Periodic solutions for n^{th} order functional differential equations*

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Abstract

In this paper, we study the existence of periodic solutions for n^{th} order functional differential equations $x^{(n)}(t) + \sum_{i=0}^{n-1} b_i[x^{(i)}(t)]^k + f(t, x(t-\tau)) = p(t)$. Some new results on the existence of periodic solutions of the equations are obtained. Our approach is based on the coincidence degree theory of Mawhin.

1 Introduction

In this paper, we are concerned with the existence of periodic solutions of the n th order functional differential equations

$$x^{(n)}(t) + \sum_{i=0}^{n-1} b_i[x^{(i)}(t)]^k + f(t, x(t-\tau)) = p(t) \tag{1.1}$$

where $b_i (i = 0, 1, \cdots, n-1)$ are constants, k is a integer, $f \in C(R^2, R)$ and $f(t+T, x) = f(t, x)$ for $\forall x \in R$, $p \in C(R, R)$ with $p(t+T) = p(t)$.

In recent years, there are many papers studying the existence of periodic solutions of first, second or third order differential equations[1, 3-4, 10-11, 13-16, 18,

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20-21, 23]. For example, in [11], Zhang and Wang studied the following differential equations

$$x'''(t) + ax''^{2k-1}(t) + bx'^{2k-1}(t) + cx^{2k-1}(t) + g(t, x(t-\tau_1, x'(t-\tau_2)) = p(t) \tag{1.2}$$

The authors established the existence of periodic solutions of Eq. (1.2) under some conditions on a, b, c and $2k-1$.

In [5-9, 12, 17, 19, 22], n, $2n$ and $2n+1$ th order differential equations of the form

$$x^{(2n)}(t) + \sum_{j=1}^{n-1} a_j x^{(2j)}(t) + (-1)^{(k+1)} g(t, x) = 0 \tag{1.3}$$

$$x^{(2n+1)}(t) + \sum_{j=1}^{n-1} a_j x^{(2j+1)}(t) + g(t, x) = 0 \tag{1.4}$$

were discussed. The authors obtained the results based on the damping terms $x^{(i)}(t)(i = 1, \cdots, n-1)$. But few of them studied the differential equations with the damping terms $[x^{(i)}(t)]^k(i = 1, \cdots, n-1)$, where $k \geq 1$.

In present paper, by using Mawhin′s continuation theorem, we will establish some theorems on the existence of periodic solutions of Eq. (1.1). The results are related to not only b_i and $f(t, x)$ but also the positive integer k. In addition, we give an example to illustrate our new results.

2 Some lemmas

We investigate the theorems based on the following Lemmas.

Lemma 2.1 If $k \geq 1$ is an integer, $x \in C^n(R, R)$, and $x(t+T) = x(t)$, then

$$(\int_0^T |x'(t)|^k dt)^{\frac{1}{k}} \leq T(\int_0^T |x''(t)|^k dt)^{\frac{1}{k}} \leq \cdots \leq T^{n-1}(\int_0^T |x^{(n)}(t)|^k dt)^{\frac{1}{k}} \tag{2.1}$$

The proof of Lemma 2.1 is easy, here we omit it.

We first introduce Mawhin′s continuation theorem.

Let X and Y be Banach spaces, $L : D(L) \subset X \to Y$ be a Fredholm operator of index zero, here $D(L)$ denotes the domain of L. $P : X \to X, Q : Y \to Y$ be projectors such that

$$ImP = KerL, KerQ = ImL, X = KerL \oplus KerP, Y = ImL \oplus ImQ.$$

It follows that

$$L|_{D(L)\cap KerP} : D(L) \cap KerP \to ImL$$

is invertible, we denote the inverse of that map by K_p. Let Ω be an open bounded subset of X, $D(L) \cap \overline{\Omega} \neq \emptyset$, the map $N : X \to Y$ will be called L-compact in $\overline{\Omega}$, if $QN(\overline{\Omega})$ is bounded and $K_p(I-Q)N : \overline{\Omega} \to X$ is compact.

Lemma 2.2 [2] Let L be a Fredholm operator of index zero and let N be L-compact on $\overline{\Omega}$. Assume that the following conditions are satisfied:
(i) $Lx \neq \lambda Nx, \forall x \in \partial\Omega \cap D(L), \lambda \in (0,1)$.
(ii) $QNx \neq 0, \forall x \in \partial\Omega \cap KerL,$
(iii) $deg\{QNx, \Omega \bigcap KerL, 0\} \neq 0,$
Then the equation $Lx = Nx$ has at least one solution in $\overline{\Omega} \bigcap D(L)$.

Now, we define $Y = \{x \in C(R,R) \mid x(t+T) = x(t)\}$ with the norm $|x|_\infty = \max_{t\in[0,T]}\{|x(t)|\}$ and $X = \{x \in C^{n-1}(R,R) \mid x(t+T) = x(t)\}$ with norm $\|x\| = \max\{|x|_\infty, |x'|_\infty \cdots, |x^{(n-1)}|_\infty\}$, we can easily see that X, Y are two Banach spaces. We also define the operators L and N as follows:

$$L : D(L) \subset X \to Y, Lx = x^{(n)}, D(L) = \{x | x \in C^n(R,R), x(t+T) = x(t)\} \tag{2.2}$$

$$N : X \to Y, Nx = -\sum_{i=1}^{n-1} b_i[x^{(i)}(t)]^k - f(t, x(t-\tau)) + p(t). \tag{2.3}$$

It is easy to see that Eq. (1.1) can be converted to the abstract equation $Lx = Nx$. Moreover, from the definition of L, we see that $kerL = R$, $dim(kerL) = 1$, $ImL = \{y | y \in Y, \int_0^T y(s)ds = 0\}$ is closed, and $dim(Y \setminus ImL) = 1$, we have $codim(ImL) = dim(kerL)$, so L is a Fredholm operator with index zero. Let

$$P : X \longrightarrow KerL, Px = x(0), Q : Y \longrightarrow Y \setminus ImL, Qy = \frac{1}{T}\int_0^T y(t)dt$$

and let

$$L|_{D(L)\cap KerP} : D(L) \cap KerP \to ImL.$$

Then $L|_{D(L)\cap KerP}$ has a unique continuous inverse K_p. One can easily find that N is L-compact in $\overline{\Omega}$, where Ω is an open bounded subset of X.

