On Modified Noor Iterations for Asymptotically Nonexpansive Mappings

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Abstract

The main aim of this paper is to introduce a modified Noor iterations scheme to provide a unified approach to the Mann and Ishikawa iteration processes. We study weak and strong convergence of the new iterations scheme of a nonself asymptotically nonexpansive map satisfying a new control condition in uniformly convex Banach spaces. Several recent results about Mann-type and Ishikawa-type iteration schemes for nonself(self) asymptotically nonexpansive maps follow directly and concurrently from our results.

1 Introduction

Let C be a nonempty convex subset of a real normed space E. The map $T : C \to C$ is said to be asymptotically nonexpansive[8] if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that $\|T^n x - T^n y\| \le k_n \|x - y\|$ for all $x, y \in C$ and for all $n \ge 1$; it becomes nonexpansive if $k_n = 1$ for all $n \ge 1$. The set of fixed points of T is denoted by $F(T) = \{x : Tx = x\}$.

Iterative techniques for approximating fixed points of nonexpansive and asymptotically nonexpansive maps have been studied by many authors using various iteration schemes including the Mann iterations scheme and Ishikawa iterations scheme(see e.g.,[2-6, 9-14, 19-26,28-29]).

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Xu and Noor[29] introduced a three-step iterative scheme, an extension of two-step iterative scheme of Ishikawa[10], as follows:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n T^n y_n + (1-\alpha_n)x_n, \\ y_n = \beta_n T^n z_n + (1-\beta_n)x_n, \\ z_n = \gamma_n T^n x_n + (1-\gamma_n)x_n, \quad n \geq 1, \end{cases} \tag{1.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are control sequences in $[0,1]$ and $T : C \to C$ is asymptotically nonexpansive.

For motivation and background of the three-step iterative scheme, the reader is referred to the fundamental work of Noor[15](see also [7]). The theory of three-step iterative scheme is very rich and is well studied in the context of one or more mappings(for example, see [3], [5], [11], [18], [21], and [26]. It has been shown in [1] that three-step method performs better than Ishikawa(two-step) and Mann(one-step) methods for solving variational inequalities. This signifies that Noor three-step methods are more efficient and robust than the Mann and Ishikawa type iterative methods to solve problems of applied sciences.

From (1.1), Ishikawa iteration scheme is obtained when $\gamma_n = 0$ for all $n \geq 1$:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n T^n y_n + (1-\alpha_n)x_n, \\ y_n = \beta_n T^n z_n + (1-\beta_n)x_n, \quad n \geq 1, \end{cases} \tag{1.2}$$

From (1.2), we get Mann iteration scheme by taking $\beta_n = 0$ for all $n \geq 1$:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = \alpha_n T^n x_n + (1-\alpha_n)x_n, \quad n \geq 1. \end{cases} \tag{1.3}$$

Recall that T is completely continuous if for every bounded sequence $\{x_n\}$ in C, $\{Tx_n\}$ has a convergent subsequence in C.

Using the scheme(1.1), Xu and Noor obtained:

Theorem 1.1[29, Theorem 2.1]. *Let E be a uniformly convex Banach space and let C be a nonempty closed bounded convex subset of E. Let T be completely continuous asymptotically nonexpansive selfmap of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}$ be control sequences in $[0,1]$ satisfying: (i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, and (ii) $0 < \liminf_{n\to\infty} \beta_n \leq \limsup_{n\to\infty} \beta_n < 1$.*
For $x_1 \in C$, generate $\{x_n\}$ by (1.1). Then $\{x_n\}$ converges strongly to a fixed point of T.

If $\gamma_n = 0$ for all $n \geq 1$, in Theorem 1.1, we obtain Ishikawa-type convergence result which is a generalization of Theorem 3 in [23]. Unfortunately, neither from Theorem 1.1 nor from Theorem 3 of Rhoades[23], one can deduce directly Mann-type convergence theorem in the presence of the condition $\liminf_{n\to\infty} \beta_n > 0$ $(1 - \beta_n < 1 - \delta$ for some $\delta > 0)$, respectively. To unify the proofs of Ishikawa-type and Mann-type convergence results, Xu and Noor[29] removed the restriction $\liminf_{n\to\infty} \beta_n > 0$ and proved:

Theorem 1.2 [29, Theorem 2.2]. *Let E be a uniformly convex Banach space and let C be a nonempty closed bounded and convex subset of E. Let T be completely continuous asymptotically nonexpansive selfmap of C with $\{k_n\}$ satisfying $k_n \geq 1$ and $\sum_{n=1}^{\infty}(k_n - 1) < \infty$. Let $\{\alpha_n\}, \{\beta_n\}$ be control sequences in $[0,1]$ satisfying: (i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, and (ii) $\limsup_{n\to\infty} \beta_n < 1$. For a given value $x_1 \in C$, generate the scheme $\{x_n\}$ given in (1.2). Then $\{x_n\}$ converges strongly to a fixed point of T.*

The choice $\beta_n = 0$ for all $n \geq 1$ in the above theorem led to a Mann-type convergence result in Theorem 2.3 of [29].

