $(P\Omega)$; for all θ iff it has both (PA) and $(P\overline{\overline{\Omega}})$.*

Let us now give a very convenient reformulation of the dual interpolation estimate which will be extensively used further.

Lemma 3. *A PLS-space X has the dual interpolation estimate if and only if the following condition holds:*

$$\begin{gathered}\forall\, N\ \exists\, M\ \forall\, K\ \exists\, n\ \forall\, m, \gamma > 0\ \exists\, k, C > 0\ \forall\, r > 0: \\ \iota_N^M B_{M,m} \subset C\Big(r^\gamma \iota_N^K B_{K,k} + \frac{1}{r} B_{N,n}\Big).\end{gathered} \tag{2}$$

If we take $\exists\, \gamma_0 > 0\ \forall\, \gamma \le \gamma_0$ then we get the dual interpolation estimate for small θ and if we take $\exists\, \gamma_0 > 0\ \forall\, \gamma \ge \gamma_0$ then we get the dual interpolation estimate for big θ.

Proof. Necessity. The right hand side of the inequality in (1) is equal (up to some constant, universal for all $x' \in X'_N$) to the minimum of the function

$$f_{x'}(r) := r^\gamma ||x||^*_{K,k} + \frac{1}{r}||x||^*_{N,n},$$

where $\theta = \frac{\gamma}{\gamma+1}$. By [2, Lemma 3.5(b)] we get (2).

Sufficiency. For every $x \in B_{M,m}$ we get $a \in B_{K,k}$, $b \in B_{N,n}$ such that

$$\iota_N^M x = C r^\gamma \iota_N^K a + \frac{C}{r} b.$$

For arbitrary $x' \in X'_N$ we have

$$|x' \circ \iota_N^M x| \le C r^\gamma |x' \circ \iota_N^K a| + \frac{1}{r}|x' b| \le C f_{x'}(r).$$

By the choice of x we obtain (1). ∎

If X and Y are PLS-spaces then it becomes essential (in view of the definition of the dual interpolation estimate) to check whether the tensor products $X\varepsilon Y$, $X\tilde{\otimes}_\varepsilon Y$, $X\tilde{\otimes}_\pi Y$are PLS-spaces or not. If $(X_N)_N$, $(Y_N)_N$ are projective spectra of LS-spaces of X and Y, respectively then by [1, Remark 4.2] $(X_N\tilde{\otimes}_\pi Y_N)'_b$ is a Fréchet-Schwartz space for all $N \in \mathbb{N}$. Therefore $(X_N\tilde{\otimes}_\pi Y_N)''$ is an LS-space for all N. By [14, 15.6.5, 15.6.8(c)] $(X_N\tilde{\otimes}_\pi Y_N)$ is reflexive therefore

$$X\tilde{\otimes}_\pi Y = \mathrm{proj}_N(X_N\tilde{\otimes}_\pi Y_N)$$

is a PLS-space. If one of the spaces X, Y is ultrabornological then by [9, Prop. 4.3, Remark 4.4] $X\varepsilon Y$ is a PLS-space and by [11, Prop. 1.2] so is its closed subspace $X\tilde{\otimes}_\varepsilon Y$. In that case

$$X\varepsilon Y = \mathrm{proj}_N\mathrm{ind}_n L(X'_{N,n}, Y_{N,n}).$$

By [3, Cor. 1.2(c)] the dual interpolation estimate implies ultrabornologicity therefore all the three tensor products are PLS-spaces and if one of the spaces X, Y is nuclear then by [14, 18.1.8(2), 21.2.1, 21.2.2] all the three tensor products coincide.

For unexplained facts and notation from functional analysis we refer the reader to [17]. For more informations on (PLS)-spaces see [7] and references therein.

3 Main results

Let X and Y be two Fréchet-Schwartz spaces. In [27, Th. 3.5] it is shown that the dual interpolation estimate for big θ is inherited by their projective tensor product. If X' and Y' have the dual interpolation estimate for all θ and either X or Y has the approximation property then by [27, Th. 2.5] also $X'\tilde{\otimes}_\pi Y'$ has the dual interpolation estimate for all θ. In the following theorem we generalize the latter statement to arbitrary LS-spaces.