3 Main result

Theorem 3.1 Suppose $n = 2m+1$, $m > 0$ an integer, k is odd, and the following conditions hold
(H_1) the function f satisfies

$$\lim_{x\to\infty} |\frac{f(t,x)}{x^k}| \leq \gamma, \tag{3.1}$$

where $\gamma \geq 0$.
(H_2)

$$|b_0| > \gamma \tag{3.2}$$

(H_3) there is a positive integer $0 < s \leq m$ such that

$$\begin{cases} b_{2s} \neq 0, \quad if \quad s = m \\ b_{2s} \neq 0, b_{2s+i} = 0, i = 1, 2, \cdots, 2m-2s, \quad if \quad 0 < s < m \end{cases} \tag{3.3}$$

(H_4)

$$\begin{cases} A_2(2s,k) + \dfrac{\gamma A_1(2s,k)}{|b_0|-\gamma} + k|b_0|T^{2s}[\dfrac{A_1(2s,k)}{|b_0|-\gamma}]^{\frac{k-1}{k}} < |b_{2s}|, & if \quad 1 < s \leq m \\ \dfrac{\gamma A_1(2,k)}{|b_0|-\gamma} + k|b_0|T^2[\dfrac{A_1(2,k)}{|b_0|-\gamma}]^{\frac{k-1}{k}} < |b_2|, & if \quad s = 1 \end{cases} \tag{3.4}$$

where $A_1(s,k) = \sum\limits_{i=1}^{s} |b_i|T^{(s-i)k}$, $A_2(s,k) = \sum\limits_{i=1}^{s-2} |b_i|T^{(s-i)k}$. Then Eq. (1.1) has at least one T-periodic solution.

Proof. Consider the equation

$$Lx = \lambda Nx, \lambda \in (0,1)$$

where L and N are defined by (2.2) and (2.3) . Let

$$\Omega_1 = \{x \in D(L)/KerL, Lx = \lambda Nx \quad for \quad some \quad \lambda \in (0,1)\}$$

for $x \in \Omega_1$, We have

$$x^{(n)}(t) = -\lambda \sum_{i=0}^{2s} b_i[x^{(i)}(t)]^k - \lambda f(t, x(t-\tau)) + \lambda p(t), \lambda \in (0,1) \tag{3.5}$$

Multiplying both sides of (3.5) by $x(t)$, and integrating them on $[0, T]$, we have for $\lambda \in (0,1)$

$$\begin{aligned} \int_0^T x^{(n)}(t)x(t)dt &= -\lambda \sum_{i=0}^{2s} b_i \int_0^T [x^{(i)}(t)]^k x(t)dt - \\ &\lambda \int_0^T f(t, x(t-\tau))x(t)dt + \lambda \int_0^T p(t)x(t)dt. \end{aligned} \tag{3.6}$$

It is easy to see that, for any positive integer i,

$$\int_0^T x^{(2i-1)}(t)x(t)dt = 0. \tag{3.7}$$

In view of $n = 2m + 1$ and k is odd, it follows from (3.3) and (3.7) that

$$b_0 \int_0^T |x(t)|^{k+1}dt = -\sum_{i=1}^{2s} b_i \int_0^T [x^{(i)}(t)]^k x(t)dt - \int_0^T f(t, x(t-\tau))x(t)dt + \int_0^T p(t)x(t)dt. \quad () \ 3.8)$$

From which it follows that

$$|b_0| \quad \int_0^T |x(t)|^{k+1}dt \leq \int_0^T |x(t)|[\sum_{i=1}^{2s} |b_i||x^{(i)}(t)|^k + |f(t, x(t-\tau))| + |p(t)|]dt \tag{3.9}$$

By using Hölder inequality and Lemma 2.1, from (3.9) , we obtain

$$
\begin{aligned}
|b_0| \int_0^T |x(t)|^{k+1}dt &\le (\int_0^T |x(t)|^{k+1}dt)^{\frac{1}{k+1}}[\sum_{i=1}^{2s}|b_i|(\int_0^T |x^{(i)}(t)|^{k+1}dt)^{\frac{k}{k+1}} \\
&+(\int_0^T |f(t,x(t-\tau))|^{\frac{k+1}{k}}dt)^{\frac{k}{k+1}} + (\int_0^T |p(t)|^{\frac{k+1}{k}}dt)^{\frac{k}{k+1}}] \\
&\le (\int_0^T |x(t)|^{k+1}dt)^{\frac{1}{k+1}}[\sum_{i=1}^{2s}|b_i|T^{(2s-i)k}(\int_0^T |x^{(2s)}(t)|^{k+1}dt)^{\frac{k}{k+1}} \\
&+(\int_0^T |f(t,x(t-\tau))|^{\frac{k+1}{k}}dt)^{\frac{k}{k+1}} + |p(t)|_\infty T^{\frac{k}{k+1}}].
\end{aligned} \tag{3.10}
$$

So

$$
\begin{aligned}
|b_0|(\int_0^T |x(t)|^{k+1}dt)^{\frac{k}{k+1}} \le A_1(2s,k)(\int_0^T |x^{(2s)}(t)|^{k+1}dt)^{\frac{k}{k+1}} \\
+ (\int_0^T |f(t,x(t-\tau))|^{\frac{k+1}{k}}dt)^{\frac{k}{k+1}} + u_1. \quad ((3.11))
\end{aligned}
$$

where u_1 is a positive constant. Choose a constant $\varepsilon > 0$ such that

$$\gamma + \varepsilon < |b_0|$$

and

$$
\begin{cases}
A_2(2s,k) + \dfrac{(\gamma+\varepsilon)A_1(2s,k)}{|b_0|-(\gamma+\varepsilon)} + k|b_0|T^{2s}[\dfrac{A_1(2s,k)}{|b_0|-(\gamma+\varepsilon)}]^{\frac{k-1}{k}} < |b_{2s}|, & if \quad 1 < s \le m \\
\dfrac{(\gamma+\varepsilon)A_1(2,k)}{|b_0|-(\gamma+\varepsilon)} + k|b_0|T^{2}[\dfrac{A_1(2,k)}{|b_0|-(\gamma+\varepsilon)}]^{\frac{k-1}{k}} < |b_2|, & if \quad s = 1
\end{cases}
$$

For the above constant $\varepsilon > 0$, we see from (3.1) that there is a constant $\delta > 0$ such that

$$|f(t,x(t-\tau))| < (\gamma+\varepsilon)|x(t-\tau)|^k, for \quad |x(t-\tau)| > \delta, t \in [0,T] \tag{3.12}$$

Denote

$$\Delta_1 = \{t \in [0,T] : |x(t-\tau)| \le \delta\}, \Delta_2 = \{t \in [0,T] : |x(t-\tau)| > \delta\}. \tag{3.13}$$

