Differences of Weighted Composition Operators on the Disk Algebra

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Abstract

We study properties of the differences of weighted composition operators on the disk algebra and will see the equivalence of the compactness, the weak compactness and the complete continuity of them. Moreover, we characterize the differences of weighted composition operators acting from the space of bounded analytic functions to the disk algebra.

1 Introduction

Let $\mathbb{D}$ be the open unit disk in the complex plane and $\overline{\mathbb{D}}$ its closure. Let $A = A(\mathbb{D})$ be the disk algebra of all continuous functions on $\overline{\mathbb{D}}$ that are analytic on $\mathbb{D}$. Then A is the Banach algebra with the supremum norm

$$\|f\|_\infty = \sup\{|f(z)|; z \in \overline{\mathbb{D}}\}.$$

And denote by $H^\infty = H^\infty(\mathbb{D})$ the set of all bounded analytic functions on $\mathbb{D}$. The norm of H^∞ also is defined by the supremum norm on $\mathbb{D}$.

The object of the study here is the operators induced by multiplying an analytic function and by the composition with an analytic self-map of $\mathbb{D}$. More precisely, for analytic functions $u, \varphi \in A$ with $\|\varphi\|_\infty \leq 1$, we define a *weighted composition operator* uC_φ on A by

$$uC_\varphi f = u\cdot(f \circ \varphi) \quad \text{for} \quad f \in A.$$

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It is clear that uC_φ is linear and bounded on A.

Multiplication and composition operators have been investigated on various function spaces by researchers. See [2], [7], and [8]. In [6], MacCluer, Zhao and the author gave an attention to the differences of composition operators on H^∞. And Hosokawa, Izuchi and the author [5] have continued the study of properties of the differences to the case of the weighted composition operators on H^∞. They gave the necessary and sufficient conditions for a difference of two weighted composition operators to be compact on H^∞ and simultaneously showed that conditions of compactness, weak compactness, and complete continuity are equivalent for a difference of operators. In this paper we consider the case of the disk algebra. To characterize the compactness, the case of the disk algebra is different from the case H^∞. Indeed we can not use the "weak convergence theorem" (for example, Proposition 3.11 of [2]) in the case of the disk algebra. In section 2 we show the equivalence of the compactness, the weak compactness and the complete continuity in the case of the disk algebra. Moreover, in section 3, we characterize the differences of weighted composition operators acting from the space of bounded analytic functions to the disk algebra.

2 Compact differences of weighted composition operators

In this section, we shall give necessary and sufficient conditions for a difference of two weighted composition operators to be compact on A and simultaneously show that conditions of compactness, weak compactness, and complete continuity are equivalent.

Let T be a bounded linear operator on a Banach space. Recall that T is said to be (*weakly*) *compact* if T maps every bounded set into relatively (weakly) compact one, and that T is said to be *completely continuous* if T maps every weakly convergent sequence into norm convergent one. In general, every compact operator is completely continuous. But the converse is not always true. A Banach space X is said to have the *Dunford-Pettis property* if every weakly compact operator on X becomes completely continuous. See [3] for more information on the Dunford-Pettis property.

Our results involve the pseudo-hyperbolic metric. For $z, w \in \mathbb{D}$, the pseudo-hyperbolic distance between z and w is given by

$$\rho(z,w) = \left| \frac{z-w}{1-\bar{z}w} \right|.$$

For $\varphi \in A$ with $\|\varphi\|_\infty \le 1$, let $\Gamma_\varphi = \{\zeta \in \partial\mathbb{D} : |\varphi(\zeta)| = 1\}$. Our main result in this section is the following.

Theorem 1. *Let $u, v \in A$ and $\varphi, \psi \in A$ with $\|\varphi\|_\infty \le 1$, $\|\psi\|_\infty \le 1$. Then the following conditions are equivalent:*

(i) *$uC_\varphi - vC_\psi$ is compact on A.*

(ii) *$uC_\varphi - vC_\psi$ is weakly compact on A.*

(iii) *$uC_\varphi - vC_\psi$ is completely continuous on A.*

(iv) *The following three conditions hold.*

(a) *If $\zeta \in \Gamma_\varphi$ and $\lim_{z\to\zeta} \rho(\varphi(z), \psi(z)) \neq 0$, then $u(\zeta) = 0$.*

(b) *If $\zeta \in \Gamma_\psi$ and $\lim_{z\to\zeta} \rho(\varphi(z), \psi(z)) \neq 0$, then $v(\zeta) = 0$.*

(c) *If $\zeta \in \Gamma_\varphi \cap \Gamma_\psi$, then $u(\zeta) = v(\zeta)$.*

Proof. The implication (i)⇒(ii) is clear. Since Chaumat [1] showed that A has the Dunford-Pettis property, the implication (ii)⇒(iii) is also clear.