Theorem 4. *The dual interpolation estimate for all θ is inherited by the projective tensor product of two Fréchet-Schwartz as well as of two LS-spaces.*

Proof. We will show the result for LS-spaces. The case of Fréchet-Schwartz spaces follows analogously if, instead of a fundamental sequence of bounded sets, we take a fundamental sequence of zero neighbourhoods. Let X and Y be two LS-spaces, i.e. duals of Fréchet-Schwartz spaces with the fundamental sequences of bounded sets $(A_n)_n$, $(B_n)_n$, respectively. By the discussion at the end of Section 2 $X\tilde{\otimes}_\pi Y$ is an LS-space and by [15, 41.4(7)] $\overline{\Gamma(A_n \otimes B_n)}$ is a fundamental sequence of bounded sets (here Γ stands for the absolutely convex hull). According to Prop. 1(ii) we assume that

$$\begin{aligned} &\exists\, n\ \forall\, m, \gamma > 0\ \exists\, k, C > 0\ \forall\, r > 0 : \\ &\qquad A_m \subset C\Big(r^\gamma A_k + \frac{1}{r}A_n\Big), \\ &\qquad B_m \subset C\Big(r^\gamma B_k + \frac{1}{r}B_n\Big). \end{aligned} \tag{3}$$

For arbitrary $\delta > 0$, $s \in (0,1]$ we take $\gamma := 2\delta + 1$, $r := \sqrt{s}$. Tensorizing the above inclusions and using the fact that $n \le k$, $r^{2\gamma} \le r^{\gamma-1}$ one gets

$$A_m \otimes B_m \subset C^2\Big(r^{2\gamma} A_k \otimes B_k + 2r^{\gamma-1} A_k \otimes B_k + \frac{1}{r^2} A_n \otimes B_n\Big) \subset$$

$$\subset 2C^2\Big(r^{\gamma-1} A_k \otimes B_k + \frac{1}{r^2} A_n \otimes B_n\Big) = 2C^2\Big(s^\delta A_k \otimes B_k + \frac{1}{s} A_n \otimes B_n\Big).$$

For $s > 1$ we easily get

$$A_m \otimes B_m \subset A_k \otimes B_k \subset s^\delta A_k \otimes B_k + \frac{1}{s} A_n \otimes B_n.$$

We have shown that

$$\forall\, s > 0: \quad \overline{\Gamma(A_m \otimes B_m)} \subset 2C^2 \overline{s^\delta \Gamma(A_k \otimes B_k) + \frac{1}{s}\Gamma(A_n \otimes B_n)}.$$

But these are bounded sets in a Schwartz space therefore compact, which gives

$$\overline{s^\delta \Gamma(A_k \otimes B_k) + \frac{1}{s}\Gamma(A_n \otimes B_n)} = s^\delta \overline{\Gamma(A_k \otimes B_k)} + \frac{1}{s}\overline{\Gamma(A_n \otimes B_n)}$$

and by Prop. 1(ii) we are done. ∎

In [16, Prop. 2.1] it is proved that if two Köthe coechelon spaces have the dual interpolation estimate for all θ then their injective tensor product has this property too. We generalize this result.

Theorem 5. *The dual interpolation estimate for all θ is inherited by the injective tensor product of two Fréchet-Schwartz as well as of two LS-spaces.*

Proof. We will show it in the case of two Fréchet-Schwartz spaces. For their duals the proof is the same if, instead of a fundamental sequence of zero neighbourhoods, we take a fundamental sequence of bounded sets. Let X and Y be two Fréchet-Schwartz spaces with the bases of zero neighbourhoods $(U_n)_n$, $(V_n)_n$, respectively. By [15, 44.2(5)], [14, 16.4.3] and [17, prop. 24.18] $X\tilde{\otimes}_\varepsilon Y$ is a Fréchet-Schwartz space and by [15, 44.2(3)] $((U_n^\circ \otimes V_n^\circ)^\circ)_n$ is a basis of zero neighbourhoods (the polar of $U_n^\circ \otimes V_n^\circ$ taken in $X\tilde{\otimes}_\varepsilon Y$). According to Prop. 1(i) and [24, Remark after Th. 1.1] we assume that

$$\begin{aligned} &\forall\, n\ \exists\, m\ \forall\, k, \gamma > 0\ \exists\, C > 0\ \forall\, r > 0: \\ &\qquad U_m \subset C\Big(rU_k + \frac{1}{r^\gamma}U_n\Big), \\ &\qquad V_m \subset C\Big(rV_k + \frac{1}{r^\gamma}V_n\Big). \end{aligned} \tag{4}$$

Taking polars in the above inclusions, tensorizing them and again taking polars one gets

$$\begin{aligned} (U_m^\circ \otimes V_m^\circ)^\circ \subset{}& C^2\Big(\big(rU_k + \frac{1}{r^\gamma}U_n\big)^\circ \otimes \big(rV_k + \frac{1}{r^\gamma}V_n\big)^\circ\Big)^\circ \subset \\ & 4C^2\Big((rU_k)^\circ \otimes (rV_k)^\circ \cap (rU_k)^\circ \otimes (r^{-\gamma}V_n)^\circ \cap (r^{-\gamma}U_n)^\circ \\ & \otimes (rV_k)^\circ \cap (r^{-\gamma}U_n)^\circ \otimes (r^{-\gamma}V_n)^\circ\Big)^\circ. \end{aligned}$$