Since

$$
\begin{aligned}
\int_0^T |f(t,x(t-\tau))|^{\frac{k+1}{k}}dt &\le \int_{\Delta_1} |f(t,x(t-\tau))|^{\frac{k+1}{k}}dt + \int_{\Delta_2} |f(t,x(t-\tau))|^{\frac{k+1}{k}}dt \\
&\le (f_\delta)^{\frac{k+1}{k}}T + (\gamma+\varepsilon)^{\frac{k+1}{k}}\int_0^T |x(t-\tau)|^{k+1}dt \\
&= (f_\delta)^{\frac{k+1}{k}}T + (\gamma+\varepsilon)^{\frac{k+1}{k}}\int_0^T |x(t)|^{k+1}dt
\end{aligned} \tag{3.14}
$$

where $f_\delta = \max_{t\in[0,T],|x|\le\delta}|f(t,x)|$. Using inequality

$$(a+b)^l \le a^l + b^l \quad for \;\; a \ge 0, b \ge 0 \;\; and \;\; 0 \le l \le 1 \tag{3.15}$$

it follows from (3.14) that

$$(\int_0^T |f(t,x(t-\tau))|^{\frac{k+1}{k}}dt)^{\frac{k}{k+1}} \le f_\delta T^{\frac{k}{k+1}} + (\gamma+\varepsilon)(\int_0^T |x(t)|^{k+1}dt)^{\frac{k}{k+1}} \quad (3.16)$$

Substituting the above formula into (3.11) , we have

$$[|b_0| \quad -(\gamma+\varepsilon)](\int_0^T |x(t)|^{k+1}dt)^{\frac{k}{k+1}} \le A_1(2s,k)(\int_0^T |x^{(2s)}(t)|^{k+1}dt)^{\frac{k}{k+1}} + u_2. \quad (3.17)$$

where u_2 is a positive constant.
That is

$$(\int_0^T |x(t)|^{k+1}dt)^{\frac{k}{k+1}} \le \frac{A_1(2s,k)}{|b_0|-(\gamma+\varepsilon)}(\int_0^T |x^{(2s)}(t)|^{k+1}dt)^{\frac{k}{k+1}} + u_3. \quad (3.18)$$

where u_3 is a positive constant.

On the other hand, multiplying both sides of (3.5) by $x^{(2s)}(t)$, and integrating on $[0,T]$, we have

$$\begin{aligned}\int_0^T x^{(n)}(t)x^{(2s)}(t)dt &= -\sum_{i=0}^{2s} b_i \int_0^T [x^{(i)}(t)]^k x^{(2s)}(t)dt \\ &- \int_0^T f(t,x(t-\tau))x^{(2s)}(t)dt + \int_0^T p(t)x^{(2s)}(t)dt\end{aligned} \quad (3.19)$$

If $1 < s \le m$, since

$$\int_0^T x^{(2m+1)}(t)x^{(2s)}(t)dt = 0, \int_0^T [x^{(2s-1)}(t)]^k x^{(2s)}(t)dt = 0, \quad (3.20)$$

and

$$\int_0^T [x(t)]^k x^{(2s)}(t)dt = -k\int_0^T [x(t)]^{k-1}x^{(2s-1)}(t)x^{'}(t)dt \quad (3.21)$$

by using Hölder inequality and Lemma 2.1, from (3.19) , we have

$$\begin{aligned}|b_{2s}|\int_0^T |x^{(2s)}(t)|^{k+1}dt \\ &\le \int_0^T |x^{(2s)}(t)|[\sum_{i=1}^{2s-2}|b_i||x^{(i)}(t)|^k + |f(t,x(t-\tau))| + |p(t)|]dt \\ &\quad + k|b_0|\int_0^T |x(t)|^{k-1}|x^{(2s-1)}(t)||x^{'}(t)|dt \\ &\le (\int_0^T |x^{(2s)}(t)|^{k+1}dt)^{\frac{1}{k+1}}[\sum_{i=1}^{2s-2}|b_i|T^{(2s-i)k}(\int_0^T |x^{(2s)}(t)|^{k+1}dt)^{\frac{k}{k+1}} + \\ &\quad (\int_0^T |f(t,x(t-\tau))|^{\frac{k+1}{k}}dt)^{\frac{k}{k+1}} + |p(t)|_\infty T^{\frac{k}{k+1}}] + \\ &\quad k|b_0||x^{'}(t)|_\infty \int_0^T |x(t)|^{k-1}||x^{(2s-1)}(t)|dt\end{aligned} \quad () 3.22)$$

Since $x(0) = x(T)$, there exists $\xi \in [0, T]$ such that $x^{'}(\xi) = 0$. Hence for $t \in [0, T]$

$$x^{'}(t) = x^{'}(\xi) + \int_{\xi}^{t} x^{''}(\sigma)d\sigma$$

Using Hölder inequality and Lemma 2.1, we have

$$\begin{aligned} |x^{'}(t)|_{\infty} &\le \int_0^T |x^{''}(t)|dt \le T^{\frac{k}{k+1}}(\int_0^T |x^{''}(t)|^{k+1}dt)^{\frac{1}{k+1}} \\ &\le T^{2s-1-\frac{1}{k+1}}(\int_0^T |x^{(2s)}(t)|^{k+1}dt)^{\frac{1}{k+1}} \end{aligned} \quad () \ 3.23)$$

Using inequality

$$(\frac{1}{T}\int_0^T |x(t)|^r|)^{\frac{1}{r}} \le (\frac{1}{T}\int_0^T |x(t)|^l|)^{\frac{1}{l}} \quad for \quad 0 \le r \le l \ \ and \ \ \forall x \in R. \qquad (3.24)$$

and applying Hölder inequality, we obtain from Lemma 2.1

$$\begin{aligned} \int_0^T |x(t)|^{k-1}||x^{(2s-1)}(t)|dt &\le (\int_0^T |x(t)|^k dt)^{\frac{k-1}{k}}(\int_0^T |x^{(2s-1)}(t)|^k dt)^{\frac{1}{k}} \\ &\le T^{\frac{1}{k+1}}(\int_0^T |x(t)|^{k+1}dt)^{\frac{k-1}{k+1}}(\int_0^T |x^{(2s-1)}(t)|^{k+1}dt)^{\frac{1}{k+1}} \\ &\le T^{1+\frac{1}{k+1}}(\int_0^T |x(t)|^{k+1}dt)^{\frac{k-1}{k+1}}(\int_0^T |x^{(2s)}(t)|^{k+1}dt)^{\frac{1}{k+1}} \end{aligned} \qquad (3.25)$$