This reveals that a unified approach to iterative construction of fixed points hinges on the control sequences used in the process. Indeed, a survey of the literature about approximation of fixed points of some nonlinear maps through convergence of the iterative schemes reflects that conditions on the iteration control sequences play a vital role to establish the convergence results (see [1-6, 8-29, 31]).

In the schemes(1.1-1.3), T is a selfmap on C.However, if T is from C to E, the iteration processes (1.1-1.3) may fail to be well defined.

The purpose of this paper is two fold:
(1) To construct an iteration scheme of nonself asymptotically nonexpansive maps.
(2) To provide a unified approach to the two well-known iteration processes, namely, Mann iteration and Ishikawa iteration.

2 Preliminaries

Let E be a real Banach space. A subset C of E is said to be a retract of E if there exists a continuous map $P : E \to C$ such that $Px = x$ for all $x \in C$. A map $P : E \to E$ is a retraction if $P^2 = P$. For nonself nonexpansive maps, some authors have studied strong and weak convergence in Hilbert spaces or uniformly convex Banach spaces (see, e.g.[13]).

The concept of nonself asymptotically nonexpansive maps has been introduced in 2003 by Chidume, Ofoedu and Zegeye[2] as the generalization of asymptotically nonexpansive selfmaps as follows:

Let $P : E \to C$ be the nonexpansive retraction of E onto C. A map $T : C \to E$ is asymptotically nonexpansive if there exists a sequence $\{k_n\} \subset [1, \infty)$ with $\lim_{n\to\infty} k_n = 1$ such that

$$\left\| T(PT)^{n-1}x - T(PT)^{n-1}y \right\| \leq k_n \|x - y\|$$

for all $x, y \in C$ and for all $n \geq 1$.

Using the iteration process:

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P((1-\alpha_n)x_n + \alpha_n T(PT)^{n-1}x_n), \qquad n \geq 1, \end{cases}$$

Chidume, Ofoedu and Zegeye[2] obtained some convergence theorems for nonself asymptotically nonexpansive maps in uniformly convex Banach spaces.

We give the following nonself version of (1.1).

$$\begin{cases} x_1 \in C, \\ x_{n+1} = P(\alpha_n T(PT)^{n-1} y_n + (1-\alpha_n)x_n), \\ y_n = P(\beta_n T(PT)^{n-1} z_n + (1-\beta_n)x_n), \\ z_n = P(\gamma_n T(PT)^{n-1} x_n + (1-\gamma_n)x_n), \quad n \geq 1, \end{cases} \tag{2.1}$$

where $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ are control sequences in $[0,1]$, $T : C \to E$ asymptotically nonexpansive map and P is the nonexpansive retraction as defined above.

Ishikawa type-iteration scheme and Mann type-iteration scheme follow immediately from (2.1). When T is a selfmap, P becomes the identity map and hence (2.1) coincides with (1.1).

In this note, we focus on weak and strong convergence of the scheme(2.1) to a fixed point of a nonself asymptotically nonexpansive map on an unbounded domain under a new and more flexible condition on a control sequence and deduce Ishikawa-type convergence and Mann-type convergence results simultaneously as a special case of our results.

In the sequel, we need the following definitions and results:

Definition 2.1(cf.[19]). *A normed space E is said to satisfy Opial's condition if for any sequence $\{x_n\}$ in E, $x_n \rightharpoonup x$* (weak convergence of x_n to x) *implies that* $\limsup_{n\to\infty} \|x_n - x\| < \limsup_{n\to\infty} \|x_n - y\|$ *for all $y \in E$ with $y \neq x$.*

Definition 2.2(cf.[19]). *A map $T : C \to E$ is called demiclosed with respect to $y \in E$ if for each sequence $\{x_n\}$ in C and each $x \in E$, $x_n \rightharpoonup x$ and $Tx_n \to y$ imply that $x \in C$ and $Tx = y$.*

Definition 2.3 (cf.[12]). *A map $T : C \to E$ is said to satisfy condition (A) if there exists a nondecreasing function $f : [0,\infty) \to [0,\infty)$ with $f(0) = 0$, $f(r) > 0$ for all $r \in (0,\infty)$ such that $\|x - Tx\| \geq f(d(x,F))$ for all $x \in C$ where $d(x,F) =$* $\inf_{p\in F(T)} \|x - p\|$.