Suppose that (iii) holds. We shall prove (iv). To prove (a), we can find out a sequence $\{z_n\}$ in $\overline{\mathbb{D}}$ satisfying the following properties: as $n \to \infty$,

(1) $z_n \to \zeta \in \Gamma_\varphi$.

(2) $\varphi(z_n) \to \varphi(\zeta)$.

(3) $\overline{\varphi(\zeta)} \dfrac{\varphi(z_n) - \varphi(\zeta)}{1 - \overline{\varphi(z_n)}\varphi(\zeta)} \to \lambda$ for some constant λ with modulus one.

(4) $\dfrac{\varphi(z_n) - \psi(z_n)}{1 - \overline{\varphi(z_n)}\psi(z_n)} \longrightarrow \sigma \neq 0$.

Then define the functions $\{f_n(z)\}$ on $\overline{\mathbb{D}}$ by

$$f_n(z) = \left(\overline{\varphi(\zeta)} \frac{\varphi(z_n) - z}{1 - \overline{\varphi(z_n)}z} - 1 \right) \left(\overline{\varphi(\zeta)} \frac{\varphi(z_n) - z}{1 - \overline{\varphi(z_n)}z} - \lambda \right) \times \left(\frac{\varphi(z_n) - z}{1 - \overline{\varphi(z_n)}z} - \sigma \right).$$

Obviously $f_n \in A$ and $\|f_n\|_\infty \le 8$.

We can easily check that $f_n(z) \to 0$ for $z \in \overline{\mathbb{D}}$ as $n \to \infty$. So, by [4: Exercise IV.13.37], f_n converges weakly to 0 in A.

Thus we have

$$\begin{aligned} \|(uC_\varphi - vC_\psi)f_n\|_\infty &\ge |(uC_\varphi - vC_\psi)f_n(z_n)| \\ &= |u(z_n)\lambda\sigma - v(z_n)f_n(\psi(z_n))|. \end{aligned}$$

Since $C_\varphi - C_\psi$ is completely continuous on A and $f_n(\psi(z_n)) \to 0$ as $n \to \infty$, then $u(z_n) \to 0 = u(\zeta)$. We obtain the condition (a).

A similar argument follows that the condition (b) holds.

Next suppose $\zeta \in \Gamma_\varphi \cap \Gamma_\psi$. We can find out a sequence $\{w_n\}$ in $\overline{\mathbb{D}}$ satisfying the following properties: as $n \to \infty$,

(5) $w_n \to \zeta$.

(6) $\varphi(w_n) \to \varphi(\zeta)$.

(7) $\overline{\varphi(\zeta)} \dfrac{\varphi(w_n) - \varphi(\zeta)}{1 - \overline{\varphi(w_n)} p(\zeta)} \to \lambda$ for some constant λ with modulus one.

By the conditions (a) and (b), it is sufficient to consider only the case that $\rho(\varphi(w_n), \psi(w_n)) \to 0$ as $w_n \to \zeta$. For the above sequence $\{w_n\}$, define the functions

$$f_n(z) = \left(\overline{\varphi(\zeta)} \frac{\varphi(w_n) - z}{1 - \overline{\varphi(w_n)} z} - 1 \right) \left(\overline{\varphi(\zeta)} \frac{\varphi(w_n) - z}{1 - \overline{\varphi(w_n)} z} - \lambda \right).$$

Then $f_n \in A$ and $\|f_n\|_\infty \leq 4$. And we also can obtain that $f_n(z) \to 0$ for each $z \in \overline{\mathbb{D}}$ and so f_n converges weakly to 0 in A. Since $uC_\varphi - vC_\psi$ is completely continuous on A, then $\|(uC_\varphi - vC_\psi)f_n\|_\infty \to 0$ as $n \to \infty$.