Substituting the above formula, (3.16) and (3.23) into (3.22) , we have

$$\begin{aligned} |b_{2s}| \ & \textstyle\int_0^T |x^{(2s)}(t)|^{k+1}dt \\ & \le (\int_0^T |x^{(2s)}(t)|^{k+1}dt)^{\frac{1}{k+1}}[A_2(2s,k)(\int_0^T |x^{(2s)}(t)|^{k+1}dt)^{\frac{k}{k+1}} \\ & +(\gamma+\varepsilon)(\int_0^T |x(t)|^{k+1}dt)^{\frac{k}{k+1}} + (|p(t)|_{\infty} + f_{\delta})T^{\frac{k}{k+1}}] \\ & +k|b_0|T^{2s}(\int_0^T |x^{(2s)}(t)|^{k+1}dt)^{\frac{2}{k+1}}(\int_0^T |x(t)|^{k+1}|dt)^{\frac{k-1}{k+1}} \end{aligned} \qquad (3.26)$$

Then, we have

$$\begin{aligned} (|b_{2s}| \ & -A_2(2s,k))(\int_0^T |x^{(2s)}(t)|^{k+1}dt)^{\frac{k}{k+1}} \\ & \le k|b_0|T^{2s}(\int_0^T |x^{(2s)}(t)|^{k+1}dt)^{\frac{1}{k+1}}(\int_0^T |x(t)|^{k+1}|dt)^{\frac{k-1}{k+1}} \\ & +(\gamma+\varepsilon)(\int_0^T |x(t)|^{k+1}dt)^{\frac{k}{k+1}} + u_4 \end{aligned} \qquad (3.27)$$

where u_4 is a positive constant.
Using inequality

$$(a+b)^l \le a^l + b^l \quad for \ \ a \ge 0, b \ge 0 \ \ and \ \ 0 \le l \le 1 \qquad (3.28)$$

it follows from (3.18) that

$$\int_0^T |x(t)|^{k+1}dt)^{\frac{k-1}{k+1}} \le [\frac{A_1(2s,k)}{|b_0| - (\gamma+\varepsilon)}]^{\frac{k-1}{k}} \int_0^T |x^{(2s)}(t)|^{k+1}dt)^{\frac{k-1}{k+1}} + u_5 \qquad (3.29)$$

where u_5 is a positive constant.
Substituting the above formula and (3.18) into (3.27) , we have

$$\{|b_{2s}| - A_2(2s,k) - \frac{(\gamma+\varepsilon)A_1(2s,k)}{|b_0|-(\gamma+\varepsilon)} - k|b_0|T^{2s}[\frac{A_1(2s,k)}{|b_0|-(\gamma+\varepsilon)}]^{\frac{k-1}{k}}\}(\int_0^T |x^{(2s)}(t)|^{k+1}dt)^{\frac{k}{k+1}} \leq u_5 k|b_0|T^{2s}(\int_0^T |x^{(2s)}(t)|^{k+1}dt)^{\frac{1}{k+1}} + u_6 \tag{3.30}$$

where u_6 is a positive constant.
If $s=1$, since $\int_0^T [x'(t)]^k x''(t)dt = 0$, $\int_0^T [x(t)]^k x''(t)dt = -k\int_0^T [x(t)]^{k-1}[x'(t)]^2 dt$, from (3.19) , we have

$$b_2\int_0^T [x''(t)]^{k+1}dt = -kb_0\int_0^T [x(t)]^{k-1}[x'(t)]^2dt - \int_0^T f(t,x(t-\tau))x'(t)dt + \int_0^T p(t)x'(t)dt \quad () \ 3.31)$$

Applying the above method, we have

$$\{|b_2| \quad - \frac{(\gamma+\varepsilon)A_1(2,k)}{|b_0|-(\gamma+\varepsilon)} - k|b_0|T^2[\frac{A_1(2,k)}{|b_0|-(\gamma+\varepsilon)}]^{\frac{k-1}{k}}\}(\int_0^T |x''(t)|^{k+1}dt)^{\frac{k}{k+1}} \leq u_7 k|b_0|T^2(\int_0^T |x''(t)|^{k+1}dt)^{\frac{1}{k+1}} + u_8 \tag{3.32}$$

where u_7, u_8 is a positive constant.
Hence there is a constant $M_1, M_2 > 0$ such that

$$\int_0^T |x^{(2s)}(t)|^{k+1}dt \leq M_1 \tag{3.33}$$

and

$$\int_0^T |x(t)|^{k+1}dt \leq M_2 \tag{3.34}$$

From (3.5), using Hölder inequality and Lemma 2.1, we have

$$\begin{aligned}\int_0^T |x^{(n)}(t)|dt &\leq \sum_{i=1}^{2s}|b_i|\int_0^T |x^{(i)}(t)|^k dt + |b_0|\int_0^T |x(t)|^k dt + \int_0^T |f(t,x(t-\tau))|dt + \int_0^T |p(t)|dt \\ &\leq \sum_{i=1}^{2s}|b_i|T^{(2s-i)k+\frac{1}{k+1}}(\int_0^T |x^{(2s)}(t)|^{k+1}dt)^{\frac{k}{k+1}} + |b_0|T^{\frac{1}{k+1}}(\int_0^T |x(t)|^{k+1}dt)^{\frac{k}{k+1}} \\ &\quad + (\gamma+\varepsilon)T^{\frac{1}{k+1}}(\int_0^T |x(t)|^{k+1}dt)^{\frac{k}{k+1}} + (|p(t)|_\infty + f_\delta)T \\ &\leq \sum_{i=1}^{2s}|b_i|T^{(2s-i)k+\frac{1}{k+1}}(M_1)^{\frac{k}{k+1}} + |b_0|T^{\frac{1}{k+1}}(M_2)^{\frac{k}{k+1}} \\ &\quad + (\gamma+\varepsilon)T^{\frac{1}{k+1}}(M_2)^{\frac{k}{k+1}} + (|p(t)|_\infty + f_\delta)T = M \quad () \ 3.35)\end{aligned}$$

where M is a positive constant. We claim that

$$|x^{(i)}(t)| \leq T^{n-i-1}\int_0^T |x^{(n)}(t)|dt, (i = 1, 2, \cdots, n-1) \quad (3.36)$$

In fact, noting that $x^{(n-2)}(0) = x^{(n-2)}(T)$, there must be a constant $\xi_1 \in [0, T]$ such that $x^{(n-1)}(\xi_1) = 0$, we obtain

$$|x^{(n-1)}(t)| = |x^{(n-1)}(\xi_1) + \int_{\xi_1}^t x^{(n)}(s)ds| \leq |x^{(n-1)}(\xi_1)| + \int_0^T |x^{(n)}(t)|dt = \int_0^T |x^{(n)}(t)|dt. \quad () 3.37)$$

Similarly, since $x^{(n-3)}(0) = x^{(n-3)}(T)$, there must be a constant $\xi_2 \in [0, T]$ such that $x^{(n-2)}(\xi_2) = 0$, from (3.37) we get

$$|x^{(n-2)}(t)| = |x^{(n-2)}(\xi_2) + \int_{\xi_2}^t x^{(n-1)}(s)ds| \leq \int_0^T |x^{(n-1)}(t)|dt \leq T\int_0^T |x^{(n)}(t)|dt. \quad (3.38)$$