Lemma 2.1[30, Theorem 2]. *Let $r > 0$ be a fixed real number. Then a Banach space E is uniformly convex if and only if there is a continuous strictly increasing convex map $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ such that for all $x, y \in B_r[0] = \{x \in E : \|x\| \leq r\}$,*

$$\|\lambda x + (1-\lambda)y\|^2 \leq \lambda \|x\|^2 + (1-\lambda) \|y\|^2 - \lambda(1-\lambda)g(\|x-y\|)$$

for all $\lambda \in [0,1]$.

Lemma 2.2 [27, Lemma 2.2]. *Let $g : [0,\infty) \to [0,\infty)$ with $g(0) = 0$ be a continuous strictly increasing map. If a sequence $\{x_n\}$ in $[0,\infty)$ satisfies* $\lim_{n\to\infty} g(x_n) = 0$, *then* $\lim_{n\to\infty} x_n = 0$.

Lemma 2.3[11, Lemma 1.1]. *Let $\{r_n\}$ and $\{s_n\}$ be two nonnegative real sequences such that*

$$r_{n+1} \leq (1 + s_n)r_n, \quad \text{for all } n \geq 1.$$

If $\sum_{n=1}^{\infty} s_n < \infty$, then $\lim_{n\to\infty} r_n$ *exists.*

3 Weak and Strong Convergence Results

We prove a pair of lemmas to establish our weak and strong convergence theorems.

Lemma 3.1. *Let C be a nonempty closed convex subset of a normed space E* which is also a nonexpansive retract of E with nonexpansive retraction P. *Let $T : C \to E$ be asymptotically nonexpansive map with $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \phi$. Then for the iterative scheme $\{x_n\}$ given in (2.1), we have that: (a) $\lim_{n\to\infty} \|x_n - q\|$ exists for any $q \in F(T)$ and (b) $\lim_{n\to\infty} d(x_n, F)$ exists.*

Proof. Let $q \in F(T)$. Using (2.1), we have the following estimates:

$$\begin{aligned} \|y_n - q\| &= \left\| P(\beta_n T(PT)^{n-1} z_n + (1-\beta_n)x_n) - Pq \right\| \\ &\leq \left\| \beta_n T(PT)^{n-1} z_n + (1-\beta_n)(x_n - q) \right\| \\ &= \left\| \beta_n (T(PT)^{n-1} z_n - q) + (1-\beta_n)(x_n - q) \right\| \\ &\leq \beta_n k_n \|z_n - q\| + (1-\beta_n)\|x_n - q\| . \end{aligned} \tag{3.1}$$

$$\begin{aligned} \|z_n - q\| &= \left\| P(\gamma_n T(PT)^{n-1} x_n + (1-\gamma_n)x_n) - Pq \right\| \\ &\leq \left\| \gamma_n T(PT)^{n-1} x_n + (1-\gamma_n)(x_n - q) \right\| \\ &= \left\| \gamma_n (T(PT)^{n-1} x_n - q) + (1-\gamma_n)(x_n - q) \right\| \\ &\leq \gamma_n k_n \|x_n - q\| + (1-\gamma_n)\|x_n - q\| . \end{aligned} \tag{3.2}$$

$$\begin{aligned} \|x_{n+1} - q\| &= \left\| P(\alpha_n T(PT)^{n-1} y_n + (1-\alpha_n)x_n) - Pq \right\| \\ &\leq \left\| \alpha_n (T(PT)^{n-1} y_n + (1-\alpha_n)(x_n - q) \right\| \\ &= \left\| \alpha_n ((T(PT)^{n-1} y_n - q) + (1-\alpha_n)(x_n - q) \right\| \\ &\leq \alpha_n k_n \|y_n - q\| + (1-\alpha_n)\|x_n - q\| . \end{aligned} \tag{3.3}$$

Combining the estimates in (3.1),(3.2) and (3.3), we have

$$\begin{aligned} \|x_{n+1} - q\| &\leq \alpha_n \beta_n \gamma_n k_n^3 \|x_n - q\| + \alpha_n \beta_n (1-\gamma_n) k_n^2 \|x_n - q\| \\ &\quad + \alpha_n k_n (1-\beta_n)\|x_n - q\| + (1-\alpha_n)\|x_n - q\| \\ &\leq k_n^3 \|x_n - q\| . \end{aligned}$$

That is,

$$\|x_{n+1} - q\| \leq k_n^3 \|x_n - q\| . \tag{3.4}$$

As $\sum_{n=1}^{\infty}(k_n^3 - 1) < \infty$, therefore by Lemma 2.3, we get that $\lim_{n\to\infty} \|x_n - q\|$ exists, which proves (a).