Now for arbitrary $\delta > 0$, $s > 1$ we take $\gamma := 2\delta + 1$, $r := \sqrt{s}$. Then $r^{2\gamma} > r^{\gamma-1}$ and together with $n \le k$ we obtain

$$(U_m^\circ \otimes V_m^\circ)^\circ \subset 4C^2\Big(r^{-2}(U_k^\circ \otimes V_k^\circ) \cap r^{\gamma-1}(U_n^\circ \otimes V_n^\circ)\Big)^\circ \subset$$

$$\subset 4C^2\Big(s^{-1}\overline{\Gamma(U_k^\circ \otimes V_k^\circ)} \cap s^{\delta}\overline{\Gamma(U_n^\circ \otimes V_n^\circ)}\Big)^\circ.$$

By [23, Ch. IV, 1.5, Cor. 2] this gives

$$(U_m^\circ \otimes V_m^\circ)^\circ \subset 4C^2\overline{\Gamma(s(U_k^\circ \otimes V_k^\circ)^\circ \cup s^{-\delta}(U_n^\circ \otimes V_n^\circ)^\circ)} \subset$$

$$\subset 4C^2\overline{s(U_k^\circ \otimes V_k^\circ)^\circ + s^{-\delta}(U_n^\circ \otimes V_n^\circ)^\circ}.$$

But the sets being considered are zero neighbourhoods therefore we get

$$(U_m^\circ \otimes V_m^\circ)^\circ \subset (4C^2+1)\Big(s(U_k^\circ \otimes V_k^\circ)^\circ + \frac{1}{s^\delta}(U_n^\circ \otimes V_n^\circ)^\circ\Big).$$

Obviously for $s \in (0,1)$ one has

$$(U_m^\circ \otimes V_m^\circ)^\circ \subset (U_m^\circ \otimes V_m^\circ)^\circ \subset (4C^2+1)\frac{1}{s^\delta}(U_n^\circ \otimes V_n^\circ)^\circ$$

which proves

$$\forall\, s > 0\colon \quad (U_m^\circ \otimes V_m^\circ)^\circ \subset (4C^2+1)\Big(s(U_k^\circ \otimes V_k^\circ)^\circ + \frac{1}{s^\delta}(U_n^\circ \otimes V_n^\circ)^\circ\Big)$$

and by Prop. 1(i) and [24, Remark after Th. 1.1] finishes the proof. ∎

Now we proceed to the case when one of the spaces is a nuclear Fréchet space or its dual and the other one is an arbitrary PLS-space. The proofs are based on the idea of Vogt used in [26].

Theorem 6. *Let X be a nuclear Fréchet space and Y an arbitrary PLS-space. If they both have the dual interpolation estimate for all (big, small) θ then their completed tensor product has the same sort of the dual interpolation estimate.*

Proof. We will show the above statement in the case of all θ. The other two are proved in the same way. Let X and Y have the following representations:

$$X := \mathrm{proj}_N X_N, \qquad Y := \mathrm{proj}_N \mathrm{ind}_n Y_{N,n}.$$

Let moreover $B(Y'_{N,n}, X_N)$ denote the closed unit ball in $L(Y'_{N,n}, X_N)$ and $U_{N,n}$ the closed unit ball in $Y_{N,n}$. By $\rho_N^M\colon X_M \to X_N$, $\iota_N^M\colon Y_M \to Y_N$ $(M \ge N)$ we mean the linking maps in X and Y, respectively. We may assume that all the spaces X_N are Hilbert and all the maps ρ_N^M are nuclear. Moreover we may assume that for all $N, n \in \mathbb{N}$ the operator ι_N^{N+1} acts from $Y_{N+1,n}$ into $Y_{N,n}$. By [13, Th. 1.3] we may assume that all the spaces $Y_{N,n}$ are reflexive. We know that $X\tilde{\otimes}Y = \mathrm{proj}_N \mathrm{ind}_n L(Y'_{N,n}, X_N)$ and by Lemma 3 we have to prove that

$$\begin{aligned}&\forall N \ \exists M \ \forall K \ \exists n \ \forall m, \gamma > 0 \ \exists k, C > 0 \ \forall r > 0 : \\ &\rho_N^M B(Y'_{M,m}, X_M)(\iota_N^M)' \subset C\Big(r^\gamma \rho_N^K B(Y'_{K,k}, X_K)(\iota_N^K)' + \frac{1}{r} B(Y'_{N,n}, X_N)\Big).\end{aligned} \tag{5}$$

This inclusion is understood in the following sense: for every operator $T \in B(Y'_{M,m}, X_M)$ there exist operators $S \in Cr^\gamma B(Y'_{K,k}, X_K)$ and $R \in \frac{C}{r} B(Y'_{N,n}, X_N)$ such that for all $y' \in Y'_N$ we have

$$\rho_N^M \circ T(y' \circ \iota_N^M) = \rho_N^K \circ S(y' \circ \iota_N^K) + R(y').$$