By induction, we have

$$|x^{(i)}(t)| \leq T^{n-i-1}\int_0^T |x^{(n)}(t)|dt, (i = 1, 2, \cdots, n-1) \quad (3.39)$$

Furthermore, we have

$$|x^{(i)}(t)|_\infty \leq T^{n-i-1}\int_0^T |x^{(n)}(t)|dt \leq T^{n-i-1}M, (i = 1, 2, \cdots, n-1) \quad (3.40)$$

From (3.34) it follows that there exists a $\xi \in [0, T]$ such that $|x(\xi)| \leq M_2^{\frac{1}{k+1}}$. Applying Lemma 2.1, we get

$$\begin{aligned} |x(t)|_\infty &\leq x(\xi) + \int_\xi^t x'(t)dt \leq M_2^{\frac{1}{k+1}} + T^{\frac{k}{k+1}}(\int_0^T |x'(t)|^{k+1}dt)^{\frac{1}{k+1}} \\ &\leq M_2^{\frac{1}{k+1}} + T^{2s-1+\frac{k}{k+1}}(\int_0^T |x^{(2s)}(t)|^{k+1}dt)^{\frac{1}{k+1}} = M_2^{\frac{1}{k+1}} + T^{2s-1+\frac{k}{k+1}}M_1^{\frac{1}{k+1}} \end{aligned} \quad (3.41)$$

It follows that there is a constant $A > 0$ such that $\|x\| \leq A$, Thus Ω_1 is bounded.

Let $\Omega_2 = \{x \in KerL, QNx = 0\}$. Suppose $x \in \Omega_2$, then $x(t) = d \in R$ and satisfies

$$QNx = \frac{1}{T}\int_0^T [-b_0 d^k - f(t, d) + p(t)]dt = 0, \quad (3.42)$$

We will prove that there exists a constant $B > 0$ such that $|d| \leq B$. If $|d| \leq \delta$, taking $\delta = B$, we get $|d| \leq B$. If $|d| > \delta$, from (3.42) , we have

$$\begin{aligned} |b_0||d|^k &= |\frac{1}{T}\int_0^T [-f(t, d) + p(t)]dt| \\ &\leq \frac{1}{T}\int_0^T |f(t, d)|dt + |p(t)|_\infty \leq (\gamma + \varepsilon)|d|^k + |p(t)|_\infty \end{aligned} \quad (3.43)$$

Thus

$$|d| \le [\frac{|p(t)|_\infty}{|b_0| - (\gamma + \varepsilon)}]^{\frac{1}{k}} \tag{3.44}$$

Taking $[\frac{|p(t)|_\infty}{|b_0|-(\gamma+\varepsilon)}]^{\frac{1}{k}} = B$, we have $|d| \le B$, which implies Ω_2 is bounded. Let Ω be a non-empty open bounded subset of X such that $\Omega \supset \overline{\Omega_1} \cup \overline{\Omega_2}$. We can easily see that L is a Fredholm operator of index zero and N is L-compact on $\overline{\Omega}$. Then by the above argument we have

(i) $Lx \ne \lambda Nx, \forall x \in \partial\Omega \cap D(L), \lambda \in (0,1)$.

(ii) $QNx \ne 0, \forall x \in \partial\Omega \cap KerL$.

At last we will prove that condition (iii) of Lemma 2.2 is satisfied. We take

$$\begin{aligned} &H : (\Omega \cap KerL) \times [0,1] \to KerL \\ &H(d,\mu) = sgn(-b_0)\mu d + \frac{1-\mu}{T}\int_0^T [-b_0 d^k - f(t,d) + p(t)]dt \end{aligned} \tag{3.45}$$

From assumptions (H_1) and (H_2), we can easily obtain $H(d,\mu) \ne 0, \forall (d,\mu) \in \partial\Omega \cap KerL \times [0,1]$, which results in

$$deg\{QN, \Omega \cap KerL, 0\} = deg\{H(\cdot,0), \Omega \cap KerL, 0\} = deg\{H(\cdot,1), \Omega \cap KerL, 0\} \ne 0 \tag{3.46}$$

Hence, by using Lemma 2.2, we know that Eq. (1.1) has at least one T-periodic solution.

Theorem 3.2 Suppose $n = 4m+1$, $m > 0$ an integer, k is odd, conditions (H_1), (H_2) hold. If

(H_5) there is a positive integer $0 < s \le m$ such that

$$b_{4s-3} \ne 0, b_{4s-3+i} = 0, i = 1, 2, \cdots, 4m - 4s + 3, \tag{3.47}$$

(H_6)

$$\begin{cases} A_2(4s-3,k) + \dfrac{\gamma A_1(4s-3,k)}{|b_0| - \gamma} + k|b_0|T^{4s-3}[\dfrac{A_1(4s-3,k)}{|b_0|-\gamma}]^{\frac{k-1}{k}} < b_{4s-3}, & if \quad 1 < s \le m \\ \dfrac{\gamma A_1(1,k)}{|b_0| - \gamma} < b_1, & if \quad s = 1 \end{cases} \tag{3.48}$$

Then Eq. (1.1) has at least one T-periodic solution.

Proof From the proof of Theorem 3.1, we have

$$(\int_0^T |x(t)|^{k+1}dt)^{\frac{k}{k+1}} \le \frac{A_1(4s-3,k)}{|b_0| - (\gamma+\varepsilon)}(\int_0^T |x^{(4s-3)}(t)|^{k+1}dt)^{\frac{k}{k+1}} + u_9. \tag{3.49}$$

where u_9 is a positive constant.

Multiplying both sides of (3.5) by $x^{(4s-3)}(t)$, and integrating on $[0,T]$, we have

$$\begin{aligned} \int_0^T x^{(n)}(t)x^{(4s-3)}(t)dt &= -\lambda\sum_{i=0}^{4s-3} b_i \int_0^T [x^{(i)}(t)]^k x^{(4s-3)}(t)dt \\ &\quad -\lambda\int_0^T f(t,x(t-\tau))x^{(4s-3)}(t)dt + \lambda\int_0^T p(t)x^{(4s-3)}(t)dt \end{aligned} \tag{3.50}$$

Since

$$\int_0^T x^{(4m+1)}(t)x^{(4s-3)}(t)dt = (-1)^{2m-2s+2}\int_0^T [x^{(2m+2s-1)}(t)]^2 dt \tag{3.51}$$

Then from (3.50) (3.51) it follows that

$$\begin{aligned} b_{4s-3}\int_0^T |x^{(4s-3)}(t)|^{k+1}dt \\ \leq -\sum_{i=0}^{4s-4} b_i \int_0^T [x^{(i)}(t)]^k x^{(4s-3)}(t)dt - \int_0^T f(t,x(t-\tau))x^{(4s-3)}(t)dt \\ + \int_0^T p(t)x^{(4s-3)}(t)dt \quad () \; 3.52) \end{aligned}$$

By using the same way as in the proof of Theorem 3.1, the following theorems can be proved in case $1 < s \leq m$ or $s = 1$.