Further, the inequality (3.4) gives that

$$\inf_{q\in F(T)} \|x_{n+1} - q\| \leq k_n^3 \inf_{q\in F(T)} \|x_n - q\| .$$

That is,

$$d(x_{n+1}, F) \leq k_n^3 d(x_n, F)$$

and hence by an argument similar to the one above, we conclude that $\lim_{n\to\infty} d(x_n, F)$ exists. So (b) holds.

The condition $\limsup_{n\to\infty} \beta_n k_n(1 + \gamma_n k_n)$, in the lemma to follow, is a new one (cf.(ii) in Theorem 1.1).

Lemma 3.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E* which is also a nonexpansive retract of E with nonexpansive retraction P. *Let $T : C \to E$ be asymptotically nonexpansive map with $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \phi$. For a given $x_1 \in C$, let $\{x_n\}$ be the sequence given in (2.1). If $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, then* $\lim_{n\to\infty} \left\|x_n - T(PT)^{n-1}y_n\right\| = 0$. *Further, if* $\limsup_{n\to\infty} \beta_n k_n(1 + \gamma_n k_n) < 1$, *then* $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Proof. For any $q \in F(T)$, $\{x_n - q\}$ is bounded. Therefore $\{ x_n - q, T(PT)^{n-1}y_n - q\} \subset B_r[0] \cap C$ for some $r > 0$.

Using Lemma 2.1 and the scheme(2.1), we obtain the estimates:

$$\begin{aligned}
\|z_n - q\|^2 &= \left\|P(\gamma_n T(PT)^{n-1}x_n + (1-\gamma_n)x_n) - Pq\right\|^2 \\
&\leq \left\|\gamma_n T(PT)^{n-1}x_n + (1-\gamma_n)x_n - q\right\|^2 \\
&= \left\|\gamma_n (T(PT)^{n-1}x_n - q) + (1-\gamma_n)(x_n - q)\right\|^2 \\
&\leq \gamma_n \left\|T(PT)^{n-1}x_n - q\right\|^2 + (1-\gamma_n)\|x_n - q\|^2 \\
&- \gamma_n(1-\gamma_n)g(\left\|x_n - T(PT)^{n-1}x_n\right\|) \\
&\leq \gamma_n k_n^2 \|x_n - q\|^2 + (1-\gamma_n)\|x_n - q\|^2 \\
&\leq k_n^2 \|x_n - q\|^2 .
\end{aligned} \tag{3.5}$$

$$\begin{aligned}
\|y_n - q\|^2 &= \left\|P(\beta_n T(PT)^{n-1}z_n + (1-\beta_n)x_n) - Pq\right\|^2 \\
&\leq \left\|\beta_n T(PT)^{n-1}z_n + (1-\beta_n)x_n - q\right\|^2 \\
&= \left\|\beta_n (T(PT)^{n-1}z_n - q) + (1-\beta_n)(x_n - q)\right\|^2 \\
&\leq \beta_n \left\|T(PT)^{n-1}z_n - q\right\|^2 + (1-\beta_n)\|x_n - q\|^2 \\
&- \beta_n(1-\beta_n)g(\left\|x_n - T(PT)^{n-1}z_n\right\|) \\
&\leq \beta_n k_n^2 \|z_n - q\|^2 + (1-\beta_n)\|x_n - q\|^2 .
\end{aligned} \tag{3.6}$$

$$\begin{aligned}
\|x_{n+1}-q\|^2 &= \left\|P(\alpha_n T(PT)^{n-1}y_n+(1-\alpha_n)x_n)-Pq\right\|^2 \\
&\le \left\|\alpha_n T(PT)^{n-1}y_n+(1-\alpha_n)x_n-q\right\|^2 \\
&= \left\|\alpha_n(T(PT)^{n-1}y_n-q)+(1-\alpha_n)(x_n-q)\right\|^2 \\
&\le \alpha_n\left\|T(PT)^{n-1}y_n-q\right\|^2+(1-\alpha_n)\|x_n-q\|^2 \\
&- \alpha_n(1-\alpha_n)g(\left\|x_n-T(PT)^{n-1}y_n\right\|) \\
&\le \alpha_n k_n^2\|y_n-q\|^2+(1-\alpha_n)\|x_n-q\|^2 \\
&- \alpha_n(1-\alpha_n)g(\left\|x_n-T(PT)^{n-1}y_n\right\|).
\end{aligned} \tag{3.7}$$