Since X and Y have the dual interpolation estimate we obtain (with the quantifiers in mind)

$$||x' \circ \rho_{N+2}^M||_M^* \le C(||x' \circ \rho_{N+2}^{K+2}||_{K+2}^*)^{1-\theta}(||x'||_{N+2}^*)^\theta \quad \forall\, x' \in X'_{N+2},$$

$$||y' \circ \iota_N^M||_{M,m}^* \le C(||y' \circ \iota_N^K||_{K,k}^*)^{1-\theta}(||y'||_{N,n}^*)^\theta \quad \forall\, y' \in Y'_N.$$

Multiplying these inequalities and proceeding as in Lemma 3 we get for $\gamma = \frac{\theta}{1-\theta}$ and all $r > 0$

$$\begin{aligned}&||x' \circ \rho_{N+2}^M||_M^* ||y' \circ \iota_N^M||_{M,m}^* \le \\ &\le C^2\Big(r^\gamma ||x' \circ \rho_{N+2}^{K+2}||_{K+2}^* ||y' \circ \iota_N^K||_{K,k}^* + \frac{1}{r}||x'||_{N+2}^* ||y'||_{N,n}^*\Big).\end{aligned}$$

By polarization as in [31, Lemma 2.1] this gives

$$||x' \circ \rho_{N+2}^M||_M^* U_{M,m} \subset C^2\Big(r^\gamma ||x' \circ \rho_{N+2}^{K+2}||_{K+2}^* U_{K,k} + \frac{1}{r}||x'||_{N+2}^* U_{N,n}\Big). \tag{6}$$

Since the operator ρ_{N+1}^{K+1} is nuclear by the Spectral Theorem [5, Th. 4.1] we find its representation

$$\rho_{N+1}^{K+1} x = \sum_{j=1}^{+\infty} a_j \langle x, e_j \rangle_{K+1} f_j,$$

where $(a_j)_j$ is a sequence of positive numbers, $(e_j)_j$ is an orthonormal sequence in X_{K+1} and $(f_j)_j$ is an orthonormal basis in X_{N+1}. Let x_j^* be the j-th evaluation functional with respect to $(f_j)_j$, i.e.

$$x_j^*(x) := \langle x, f_j \rangle_{N+1} \quad \forall\, x \in X_{N+1}.$$

For an arbitrary operator $A \in B(Y'_{M,m}, X_M)$ we define elements

$$\phi_j := x_j^* \circ \rho_{N+1}^M \circ A \in Y''_{M,m} = Y_{M,m} \quad \forall\, j \in \mathbb{N}.$$

As can be easily calculated, $||\phi_j||_{M,m} \le ||x_j^* \circ \rho_{N+1}^M||_M^*$. Therefore $\phi_j \in ||x_j^* \circ \rho_{N+1}^M||_M^* U_{M,m}$ for all $j \in \mathbb{N}$. Applying (6) to the elements $x_j^* \circ \rho_{N+1}^{N+2} \in X'_{N+2}$ we obtain

$$\xi_j \in C^2 r^\gamma ||x' \circ \rho_{N+1}^{K+2}||_{K+2}^* U_{K,k},$$

$$\eta_j \in \frac{C^2}{r} ||x_j^* \circ \rho_{N+1}^{N+2}||_{N+2}^* U_{N,n}$$

such that

$$\iota_N^M \phi_j = \iota_N^K \xi_j + \eta_j \quad \forall\, j \in \mathbb{N}. \tag{7}$$

Now we define linear operators

$$B\colon Y'_{K,k} \to X_K, \quad By' := \sum_{j=1}^{+\infty} \frac{1}{a_j} y'(\xi_j) \rho_K^{K+1} e_j,$$

$$D\colon Y'_{N,n} \to X_N, \quad Dy' := \sum_{j=1}^{+\infty} y'(\eta_j) \rho_N^{N+1} f_j.$$

In fact they are continuous. To see this we need some calculations. We start with the operator B. First of all

$$\rho_K^{K+1} e_j \neq 0 \quad \forall\, j \in \mathbb{N}. \tag{8}$$

If not then

$$a_j f_j = \rho_{N+1}^{K+1} e_j = \rho_{N+1}^{K} \rho_K^{K+1} e_j = 0 - \text{a contradiction.}$$

If for some $j \in \mathbb{N}\colon\ (\rho_{K+1}^{K+2})' e_j = 0$ then for all $x \in X_{K+2}$ we have

$$x_j^*(\rho_{N+1}^{K+2} x) = \langle \rho_{N+1}^{K+2} x, f_j \rangle_{N+1} = a_j \langle \rho_{K+1}^{K+2} x, e_j \rangle_{K+1} = 0.$$