Theorem 3.3 Suppose $n = 4m+1$, $m > 0$ for a positive integer, k is odd, conditions $(H_1), (H_2)$ hold. If
(H_7) there is a positive integer $0 < s \leq m$ such that

$$b_{4s-1} \neq 0, b_{4s-1+i} = 0, i = 1, 2, \cdots, 4m-4s+1 \tag{3.53}$$

(H_8)

$$A_2(4s-1,k) + \frac{\gamma A_1(4s-1,k)}{|b_0|-\gamma} + k|b_0|T^{4s-1}[\frac{A_1(4s-1,k)}{|b_0|-\gamma}]^{\frac{k-1}{k}} < -b_{4s-1} \tag{3.54}$$

Then Eq. (1.1) has at least one T-periodic solution.

Theorem 3.4 Suppose $n = 4m+3$, $m \geq 0$ an integer, k is odd, conditions $(H_1) - (H_2)$ hold. If
(H_9) there is a positive integer $0 \leq s \leq m$ such that

$$b_{4s+1} \neq 0, b_{4s+1+i} = 0, i = 1, 2, \cdots, 4m-4s+1 \tag{3.55}$$

(H_{10})

$$\begin{cases} A_2(4s+1,k) + \dfrac{\gamma A_1(4s+1,k)}{|b_0|-\gamma} + k|b_0|T^{4s+1}[\dfrac{A_1(4s+1,k)}{|b_0|-\gamma}]^{\frac{k-1}{k}} < -b_{4s+1}, & if \quad 0 < s \leq m \\ \dfrac{\gamma A_1(1,k)}{|b_0|-\gamma} < -b_1, \quad if \quad s = 0 \end{cases} \tag{3.56}$$

Then Eq. (1.1) has at least one T-periodic solution.

Theorem 3.5 Suppose $n = 4m+3$, $m > 0$ an integer, k is odd, conditions $(H_1), (H_2)$ hold If
(H_{11}) there is a positive integer $0 < s \leq m$ such that

$$b_{4s-1} \neq 0, b_{4s-1+i} = 0, i = 1, 2, \cdots, 4m-4s+3 \tag{3.57}$$

(H_{12})

$$A_2(4s-1,k)+\frac{\gamma A_1(4s-1,k)}{|b_0|-\gamma}+k|b_0|T^{4s-1}[\frac{A_1(4s-3,k)}{|b_0|-\gamma}]^{\frac{k-1}{k}}<b_{4s-1} \quad (3.58)$$

Then Eq. (1.1) has at least one T-periodic solution.

Theorem 3.6 Suppose $n=4m$, $m>0$ an integer, k is odd, conditions (H_1) hold. If
(H_{13})

$$b_0>\gamma \quad (3.59)$$

(H_{14}) there is a positive integer $0<s\leq 2m$ such that

$$\begin{cases} b_{2s-1}\neq 0, & if \quad s=2m \\ b_{2s-1}\neq 0, b_{2s-1+i}=0, i=1,2,\cdots,4m-2s, & if \quad 0<s<2m \end{cases} \quad (3.60)$$

(H_{15})

$$\begin{cases} A_2(2s-1,k)+\frac{\gamma A_1(2s-1,k)}{|b_0|-\gamma}+k|b_0|T^{2s-1}[\frac{A_1(2s-1,k)}{|b_0|-\gamma}]^{\frac{k-1}{k}}<|b_{2s-1}|, & if \quad 1<s\leq 2m \\ \frac{\gamma A_1(1,k)}{|b_0|-\gamma}<|b_1|, & if \quad s=1 \end{cases} \quad (3.61)$$

Then Eq. (1.1) has at least one T-periodic solution.

Theorem 3.7 Suppose $n=4m+2$, $m>0$ an integer, k is odd, conditions (H_1) hold. If
(H_{16})

$$-b_0>\gamma \quad (3.62)$$

(H_{17}) there is a positive integer $0<s\leq 2m+1$ such that

$$\begin{cases} b_{2s-1}\neq 0, & if \quad s=2m+1 \\ b_{2s-1}\neq 0, b_{2s-1+i}=0, i=1,2,\cdots,4m-2s, & if \quad 0<s<2m+1 \end{cases} \quad (3.63)$$

(H_{18})

$$\begin{cases} A_2(2s-1,k)+\frac{\gamma A_1(2s-1,k)}{|b_0|-\gamma}+k|b_0|T^{2s-1}[\frac{A_1(2s-1,k)}{|b_0|-\gamma}]^{\frac{k-1}{k}}<|b_{2s-1}|, & if \quad 1<s\leq 2m+1 \\ \frac{\gamma A_1(1,k)}{|b_0|-\gamma}<|b_1|, & if \quad s=1 \end{cases} \quad (3.64)$$

Then Eq. (1.1) has at least one T-periodic solution.

Theorem 3.8 Suppose $n=4m$, $m>0$ an integer, k is odd, conditions (H_1), (H_{13}) hold. If
(H_{19}) there is a positive integer $0<s\leq m$ such that

$$b_{4s-2}\neq 0, b_{4s-2+i}=0, i=1,2,\cdots,4m-4s+1 \quad (3.65)$$

(H_{20})

$$\begin{cases} A_2(4s-2,k) + \frac{\gamma A_1(4s-2,k)}{|b_0|-\gamma} + k|b_0|T^{4s-2}[\frac{A_1(4s-2,k)}{|b_0|-\gamma}]^{\frac{k-1}{k}} < -b_{4s-2}, & \text{if} \quad 1 < s \le m \\ \frac{\gamma A_1(2,k)}{|b_0|-\gamma} + k|b_0|T^2[\frac{A_1(2,k)}{|b_0|-\gamma}]^{\frac{k-1}{k}} < |b_2|, \quad \text{if} \quad s = 1 \end{cases} \tag{3.66}$$

Then Eq. (1.1) has at least one T-periodic solution.