Combining (3.5),(3.6) and (3.7) and simplifying, we have

$$\begin{aligned}
\|x_{n+1}-q\|^2 \le \|x_n-p\|^2+(k_n^6-1)\|x_n-p\|^2 \\
- \alpha_n(1-\alpha_n)g(\left\|x_n-T(PT)^{n-1}y_n\right\|).
\end{aligned}$$

Since $\{\|x_n-p\| : n\ge 1\}$ is bounded, there exists $\epsilon>0$ such that $\|x_n-p\|^2\le \epsilon$. Therefore, the above inequality becomes:

$$\begin{aligned}
\|x_{n+1}-p\|^2 \le \|x_n-p\|^2+(k_n^6-1)\epsilon \\
- \alpha_n(1-\alpha_n)g(\left\|x_n-T(PT)^{n-1}y_n\right\|).
\end{aligned} \tag{3.8}$$

Suppose that $0<\liminf_{n\to\infty}\alpha_n\le\limsup_{n\to\infty}\alpha_n<1$. Then there exist some real number $\delta>0$ and a natural number n_0 such that $\alpha_n(1-\alpha_n)\ge\delta$ for all $n\ge n_0$. Let m be a positive integer such that $m\ge n_0$. Then from the inequality (3.8), we have

$$\begin{aligned}
\sum_{n=n_0}^{m}\delta g(\left\|x_n-T(PT)^{n-1}y_n\right\|) &\le \|x_1-p\|^2-\|x_{m+1}-p\|^2 \\
&+\sum_{n=n_0}^{m}(k_n^6-1)\epsilon \\
&\le \|x_1-p\|^2+\sum_{n=n_0}^{m}(k_n^6-1)\epsilon.
\end{aligned}$$

When $m\to\infty$ in the above inequality, we get

$$\sum_{n=n_0}^{\infty}\delta g(\left\|x_n-T(PT)^{n-1}y_n\right\|)<\infty$$

and therefore $\lim_{n\to\infty}g(\left\|x_n-T(PT)^{n-1}y_n\right\|)=0$.

By Lemma 2.2, it follows that

$$\lim_{n\to\infty}\left\|x_n-T(PT)^{n-1}y_n\right\|=0. \tag{3.9}$$

Now consider

$$\begin{aligned}
\left\|T(PT)^{n-1}x_n - x_n\right\| &\leq \left\|T(PT)^{n-1}y_n - T(PT)^{n-1}x_n\right\| \\
&+ \left\|T(PT)^{n-1}y_n - x_n\right\| \\
&\leq k_n \left\|y_n - x_n\right\| + \left\|T(PT)^{n-1}y_n - x_n\right\| \\
&\leq \beta_n k_n \left\|x_n - T(PT)^{n-1}z_n\right\| \\
&+ \left\|T(PT)^{n-1}y_n - x_n\right\| \\
&\leq \beta_n k_n \left\|T(PT)^{n-1}z_n - T(PT)^{n-1}x_n\right\| \\
&+ \beta_n k_n \left\|T(PT)^{n-1}x_n - x_n\right\| \\
&+ \left\|T(PT)^{n-1}y_n - x_n\right\| \\
&\leq \beta_n \gamma_n k_n^2 \left\|x_n - T(PT)^{n-1}x_n\right\| \\
&+ \beta_n k_n \left\|T(PT)^{n-1}x_n - x_n\right\| \\
&+ \left\|T(PT)^{n-1}y_n - x_n\right\| \\
&\leq \beta_n k_n(1 + \gamma_n k_n) \left\|T(PT)^{n-1}x_n - x_n\right\| \\
&+ \left\|T(PT)^{n-1}y_n - x_n\right\|
\end{aligned}$$

and hence it follows that

$$\left\|T(PT)^{n-1}x_n - x_n\right\| \leq \frac{1}{1 - \beta_n k_n(1 + \gamma_n k_n)} \left\|T(PT)^{n-1}y_n - x_n\right\|. \tag{3.10}$$