On the other hand,

$$\begin{aligned}
&\|(uC_\varphi - vC_\psi)f_n\|_\infty \\
&\geq |u(w_n) f_n(\varphi(w_n)) - v(w_n) f_n(\psi(w_n))| \\
&\geq |u(w_n) - v(w_n)||\lambda| - 2\|v\|_\infty \|f_n\|_\infty \rho(\varphi(w_n), \psi(w_n)).
\end{aligned}$$

Thus if $n \to \infty$, we obtain $u(w_n) - v(w_n) \to 0$, that is, the condition (c).

Next we show the implication (iv)⇒(i) holds. Let $f_n \in A$ and $\|f_n\|_\infty = 1$. By the normal family argument, there exist a subsequence $\{f_{n_k}\}$ of $\{f_n\}$ and a function g analytic on $\mathbb{D}$ such that f_{n_k} converges to g uniformly on compact subsets of $\mathbb{D}$. Here we have

$$\sup_{z \in \mathbb{D}} |g(z)| \leq 1.$$

Now define a function G on $\overline{\mathbb{D}}$ by setting

$$G(z) = \begin{cases} -v(z) C_\psi g(z) & z \in \Gamma_\varphi \setminus \Gamma_\psi \\ u(z) C_\varphi g(z) & z \in \Gamma_\psi \setminus \Gamma_\varphi \\ 0 & z \in \Gamma_\varphi \cap \Gamma_\psi \\ (uC_\varphi - vC_\psi) g(z) & \text{otherwise.} \end{cases}$$

We first show that G is continuous on $\overline{\mathbb{D}}$. Indeed G is continuous on $\overline{\mathbb{D}} \setminus \Gamma_\varphi \cup \Gamma_\psi$.

For $\zeta \in \Gamma_\varphi \cap \Gamma_\psi$, let $\{z_n\}$ be a sequence in $\overline{\mathbb{D}}$ converging to ζ such that $|\varphi(z_n)| \to 1, |\psi(z_n)| \to 1$. Then we have

$$\begin{aligned}
&|(uC_\varphi - vC_\psi) G(z_n)| \\
&\leq |u(z_n) - v(z_n)| \|G\|_\infty + 2\|v\|_\infty \|G\|_\infty \rho(\varphi(z_n), \psi(z_n)).
\end{aligned}$$

So, if $\rho(\varphi(z_n), \psi(z_n)) \to 0$, then we obtain $\lim_{z_n \to \zeta} (uC_\varphi - vC_\psi) G(z_n) = 0 = G(\zeta)$.

If $\rho(\varphi(z_n), \psi(z_n)) \not\to 0$, by the conditions (a) and (b), $u(\zeta) = v(\zeta) = 0$. Thus we also have $\lim_{z_n \to \zeta} (uC_\varphi - vC_\psi) G(z_n) = 0 = G(\zeta)$.

For $\zeta \in \Gamma_\varphi \setminus \Gamma_\psi$, let $\{z_n\}$ be a sequence in $\overline{\mathbb{D}}$ converging to ζ such that $|\varphi(z_n)| \to 1, |\psi(z_n)| \not\to 1$. So $\rho(\varphi(z_n), \psi(z_n)) \not\to 0$. Then we have

$$\begin{aligned}&|(uC_\varphi - vC_\psi)G(z_n) + v(\zeta)C_\psi g(\zeta)| \\ &\leq |u(z_n)|\|G\|_\infty + |v(z_n)C_\psi G(z_n) - v(\zeta)C_\psi g(\zeta)|.\end{aligned}$$

So by (a), we obtain $\lim_{z_n \to \zeta}(uC_\varphi - vC_\psi)G(z_n) = -v(\zeta)C_\psi g(\zeta)$.

For $\zeta \in \Gamma_\psi \setminus \Gamma_\varphi$, we can prove the continuity of G by the similar way as the above. After all, G is continuous on $\overline{\mathbb{D}}$ and so $G \in A$.