This gives $||x_j^* \circ \rho_{N+1}^{K+2}||_{K+2}^* = 0$ and implies $\xi_j = 0$ therefore we may assume that

$$(\rho_{K+1}^{K+2})' e_j \neq 0 \quad \forall\, j \in \mathbb{N}. \tag{9}$$

By (8) and (9) we define, for all $j \in \mathbb{N}$, positive numbers

$$\lambda_j^K := ||(\rho_{K+1}^{K+2})' e_j||_{K+2} ||\rho_K^{K+1} e_j||_K,$$

$$\gamma_j^K := \left(||(\rho_{K+1}^{K+2})' e_j||_{K+2} \right)^{-1}.$$

Now for any $j \in \mathbb{N}$, $x \in X_{K+2}$ we obtain

$$\gamma_j^K |\langle \rho_{K+1}^{K+2} x, e_j \rangle_{K+1}| \leq ||x||_{K+2}$$

Using the definition of x_j^* and ρ_{N+1}^{K+1} we get

$$\frac{1}{a_j} \gamma_j^K |x_j^*(\rho_{N+1}^{K+2} x)| = \gamma_j^K |\langle \rho_{K+1}^{K+2} x, e_j \rangle_{K+1}| \leq ||x||_{K+2}.$$

This gives

$$\sup_j \frac{1}{a_j} \gamma_j^K ||x_j^* \circ \rho_{N+1}^{K+2}||_{K+2}^* \leq 1. \tag{10}$$

Let us now focus on the operator D. If for some $j \in \mathbb{N}$ we have $\rho_N^{N+1} f_j = 0$ then this particular summand doesn't influence the estimation of the norm of D. Therefore we may assume that

$$\rho_N^{N+1} f_j \neq 0 \quad \forall j \in \mathbb{N}. \tag{11}$$

Moreover

$$(\rho_{N+1}^{N+2})' f_j \neq 0 \quad \forall j \in \mathbb{N}. \tag{12}$$

If not then

$$a_j = \langle \rho_{N+1}^{K+1} e_j, f_j \rangle_{N+1} = \langle \rho_{N+1}^{N+2} \rho_{N+2}^{K+1} e_j, f_j \rangle_{N+1} = 0 \text{ – a contradiction.}$$

By (11) and (12) we define, for all $j \in \mathbb{N}$, positive numbers

$$\lambda_j^N := ||(\rho_{N+1}^{N+2})' f_j||_{N+2} ||\rho_N^{N+1} f_j||_N,$$

$$\gamma_j^K := \left(||(\rho_{N+1}^{N+2})' f_j||_{N+2}\right)^{-1}.$$

Similarly as above we obtain

$$\gamma_j^N |x_j^*(\rho_{N+1}^{N+2} x)| \leq ||x||_{N+2}$$

which gives

$$\sup_j \gamma_j^N ||x_j^* \circ \rho_{N+1}^{N+2}||_{N+2}^* \leq 1. \tag{13}$$

Since every nuclear operator between Hilbert spaces is Hilbert-Schmidt we obtain for $\nu := N$, $g_j := f_j$ and $\nu := K$, $g_j := e_j$ the following inequalities:

$$\sum_j \lambda_j^\nu = \sum_j ||(\rho_{\nu+1}^{\nu+2})' g_j||_{\nu+2} ||\rho_\nu^{\nu+1} g_j||_\nu \leq \sigma(\rho_{\nu+1}^{\nu+2}) \sigma(\rho_\nu^{\nu+1}) =: C(\nu) < +\infty,$$

where $\sigma(\cdot)$ denotes the Hilbert-Schmidt norm. Finally, recalling the choice of ξ_j, η_j and using (10), (13) and the above estimation we calculate the norms of the operators B and D:

$$||By'||_K \leq \sum_j \frac{1}{a_j} |y'(\xi_j)| ||\rho_K^{K+1} e_j||_K = \sum_j \frac{1}{a_j} |y'(\xi_j)| \gamma_j^K \lambda_j^K \leq$$

$$\leq \sum_j \lambda_j^K \sup_i \frac{1}{a_j} \gamma_j^K ||\xi_j||_{K,k} ||y'||_{K,k}^* \leq$$

$$\leq C(K) \sup_j \frac{1}{a_j} \gamma_j^K C^2 r^\gamma ||x_j^* \circ \rho_{N+1}^{K+2}||_{K+2}^* ||y'||_{K,k} \leq$$

$$\leq C^2 C(K) r^\gamma ||y'||_{K,k};$$

$$||Dy'||_N \leq \sum_j |y'(\eta_j)| ||\rho_N^{N+1} f_j||_N = \sum_j |y'(\eta_j) \gamma_j^N \lambda_j^N \leq$$

$$\leq C(N)\sup_j \gamma_j^N ||\eta_j||_{N,n}||y'||^*_{N,n} \leq C(N)C^2\frac{1}{r}||y'||^*_{N,n}.$$

Of course, the above estimations are valid for all $r > 0$. It remains to show that

$$\rho_N^M \circ A(y' \circ \iota_N^M) = \rho_N^K \circ B(y' \circ \iota_N^K) + D(y') \quad \forall y' \in Y'_N.$$