Since $\liminf_{n\to\infty}(1 - \beta_n k_n(1 + \gamma_n k_n)) = 1 - \limsup_{n\to\infty} \beta_n k_n(1 + \gamma_n k_n) > 0$, so we apply lim sup on both sides of the inequality (3.10) and get

$$\lim_{n\to\infty} \left\|T(PT)^{n-1}x_n - x_n\right\| = 0. \tag{3.11}$$

Denote by $c_n = \left\|T(PT)^{n-1}x_n - x_n\right\|$. Then

$$\begin{aligned}
\|x_{n+1} - Tx_{n+1}\| &\leq c_{n+1} + \|T(PT)^n x_{n+1} - Tx_{n+1}\| \\
&\leq c_{n+1} + k_1 \left\|T(PT)^{n-1}x_{n+1} - x_{n+1}\right\| \\
&\leq c_{n+1} + k_1\|x_{n+1} - x_n\| + k_1 \left\|x_n - T(PT)^{n-1}x_n\right\| \\
&+ k_1 k_n \left\|x_n - x_{n+1}\right\| \\
&\leq c_{n+1} + \alpha_n(k_1 + 1)k_n \left\|x_n - T(PT)^{n-1}y_n\right\| \\
&+ k_1 \left\|x_n - T(PT)^{n-1}x_n\right\|.
\end{aligned}$$

This, together with (3.9) and (3.11), shows that

$$\lim_{n\to\infty} \|Tx_n - x_n\| = 0.$$

Now we establish our weak convergence theorem.

Theorem 3.1. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E* which is also a nonexpansive retract of E with nonexpansive retraction *P. Let $T : C \to E$ be asymptotically nonexpansive map with $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \phi$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be control sequences in $[0, 1]$ satisfying: (i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ and (ii) $\limsup_{n\to\infty} \beta_n k_n(1 + \gamma_n k_n) < 1$. For a given $x_1 \in C$, generate $\{x_n\}$ as given in (2.1). Then $\{x_n\}$ converges weakly to a fixed point of T.*

Proof. Let p be a fixed point of T. Then $\lim_{n\to\infty} \|x_n - p\|$exists as proved in Lemma 3.1. We prove that $\{x_n\}$ has a unique weak subsequential limit in $F(T)$. Suppose that w_1 and w_2 are weak limits of the sequences $\{x_{n_i}\}$ and $\{x_{n_j}\}$, respectively. Now $\lim_{n\to\infty} \|Tx_n - x_n\| = 0$ gives that $\lim_{n\to\infty}(I - T)(x_{n_i}) = 0$. Then by the demiclosedness of $I - T$, we obtain $T(w_1) = w_1$. Similarly, we can prove that $T(w_2) = w_2$. Next, we establish the uniqueness. To do this, let w_1 and w_2 be distinct, then by the Opial's property,

$$\begin{aligned}
\lim_{n\to\infty} \|x_n - w_1\| &= \lim_{n_i\to\infty} \|x_{n_i} - w_1\| \\
&< \lim_{n_i\to\infty} \|x_{n_i} - w_2\| \\
&= \lim_{n\to\infty} \|x_n - w_2\| \\
&= \lim_{n_j\to\infty} \left\|x_{n_j} - w_2\right\| \\
&< \lim_{n_j\to\infty} \left\|x_{n_j} - w_1\right\| \\
&= \lim_{n\to\infty} \|x_n - w_1\|,
\end{aligned}$$

a contradiction. Hence the proof.

Using the condition(A), we establish our strong convergence result as follows:

Theorem 3.2. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E* which is also a nonexpansive retract of E with nonexpansive retraction *P. Let $T : C \to E$ be asymptotically nonexpansive map with $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \phi$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be control sequences in $[0, 1]$ satisfying: (i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ and (ii) $\limsup_{n\to\infty} \beta_n k_n(1 + \gamma_n k_n) < 1$. For a given $x_1 \in C$, generate $\{x_n\}$ as given in (2.1). If T satisfies the condition(A), then $\{x_n\}$ converges strongly to a fixed point of T.*

Proof. Lemma 3.2 gives that

$$\lim_{n\to\infty} \|x_n - Tx_n\| = 0$$

and hence the condition (A) reduces to

$$\lim_{n\to\infty} f(d(x_n, F)) = 0.$$

Since $f : [0, \infty) \to [0, \infty)$ is a nondecreasing function satisfying $f(0) = 0$, $f(r) > 0$ for all $r \in (0, \infty)$, we get that $\liminf_{n\to\infty} d(x_n, F) = 0$. In Lemma 3.1, we have shown that $\lim_{n\to\infty} d(x_n, F)$ exists, therefore $\lim_{n\to\infty} d(x_n, F) = 0$. Since