To show that $(uC_\varphi - vC_\psi)f_n$ converges uniformly to G on $\overline{\mathbb{D}}$, we may assume that for some $\varepsilon > 0$, $\|(uC_\varphi - vC_\psi)f_n - G\|_\infty > \varepsilon > 0$ for every n. Then there exists a sequence $\{z_n\}_n$ in $\mathbb{D}$ such that

$$|u(z_n)f_n(\varphi(z_n)) - v(z_n)f_n(\psi(z_n)) - G(z_n)| > \varepsilon \quad \text{for every } n. \tag{8}$$

This implies that $\max\{|\varphi(z_n)|, |\psi(z_n)|\} \to 1$ as $n \to \infty$. Here we may assume that $|\varphi(z_n)| \to 1$ and $\psi(z_n) \to w_0$ for some complex number w_0. Moreover we may assume that

$$\rho(\varphi(z_n), \psi(z_n)) \to r \text{ as } n \to \infty.$$

Suppose that $r > 0$. If $|w_0| = 1$, by (a) and (b) we have $u(z_n) \to 0$ and $v(z_n) \to 0$. On the other hand, $G(z_n) \to 0$ as $n \to \infty$ by the definition of G. This contradicts (8). If $|w_0| < 1$, by (a) we have $u(z_n) \to 0$ and $v(z_n)f_n(\psi(z_n)) \to v(\zeta)g(\psi(\zeta))$ as $n \to \infty$. So $-v(z_n)f_n(\psi(z_n)) - G(z_n) \to 0$. Also this contradicts (8). Hence we obtain $r = 0$. Then we have $|\psi(z_n)| \to 1$ as $n \to \infty$ and

$$\begin{aligned}&|u(z_n)f_n(\varphi(z_n)) - v(z_n)f_n(\psi(z_n)) - G(z_n)| \\ &\leq |u(z_n) - v(z_n)|\|f_n\|_\infty + 2\|v\|_\infty \rho(\varphi(z_n), \psi(z_n)) + |G(z_n)|.\end{aligned}$$

This shows that $u(z_n)f_n(\varphi(z_n)) - v(z_n)f_n(\psi(z_n)) - G(z_n) \to 0$. This fact also contradicts (8). Thus we get the implication (iv)⇒(i). ∎

We here give an example that both weighted composition operators are not compact but their difference is compact on A. Indeed we can present the same example functions as in [5]. That is, let $\sigma(z) = (1+z)/(1-z)$ and $\varphi(z) = (\sigma(z)^{1/2} - 1)/(\sigma(z)^{1/2} + 1)$ be a lens map. And let $\psi(z) = 1 - \sqrt{2(1-z)}$. Then $\varphi(\pm 1) = \pm 1$ and $\psi(\pm 1) = \pm 1$. Then

$$\lim_{z \to -1}\inf \rho(\varphi(z), \psi(z)) = 1.$$

On the other hand,

$$\lim_{z \to 1}\inf \rho(\varphi(z), \psi(z)) = 0.$$

We take functions u, v in the disk algebra, $u \neq v$ such that $u(1) = v(1) = 1$ and $u(-1) = v(-1) = 0$. Then neither uC_φ nor vC_ψ is compact, but by Theorem 1 $uC_\varphi - vC_\psi$ is compact on A.

3 Weighted composition operators from H^∞ to A

Suppose $T : H^\infty \to A$ is a bounded operator. Then it is known that T is weakly compact and so complete continuous. So in this section we discuss when $uC_\varphi - vC_\psi : H^\infty \to A$ is bounded.

Theorem 2. *Let $u, v \in A$ and $\varphi, \psi \in A$ with $\|\varphi\|_\infty \le 1$, $\|\psi\|_\infty \le 1$. Then the following conditions are equivalent:*

(i) *$uC_\varphi - vC_\psi : H^\infty \to A$ is bounded.*

(ii) *The following three conditions hold.*

(a) *If $\zeta \in \Gamma_\varphi$ and $\lim_{z\to\zeta} \rho(\varphi(z), \psi(z)) \neq 0$, then $u(\zeta) = 0$.*

(b) *If $\zeta \in \Gamma_\psi$ and $\lim_{z\to\zeta} \rho(\varphi(z), \psi(z)) \neq 0$, then $v(\zeta) = 0$.*