Using (7) one has

$$\rho_N^K \circ B(y' \circ \iota_N^K) + D(y') = \sum_j y'(\iota_N^K \xi_j)\rho_N^{N+1}\Big(\frac{1}{a_j}\rho_{N+1}^{K+1}e_j\Big) + \sum_j y'(\eta_j)\rho_N^{N+1}f_j =$$

$$= \sum_j y'(\iota_N^K \xi_j + \eta_j)\rho_N^{N+1}f_j = \sum_j y'(\iota_N^M \phi_j)\rho_N^{N+1}f_j =$$

$$\rho_N^{N+1}\Big(\sum_j x_j^* \circ \rho_{N+1}^M \circ A(y' \circ \iota_N^M)f_j\Big) = \rho_N^{N+1}\Big(\sum_j \langle \rho_{N+1}^M \circ A(y' \circ \iota_N^M), f_j\rangle_{N+1}f_j\Big) =$$

$$= \rho_N^{N+1} \circ \rho_{N+1}^M \circ A(y' \circ \iota_N^M) = \rho_N^M \circ A(y' \circ \iota_N^M).$$

This gives (5) with the constant $C_1 := C^2(C(N) + C(K))$ and finishes the proof. ■

In the previous results one of the spaces was an arbitrary PLS-space while the other one had a simpler structure of an LS-space or FS-space and was nuclear. Now we will show the case when still one of the PLS-spaces is arbitrary but the other one is the so called Köthe type PLS-space (see [7] for the definition).

Theorem 7. *Let X be an arbitrary PLS-space.*
(1) If X and $\Lambda^1(A)$ have the dual interpolation estimate for all (big, small) θ then their projective tensor product has the same sort of the dual interpolation estimate.
(2) If X and $\Lambda^\infty(A)$ have the dual interpolation estimate for all (big, small) θ then their injective tensor product has the same sort of the dual interpolation estimate.

Proof. We will show the case for all θ. The other two are analogous. By

$$X = \mathrm{proj}_N \mathrm{ind}_n X_{N,n},$$

$$\Lambda^1(A) = \mathrm{proj}_N \mathrm{ind}_n l^1(a_{j,N,n}),$$
$$\Lambda^\infty(A) = \mathrm{proj}_N \mathrm{ind}_n l^\infty(a_{j,N,n})$$

we denote the PLS representations of the considered spaces and by
$\iota_N^M \colon X_M \to X_N$ the linking maps.
(1): By [14, 15.4.2] and [15, 41.4(7)] we have

$$X\tilde{\otimes}_\pi \Lambda^1(A) = \mathrm{proj}_N \mathrm{ind}_n (X_{N,n}\tilde{\otimes}_\pi l^1(a_{j,N,n})).$$

By [22, Ex. 2.6] this gives

$$X\tilde{\otimes}_\pi \Lambda^1(A) = \mathrm{proj}_N \mathrm{ind}_n l^1(a_{j,N,n})(X_{N,n}),$$

where the latter is the space of sequences $(x_j)_j \subset X_{N,n}$ such that

$$\sum_{j=1}^{+\infty} ||x_j||_{N,n} a_{j,N,n} < +\infty.$$

The linking maps arise naturally from the linking maps of X therefore we omit them. If X and $\Lambda^1(A)$ have the dual interpolation estimate for all θ then

$$\begin{aligned}
&\forall\, N\ \exists\, M\ \forall\, K\ \exists\, n\ \forall\, m,\theta\in(0,1)\ \exists\, k, C>0\ \forall\, x'\in X'_N,\ j\in\mathbb{N}:\\
&||x'\circ\iota_N^M||^*_{M,m}\le C(||x'\circ\iota_N^K||^*_{K,k})^{1-\theta}(||x'||^*_{N,n})^{\theta},\\
&\frac{1}{a_{j,M,m}}\le C\Big(\frac{1}{a_{j,K,k}}\Big)^{1-\theta}\Big(\frac{1}{a_{j,N,n}}\Big)^{\theta}.
\end{aligned}\tag{14}$$

If ϕ is a functional on the N-th projective limit step of $X\tilde{\otimes}_\pi\Lambda^1(A)$ then it is uniquely determined by a sequence of functionals $(x_j^*)_j\in X'_N$ such that

$$\forall\, n\in\mathbb{N}:\quad \sup_j\frac{1}{a_{j,N,n}}||x_j^*||_{N,n}<+\infty.$$

Multiplying the inequalities in (14) we easily get

$$\sup_j\frac{1}{a_{j,M,m}}||x_j^*\circ\iota_N^M||_{M,m}\le$$

$$\le C^2\Big(\sup_j\frac{1}{a_{j,K,k}}||x_j^*\circ\iota_N^K||_{K,k}\Big)^{1-\theta}\Big(\sup_j\frac{1}{a_{j,N,n}}||x_j^*||_{N,n}\Big)^{\theta}.$$