$\{k_i\}$ is bounded and $\sum_{i=1}^{\infty}(k_i - 1) < \infty$, therefore $\sum_{i=1}^{\infty}(k_i^3 - 1) < \infty$. Fix $\delta = \exp \sum_{i=1}^{\infty}(k_i^3 - 1)$. As $\lim_{n\to\infty} d(x_n, F) = 0$,therefore for each $\epsilon > 0$, there exists an n_0 such that for all $n \geqslant n_0$

$$d(x_n, F) < \frac{\epsilon}{2\delta}.$$

That is, $\inf_{p\in F} \|x_{n_0} - p\| < \frac{\epsilon}{3\delta}$. So there must exist $p^* \in F$ such that

$$\|x_{n_0} - p^*\| < \frac{\epsilon}{2\delta}.$$

As $1 + x \leq e^x$ for $x \geq 0$, therefore for $n, m \geq n_0$, we get on the basis of inequality(3.4) that

$$\begin{aligned}
\|x_{n+m} - x_n\| &\leq \|x_{n+m} - p^*\| + \|x_n - p^*\| \\
&\leq \exp\left(\sum_{i=n_0}^{n+m-1}(k_i^3 - 1) + \sum_{i=n_0}^{n-1}(k_i^3 - 1)\right)\|x_{n_0} - p^*\| \\
&\leq 2\exp\left(\sum_{i=1}^{\infty}(k_i^3 - 1)\right)\|x_{n_0} - p^*\| \\
&< 2\delta\left(\frac{\epsilon}{2\delta}\right) = \epsilon.
\end{aligned}$$

Hence $\{x_n\}$ is a Cauchy sequence in a closed subset C of a Banach space E, so it must converge to a point of C. Let $\lim_{n\to\infty} x_n = q$. Now $\lim_{n\to\infty} d(x_n, F) = 0$ gives that $d(q, F) = 0$. Since F is closed, therefore $q \in F$, and the proof is over.

For $\gamma_n = 0$ for all $n \geq 1$, Theorem 3.1 and Theorem 3.2, reduce to the following Ishikawa-type convergence result.

Theorem 3.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E which is also a nonexpansive retract of E with nonexpansive retraction P. Let $T : C \to E$ be asymptotically nonexpansive map with $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \phi$. Suppose that $\gamma_n = 0$ for all $n \geq 1$ in $\{x_n\}$ given by (2.1) and $\{\alpha_n\}, \{\beta_n\}$ satisfy: (i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$, and (ii) $\limsup_{n\to\infty} \beta_n < 1$.*
(a) If E satisfies the Opial's property, then $\{x_n\}$ converges weakly to a fixed point of T;
(b) If T satisfies the condition(A), then $\{x_n\}$ converges strongly to a fixed point of T.

Taking $\beta_n = 0 = \gamma_n$ for all $n \geq 1$ in Theorem 3.1 and Theorem 3.2, Mann-type convergence result is obtained in the following:

Theorem 3.4. *Let C be a nonempty convex subset of a uniformly convex Banach space E which is also a nonexpansive retract of E with nonexpansive retraction P. Let $T : C \to E$ be asymptotically nonexpansive map with $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \phi$.Suppose that $\beta_n = 0 = \gamma_n$ for all $n \geq 1$ in $\{x_n\}$ given by (2.1) and $\{\alpha_n\}$ satisfies:$0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.*
(a) If E satisfies the Opial's property, then $\{x_n\}$ converges weakly to a fixed point of T;
(b) If T satisfies the condition(A), then $\{x_n\}$ converges strongly to a fixed point of T.

For a selfmap, Theorems 3.1-3.4 provide the following new convergence results.