Theorem 3.9 Suppose $n = 4m$, $m > 1$ an integer, k is odd, conditions $(H_1), (H_{13})$ hold. If
(H_{21}) there is a positive integer $1 < s \le m$ such that

$$b_{4s-4} \neq 0, b_{4s-4+i} = 0, i = 1, 2, \cdots, 4m-4s+3 \tag{3.67}$$

(H_{22})

$$A_2(4s-4,k) + \frac{\gamma A_1(4s-4,k)}{|b_0|-\gamma} + k|b_0|T^{4s-4}[\frac{A_1(4s-4,k)}{|b_0|-\gamma}]^{\frac{k-1}{k}} < -b_{4s-4} \tag{3.68}$$

Then Eq. (1.1) has at least one T-periodic solution.

Theorem 3.10 Suppose $n = 4m+2$, $m \ge 1$ an integer, k is odd, conditions $(H_1), (H_{16})$ hold, and the following conditions hold
(H_{23}) there is a positive integer $1 \le s \le m$ such that

$$b_{4s} \neq 0, b_{4s+i} = 0, i = 1, 2, \cdots, 4m-4s+1 \tag{3.69}$$

(H_{24})

$$A_2(4s,k) + \frac{\gamma A_1(4s,k)}{|b_0|-\gamma} + k|b_0|T^{4s}[\frac{A_1(4s,k)}{|b_0|-\gamma}]^{\frac{k-1}{k}} < -b_{4s} \tag{3.70}$$

Then Eq. (1.1) has at least one T-periodic solution.

Theorem 3.11 Suppose $n = 4m+2$, $m \ge 1$ an integer, k is odd, conditions $(H_1), (H_{16})$ hold. If
(H_{25}) there is a positive integer $1 \le s \le m$ such that

$$b_{4s-2} \neq 0, b_{4s-2+i} = 0, i = 1, 2, \cdots, 4m-4s+3 \tag{3.71}$$

(H_{26})

$$\begin{cases} A_2(4s-2,k) + \frac{\gamma A_1(4s-2,k)}{|b_0|-\gamma} + k|b_0|T^{4s-2}[\frac{A_1(4s-2,k)}{|b_0|-\gamma}]^{\frac{k-1}{k}} < b_{4s-2}, & \text{if} \quad 1 < s \le m \\ \frac{\gamma A_1(2,k)}{|b_0|-\gamma} + k|b_0|T^2[\frac{A_1(2,k)}{|b_0|-\gamma}]^{\frac{k-1}{k}} < b_2, \quad \text{if} \quad s = 1 \end{cases} \tag{3.72}$$

Then Eq. (1.1) has at least one T-periodic solution.
The proofs of Theorem 3.3-) 3.11 are similar to that of Theorem 3.1.

Theorem 3.12 Suppose k is even, conditions (H_1) hold. If
(H_{27}) there is an constant $c > 0$ such that $f(t,y) + b_0x^k < -|p(t)|_\infty$ $\forall t \in R$; $|x|, |y| > c$ and $f(t,0) > |p(t)|_\infty$ $\forall t \in R$.
(H_{28}) there is a positive integer $0 < s \le n-1$ such that

$$\begin{cases} b_s < 0, \quad if \quad s = n-1 \\ b_s < 0, b_{s+i} = 0, i = 1, 2, \cdots, n-1-s, \quad if \quad 0 < s < n-1 \end{cases} \tag{3.73}$$

(H_{29})

$$A_3(s,k) + \gamma T^{sk} < |b_s| \tag{3.74}$$

where $A_3(s,k) = \sum_{i=0}^{s-1} T^{(s-i)k}|b_i|$. Then Eq. (1.1) has at least one T-periodic positive solution.

Proof. For $x(t) > 0, x \in \Omega_1$, we have

$$x^{(n)}(t) = -\lambda \sum_{i=0}^{s} b_i[x^{(i)}(t)]^k - \lambda f(t, x(t-\tau)) + \lambda p(t), \qquad \lambda \in (0,1).$$

Integrating the above formula on $[0, T]$, we have

$$\int_0^T [f(t, x(t-\tau)) + b_0|x(t)|^k]dt = -\sum_{i=1}^{s} b_i \int_0^T |x^{(i)}(t)|^k dt + \int_0^T p(t)dt \tag{3.75}$$

If $s > 1$, since

$$\begin{aligned} -\sum_{i=0}^{s} b_i \int_0^T |x^{(i)}(t)|^k dt &\geq -b_s \int_0^T |x^{(s)}(t)|^k dt - \sum_{i=1}^{s-1} |b_i| \int_0^T |x^{(i)}(t)|^k dt \\ &\geq [-b_s - \sum_{i=1}^{s-1} T^{(s-i)k}|b_i|] \int_0^T |x^{(s)}(t)|^k dt \geq 0. \end{aligned} \tag{3.76}$$

it follows from (3.75) and (3.76) that we have

$$\int_0^T [f(t, x(t-\tau)) + b_0|x(t)|^k]dt \geq \int_0^T p(t)dt. \tag{3.77}$$

If $s = 1$, it is easy to see that the above inequality holds.

We can prove that there is a $t_1 \in [0, T]$ such that $|x(t_1)| < c$. Indeed, from (3.77) , there is a $t_0 \in [0, T]$ such that

$$f(t_0, x(t_0 - \tau)) + b_0|x(t_0)|^k \geq -|p(t)|_\infty \tag{3.78}$$

If $0 < x(t_0) \leq c$, then take $t_1 = t_0$ so that $0 < x(t_1) \leq c$.
If $x(t_0) > c$, it follows from assumption (H_{27}) that $0 < x(t_0 - \tau) \leq c$. Since $x(t)$ is continuous for $t \in R$ and $x(t + T) = x(t)$, so there must be an integer k and a point $t_1 \in [0, T]$ such that $t_0 - \tau = kT + t_1$. so $|x(t_1)| = |x(t_0 - \tau)| \leq c$, which implies

$$|x(t)|_\infty \leq c + \int_0^T |x'(t)|dt \leq c + T^{\frac{k-1}{k}}(\int_0^T |x'(t)|^k dt)^{\frac{1}{k}} \leq c + T^{s-\frac{1}{k}}(\int_0^T |x^{(s)}(t)|^k dt)^{\frac{1}{k}} \tag{3.79}$$

On the other hand, from (3.75) , if $s > 1$, we have

$$\begin{aligned} &b_s \int_0^T |x^{(s)}(t)|^k dt \\ &= -\sum_{i=1}^{s-1} b_i \int_0^T |x^{(i)}(t)|^k dt - b_0 \int_0^T |x(t)|^k dt - \int_0^T f(t, x(t-\tau))dt + \int_0^T p(t)dt. \end{aligned} \tag{3.80}$$