In other words,

$$||\phi||^*_{M,m}\le C^2\big(||\phi||^*_{K,k}\big)^{1-\theta}\big(||\phi||^*_{N,n}\big)^{\theta}.$$

(2): By compactness of the maps $l^\infty(a_{j,N,n})\hookrightarrow l^\infty(a_{j,N,n+1})$ we may assume that

$$\Lambda^\infty(A)=\mathrm{proj}_N\mathrm{ind}_n c_0(a_{j,N,n}).$$

The following equalities are consequences of [14, 15.4.2] and [1, Remark 4.2]:

$$\begin{aligned}
X\tilde{\otimes}_\varepsilon\Lambda^\infty(A)&=\mathrm{proj}_N(\mathrm{ind}_n X_{N,n}\tilde{\otimes}_\varepsilon\mathrm{ind}_n c_0(a_{j,N,n}))=\\
&=\mathrm{proj}_N((\mathrm{ind}_n X_{N,n})'\tilde{\otimes}_\pi(\mathrm{ind}_n c_0(a_{j,N,n}))')'=\\
&=\mathrm{proj}_N(\mathrm{proj}_n X'_{N,n}\tilde{\otimes}_\pi\mathrm{proj}_n l_1(a^{-1}_{j,N,n}))'=\\
&=\mathrm{proj}_N(\mathrm{proj}_n(X'_{N,n}\tilde{\otimes}_\pi l_1(a^{-1}_{j,N,n}))'=\\
&=\mathrm{proj}_N(\mathrm{proj}_n l_1(a^{-1}_{j,N,n})(X'_{N,n}))'=\\
&=\mathrm{proj}_N\mathrm{ind}_n l_\infty(a_{j,N,n})(X''_{N,n})=\\
&=\mathrm{proj}_N\mathrm{ind}_n c_0(a_{j,N,n})(X_{N,n}).
\end{aligned}$$

Arguing as in (1) we see that the conclusion follows form the fact that the elementary tensors satisfy the dual interpolation estimate. ■

The last problem we focus on is the case of one space being PLN. We recall (see [6]) that a PLS-space X is *deeply reduced* if any (each) of its strongly reduced representing spectra of LS-spaces (X_N,ι_N^M) satisfies the following condition:

$$\forall\, N\ \exists\, M\ \forall\, K\ \exists\, n\ \forall\, m\ \exists\, k:$$

$$\iota_N^M X_{M,m}\subset\overline{\iota_N^K X_{K,k}}^{X_{N,k}}\cap\overline{X_{N,n}}^{X_{N,k}}.$$

Proposition 8. *Dual interpolation estimate in any of its forms implies deep reducedness.*

Proof. Recall that by Lemma 3 X satisfies the dual interpolation estimate if

$$\forall N \;\exists M \;\forall K \;\exists n \;\forall m, \gamma > 0 \;\exists k, C > 0 \;\forall r > 0:$$

$$\iota_N^M B_{M,m} \subset C\Big(r^\gamma \iota_N^K B_{K,k} + \frac{1}{r} B_{N,n}\Big).$$

By the Grothendieck's Factorization Theorem [17, Th. 24.33] we find $k_0 \in \mathbb{N}$ so that $\iota_N^K\colon X_{K,k} \to X_{N,k_0}$. Therefore $\iota_N^K B_{K,k}$ is bounded in X_{N,k_0} which gives a positive constant D such that

$$\iota_N^K B_{K,k} \subset D B_{N,k_0}.$$

For arbitrary $l \in \mathbb{N}$ we find $r_l > 0$ so that $CDr_l^\gamma < \frac{1}{l}$. We obtain

$$\iota_N^M B_{M,m} \subset CDr_0^\gamma B_{N,k_0} + \frac{C}{r_0} B_{N,n} \subset X_{N,n} + \frac{1}{l} B_{N,k_0}.$$

This inclusion holds for all $l \in \mathbb{N}$ therefore

$$\iota_N^M B_{M,m} \subset \bigcap_{l \in \mathbb{N}} \Big(X_{N,n} + \frac{1}{l} B_{N,k_0}\Big) = \overline{X_{N,n}}^{X_{N,k_0}}.$$

Obviously $X_{N,n}$ embeds into $X_{N,k}$ therefore

$$B_{N,n} \subset D B_{N,k}$$

for some positive constant D. Now for arbitrary $l \in \mathbb{N}$ we take $r_l > 0$ so that $\frac{CD}{r_l} < \frac{1}{l}$ and proceeding analogously we obtain

$$\iota_N^M B_{M,m} \subset \overline{\iota_N^K X_{K,k}}^{X_{N,k}}. \qquad \blacksquare$$

The following result strengthens [6, Cor. 5.9, Cor. 5.10].