Theorem 3.5. *Let C be a nonempty closed convex subset of a uniformly connected Banach space E. Let $T : C \to C$ be asymptotically nonexpansive map with $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \phi$. Let $\{\alpha_n\}, \{\beta_n\}$ and $\{\gamma_n\}$ be control sequences in $[0, 1]$ satisfying: (i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ and (ii) $\limsup_{n\to\infty} \beta_n k_n(1 + \gamma_n k_n) < 1$. For a given $x_1 \in C$, generate $\{x_n\}$ as given in (1.1). Then $\{x_n\}$ converges weakly to a fixed point of T.*

Theorem 3.6. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $T : C \to C$ be asymptotically nonexpansive map with $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \phi$. Let $\{\alpha_n\}, \{\beta_n\}$ be control sequences in $[0, 1]$ satisfying: (i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$ and (ii) $\limsup_{n\to\infty} \beta_n < 1$. For a given $x_1 \in C$, generate $\{x_n\}$ as given in (1.2). If T satisfies the condition(A), then $\{x_n\}$ converges strongly to a fixed point of T.*

Theorem 3.7. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E. Let $T : C \to C$ be asymptotically nonexpansive map with $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \phi$. Suppose that for $\{x_n\}$ in (1.3), $\{\alpha_n\}$ satisfies: (i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$.*
(a) If E satisfies Opial's condition, then $\{x_n\}$ converges weakly to a fixed point of T;
(b) If T satisfies the condition(A), then $\{x_n\}$ converges strongly to a fixed point of T.

Remark 3.1. (i) Theorem 3.3(b) establishes: [23, Theorem 3], [25, Theorem 2.3] and Theorem 1.2 under new control conditions on parametric sequences and without " completely continuous" requirement on T.
(ii) The special cases of Theorem 3.4(b) are: ([2], Theorem 3.7),([23], Theorem 2), ([25], Theorem 1.5) and ([26], Theorem 2.6, Corollary 2.11).
(iii) Theorem 3.4(a) generalizes Theorem 2.1 in [24] while Theorem 3.3(a) extends Theorem 1 in [12] in case of one map and Theorem 1 in [20] for nonself maps.

Remark 3.2. (i) Xu and Noor[29] used the scheme(1.1) to approximate fixed points of an asymptotically nonexpansive map and obtained the Ishikawa-type convergence result as an immediate consequence of the main theorem while the Mann-type convergence could not follow directly from it. In this note, we have obtained Ishikawa-type convergence and Mann-type convergence results directly and simultaneously from our results for nonself asymptotically nonexpansive maps.
(ii) We have further analyzed the three-step iterative scheme of Xu and Noor[29] under new parametric control conditions. All of our results can be proved for three-step iterative scheme with errors in the sense of Xu[31] by making obvious and suitable changes in the statements and proofs of theorems and corollaries. We leave the details to the reader; for example, a reformulation of Lemma 3.2 would be:

Lemma 3.3. *Let C be a nonempty closed convex subset of a uniformly convex Banach space E which is also a nonexpansive retract of E with nonexpansive retraction P. Let $T : C \to E$ be asymptotically nonexpansive map with $\{k_n\} \subset [1, \infty)$ such that $\sum_{n=1}^{\infty}(k_n - 1) < \infty$ and $F(T) \neq \phi$. For a given $x_1 \in C$,let $\{x_n\}$ be defined by*

$$\begin{cases} x_{n+1} = P((1 - \alpha_n - \nu_n)x_n + \alpha_n T(PT)^{n-1} y_n + \nu_n u_n), \\ y_n = ((1 - \beta_n - \mu_n)x_n + \beta_n T(PT)^{n-1} z_n + \mu_n v_n), \\ z_n = ((1 - \gamma_n - \lambda_n)x_n + \gamma_n T(PT)^{n-1} x_n + \lambda_n w_n), \quad n \geq 1, \end{cases}$$

where $\{u_n\}, \{v_n\}, \{w_n\}$ *are bounded sequences in C and* $\{\alpha_n\}, \{\beta_n\}, \{\gamma_n\}, \{\mu_n\}$, $\{\nu_n\}, \{\lambda_n\}$ *are real sequences in* $[0,1]$ *with the following restrictions:*
(i) $0 < \liminf_{n\to\infty} \alpha_n \leq \limsup_{n\to\infty} \alpha_n < 1$,
(ii) $\limsup_{n\to\infty} \beta_n k_n(1+\gamma_n k_n) < 1$,
(iii) $\sum_{n=1}^{\infty} \nu_n < \infty, \sum_{n=1}^{\infty} \mu_n < \infty$ *and* $\sum_{n=1}^{\infty} \lambda_n < \infty$.
Then $\lim_{n\to\infty} \|x_n - Tx_n\| = 0$.

Remark 3.3. The restrictions imposed on the control sequences in the statement of Lemma 1.5[3] are inadequate for the proof. Lemma 3.3 on the one hand extends our Lemma 3.2 and on the other hand provides a correct statement as well as proof of Lemma 1.5 in [3] when T is a selfmap.

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