(c) *If $\zeta \in \Gamma_\varphi \cap \Gamma_\psi$, then $u(\zeta) = v(\zeta)$.*

Proof. Suppose that $uC_\varphi - vC_\psi : H^\infty \to A$ is bounded. For $\zeta \in \Gamma_\varphi \setminus \Gamma_\psi$, $|\varphi(\zeta)| = 1$ and $|\psi(\zeta)| < 1$. So, for $f \in H^\infty$, $vC_\psi f$ is continuous at ζ and so is $uC_\varphi f$. As $|\varphi(\zeta)| = 1$, we can a sequence $\{z_n\}$ in $\overline{\mathbb{D}}$ such that $z_n \to \zeta, \varphi(z_n) \in \mathbb{D}$ and $|\varphi(z_n)| \to 1$. Further we may assume that this sequence is an interpolating sequence. Taking an interpolating Blaschke product B with zeros $\{z_n\}$, B has an essential singularity at $\varphi(\zeta)$. Thus $u(\zeta) = 0$ because $uC_\varphi B$ is continuous at ζ.

For $\zeta \in \Gamma_\psi \setminus \Gamma_\varphi$, the similar argument implies that $v(\zeta) = 0$.

For $\zeta \in \Gamma_\varphi \cap \Gamma_\psi$, firstly assume that $\lim_{z\to\zeta} \rho(\varphi(z), \psi(z)) = 0$. Then, noting $\varphi(\zeta) = \psi(\zeta)$ in this case, for $f \in H^\infty$, we have

$$\begin{aligned}
&|(uC_\varphi - vC_\psi)f(z) - (uC_\varphi - vC_\psi)f(\zeta)| \\
&\ge |(u(z) - v(z))f(\varphi(z)) - (u(\zeta) - v(\zeta))f(\varphi(\zeta))| \\
&\qquad - |v(z)||f(\varphi(z)) - f(\psi(z))| - |v(\zeta)||f(\varphi(\zeta)) - f(\psi(\zeta))| \\
&\ge |(u(z) - v(z))f(\varphi(z)) - (u(\zeta) - v(\zeta))f(\varphi(\zeta))| \\
&\qquad - 2\|v\|_\infty \|f\|_\infty \rho(\varphi(z), \psi(z)).
\end{aligned}$$

So

$$\lim_{z\to\zeta} (u(z) - v(z))f(\varphi(z)) = (u(\zeta) - v(\zeta))f(\varphi(\zeta)).$$

Here we can take any function f which is not continuous at $\varphi(\zeta)$, so we have $u(\zeta) - v(\zeta) = 0$.

Next assume that $\lim_{z\to\zeta} \rho(\varphi(z), \psi(z)) \neq 0$. Then we have an interpolating sequence $\{\varphi(z_n)\} \cup \{\psi(z_n)\}$ and take an interpolating Blaschke product B with zeros $\{\varphi(z_n)\} \cup \{\psi(z_n)\}$. It needs $u(\zeta) = v(\zeta) = 0$ for $(uC_\varphi - vC_\psi)B$ to be continuous at $\varphi(\zeta)$.

Consequently we summarize these results to the condition (ii).

The converse implication is clear. ∎

We notice the boundedness of $uC_\varphi - vC_\psi : H^\infty \to A$ is equivalent to the compactness of $uC_\varphi - vC_\psi : A \to A$. Moreover we can easily obtain the following.

Theorem 3. *Let $u, v \in A$ and $\varphi, \psi \in A$ with $\|\varphi\|_\infty \leq 1, \|\psi\|_\infty \leq 1$. Then the following conditions are equivalent.*

(i) $uC_\varphi - vC_\psi : H^\infty \to A$ *is bounded.*

(ii) $uC_\varphi - vC_\psi : H^\infty \to A$ *is compact.*

(iii) $uC_\varphi - vC_\psi : H^\infty \to A$ *is weakly compact.*

(iv) $uC_\varphi - vC_\psi : H^\infty \to A$ *is completely continuous.*

(v) $uC_\varphi - vC_\psi : A \to A$ *is compact.*

We can prove the implication (iv)⇒(v) by the same method as in the proof of Theorem 2.2 of [5] and the implication (v)⇒(i) by the same way as in the proof of Theorem 2 above.

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