Theorem 9. *Let X and Y be two PLS-spaces with the dual interpolation estimate for all (big, small) θ. If X is a PLN-space then their completed tensor product has the same sort of the dual interpolation estimate. In particular this is so if one of the spaces is LN.*

Proof. We start with the dual interpolation estimate for X and Y separately. This means that

$$\forall N \;\exists M \;\forall K \;\exists n \;\forall m, \theta \in (0,1) \;\exists k, C > 0:$$

$$||x \circ \iota_N^M||_{M,m}^* \le C(||x \circ \iota_N^K||_{K,k}^*)^{1-\theta}(||x||_{N,n}^*)^\theta \quad \forall x \in X_N',$$

$$||y \circ \iota_N^M||_{M,m}^* \le C(||y \circ \iota_N^K||_{K,k}^*)^{1-\theta}(||y||_{N,n}^*)^\theta \quad \forall y \in Y_N'.$$

Multiplying these inequalities and using polarization techniques we obtain for all $r > 0$

$$||x \circ \iota_N^M||_{M,m}^* \iota_N^M B_{M,m} \subset C\big(r^\gamma ||x \circ \iota_N^K||_{K,k}^* \iota_N^K B_{K,k} + \frac{1}{r}||x||_{N,n}^* B_{N,n}\big), \tag{15}$$

where $\gamma := \frac{\theta}{1-\theta}$. Originally this inclusion holds for all $x \in X_N'$ but X is deeply reduced therefore by [6, Prop. 5.3] it is true for all $x \in \overline{X_{N,k}'}^{X_{N,n}' \cap X_{K,k}'}$. The next step is to obtain, for some positive constant D, the inclusion

$$\iota_N^M B(X_{M,m}', Y_{M,m}) \subset D\big(r^\gamma \iota_N^K B(X_{K,k+1}', Y_{K,k}) + \frac{1}{r} B(X_{N,n+1}', Y_{N,n})\big) \tag{16}$$

This is done exactly like in the proof of [6, Th. 5.2(b)] (compare also with the proof of [8, Th. 3.1]) but for the convenience of the reader we give a short sketch of it. We define spaces

$$H := \overline{X_{N,k+1}'}^{X_{N,n+1}' \cap X_{K,k+1}'} \subset \overline{X_{N,k}'}^{X_{N,n}' \cap X_{K,k}'},$$

$$H_1 := \big(\ker \iota_N^K |_{X_{K,k+1}}\big)^\perp,$$

$$H_0 := X_{N,n+1}', \quad U := X_{K,n}'$$

and by

$$\pi_0 \colon H_0 \to X_{N,n}', \qquad \pi_1 \colon H_1 \to X_{K,k}'$$

we denote the natural Hilbert-Schmidt injections. Moreover, $r \colon H \hookrightarrow X_{M,m}'$ is the standard continuous injection. By [8, Lemma 2.2] (compare also [18]) we obtain, for any $\varepsilon > 0$, a set I, positive weights $v, w \colon I \to \mathbb{R}$ and $(1+\varepsilon)$-isomorphisms

$$T \colon H \to l_2(I), \ V \colon H_0 \to l_2(I, v), \ W \colon H_1 \to l_2(I, w).$$

For an arbitrary operator $R \in B(X_{M,m}', Y_{M,m})$ we define

$$\phi_i := R(rT^{-1}(e_i)) \in ||rT^{-1}e_i||_{M,m}^* B_{M,m}.$$

Using (15) we find elements

$$\xi_j \in Cr^\gamma ||\pi_1 j_1 T^{-1} e_j||_{K,k}^* B_{K,k}, \quad \eta_j \in \frac{C}{r}||\pi_0 j_0 T^{-1} e_j||_{N,n}^* B_{N,n},$$

such that

$$\iota_N^M \phi_j = \iota_N^K \xi_j + \eta_j \quad \forall\, j \in I. \tag{17}$$

Now we define operators

$$\tilde{W} \colon H_1 \to Y_{K,k}, \ \tilde{W}x := \sum_j \frac{1}{w_j^2} \langle Wx, e_j \rangle_{l_2(I,w)} \xi_j,$$

$$\tilde{V} \colon H_0 \to Y_{N,n}, \ \tilde{V}x := \sum_j \frac{1}{v_j^2} \langle Vx, e_j \rangle_{l_2(I,v)} \eta_j$$

and show their continuity by estimating their norms:

$$||\tilde{W}|| \leqslant Cr^{\gamma}(1+\varepsilon)^2\nu_2(\pi_1), \quad ||\tilde{V}|| \leqslant \frac{C}{r}(1+\varepsilon)^2\nu_2(\pi_0).$$

By (17) we show that

$$\iota_N^M \circ R = \iota_N^K \circ \tilde{W} \circ j_1 + \tilde{V} \circ j_0$$

for all $x \in X_N'$. This gives (16) and by Lemma 3 proves the assertion. ∎

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