Thus, applying Lemma 2.1, we get

$$\begin{aligned}|b_s|\int_0^T|x^{(s)}(t)|^kdt &\le \sum_{i=1}^{s-1}|b_i|\int_0^T|x^{(i)}(t)|^kdt+\\ &\qquad |b_0|\int_0^T|x(t)|^kdt+\int_0^T|f(t,x(t-\tau))|dt+\int_0^T|p(t)|dt\\ &\le \sum_{i=1}^{s-1}|b_i|\int_0^T|x^{(i)}(t)|^kdt+|b_0|\int_0^T|x(t)|^kdt+(\gamma+\varepsilon)\int_0^T|x(t-\tau)|^kdt\\ &\qquad +(f_\delta+|p(t)|)T\\ &\le \sum_{i=1}^{s-1}T^{(s-i)k}|b_i|\int_0^T|x^{(s)}(t)|^kdt+[|b_0|+(\gamma+\varepsilon)]T|x(t)|_\infty^k+(f_\delta+|p(t)|)T\end{aligned} \tag{3.81}$$

We can prove that there is a constant $M_3>0$ such that

$$\int_0^T|x^{(s)}(t)|^kdt\le M_3 \tag{3.82}$$

For some nonnegative integer l, there is a constant $0<h<1$ such that

$$(1+x)^l<1+(l+1)x, x\in(0,h) \tag{3.83}$$

Now we consider two cases to finish our proof.

Case 1 If $(\int_0^T|x^{(s)}(t)|^kdt)^{\frac{1}{k}}\le\frac{c}{T^{s-\frac{1}{k}}h}$, then

$$|x(t)|_\infty\le c+T^{s-\frac{1}{k}}(\int_0^T|x^{(s)}(t)|^kdt)^{\frac{1}{k}}\le c+\frac{c}{h} \tag{3.84}$$

So substituting the above formula into (3.81) , we have

$$[|b_s|-\sum_{i=1}^{s-1}T^{(s-i)k}|b_i|]\int_0^T[x^{(s)}(t)]^kdt\le[|b_0|+(\gamma+\varepsilon)]T((c+\frac{c}{h}))^k+(f_\delta+|p(t)|)T \tag{3.85}$$

Hence there is a constant $M_3>0$ such that

$$\int_0^T|x^{(s)}(t)|^kds\le M_3 \tag{3.86}$$

Case 2 If $(\int_0^T|x^{(s)}(t)|^kdt)^{\frac{1}{k}}>\frac{c}{T^{s-\frac{1}{k}}h}$.

$$\begin{aligned}|x(t)|_\infty^k &\le [c+T^{s-\frac{1}{k}}(\int_0^T|x^{(s)}(t)|^kdt)^{\frac{1}{k}}]^k\\ &= T^{sk-1}(\int_0^T|x^{(s)}(t)|^kdt)[1+\frac{c}{T^{s-\frac{1}{k}}(\int_0^T|x^{(s)}(t)|^kdt)^{\frac{1}{k}}}]^k\\ &\le T^{sk-1}(\int_0^T|x^{(s)}(t)|^kdt)[1+\frac{c(k+1)}{T^{s-\frac{1}{k}}(\int_0^T|x^{(s)}(t)|^kdt)^{\frac{1}{k}}}]\\ &= T^{sk-1}(\int_0^T|x^{(s)}(t)|^kdt)+c(k+1)T^{s(k-1)+\frac{1}{k}-1}(\int_0^T|x^{(s)}(t)|^kdt)^{\frac{k-1}{k}}\end{aligned} \tag{3.87}$$

Substituting the above formula into (3.81) , we have

$$\begin{aligned}|b_s|\quad&\int_0^T|x^{(s)}(t)|^kdt\\ &\le\sum_{i=1}^{s-1}T^{(s-i)k}|b_i|\int_0^T|x^{(s)}(t)|^kdt+[|b_0|+(\gamma+\varepsilon)][T^{sk}(\int_0^T|x^{(s)}(t)|^kdt)\\ &+c(k+1)T^{s(k-1)+\frac{1}{k}}(\int_0^T|x^{(s)}(t)|^kdt)^{\frac{k-1}{k}}]+(f_\delta+|p(t)|)T\end{aligned} \tag{3.88}$$

Then

$$[|b_s| \quad -A_3(s,k)-(\gamma+\varepsilon)T^{sk}]\int_0^T[x^{(s)}(t)]^k dt \\ \leq c(k+1)[|b_0|+(\gamma+\varepsilon)]T^{s(k-1)}(\int_0^T|x^{(s)}(t)|^k dt)^{\frac{k-1}{k}}+(f_\delta+|p(t)|)T \tag{3.89}$$

Hence there is a constant $M_4>0$ such that

$$\int_0^T|x^{(s)}(t)|^k dd\leq M_4 \tag{3.90}$$

If $s=1$, similarly, we can prove that there is a constant $M_5>0$ such that

$$\int_0^T|x'(t)|^k dd\leq M_5 \tag{3.92}$$

The remainder can be proved in the same way as in the proof of Theorem 3.1.

Theorem 3.13 Suppose k is even, conditions (H_1) and (H_{29}) hold. If
(H_{30}) there is an constant $c>0$ such that $f(t,y)+b_0x^k>|p(t)|_\infty \quad \forall t\in R$; $|x|,|y|>c$ and $f(t,0)<-|p(t)|_\infty\forall t\in R$.
(H_{31}) there is a positive integer $0<s\leq n-1$ such that

$$\begin{cases} b_s>0, \quad if \quad s=n-1 \\ b_s>0, b_{s+i}=0, i=1,2,\cdots,n-1-s, \quad if \quad 0<s<n-1 \end{cases} \tag{3.93}$$

Then Eq. (1.1) has at least one T-periodic positive solution.

Example 3.1 Consider the following equation

$$x^{(5)}(t)+1000[x''(t)]^3+\tfrac{1}{100}[x'(t)]^3+\tfrac{1}{8000}[x(t)]^3+\tfrac{1}{40000}(\sin t)[x(t-\pi)]^3=\cos t \tag{3.94}$$

where $n=5, k=3, b_4=b_3=0, b_2=1000, b_1=\frac{1}{100}, b_0=\frac{1}{8000}, f(t,x)=\frac{1}{40000}(\sin t)x^3, p(t)=\cos t, \tau=\pi$. Thus, $T=2\pi, \gamma=\frac{1}{40000}, A_1(2,k)=|b_1|(2\pi)^3+|b_2|=\frac{1}{100}\times(2\pi)^3+1000$. Obviously assumption $(H_1)-(H_3)$ hold and

$$\tfrac{\gamma A_1(2,k)}{|b_0|-\gamma}+k|b_0|(2\pi)^2[\tfrac{A_1(2,k)}{|b_0|-\gamma}]^{\frac{k-1}{k}}<|b_2| \tag{3.95}$$

By Theorem 3.1, we know that Eq. (3.94) has at least one 2π-periodic solution.

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