The cup-length of the oriented Grassmannians vs a new bound for zero-cobordant manifolds*

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Abstract

We derive an inequality for the $\mathbb{Z}_2$-cup-length of any smooth closed connected manifold unorientedly cobordant to zero. In relation to this, we introduce a new numerical invariant of a smooth closed connected manifold, called the characteristic rank. In particular, our inequality yields strong upper bounds for the cup-length of the oriented Grassmann manifolds $\tilde{G}_{n,k} \cong SO(n)/SO(k) \times SO(n-k)$ $(6 \leq 2k \leq n)$ if n is odd; if n is even, we obtain new upper bounds in a different way. We also derive lower bounds for the cup-length of $\tilde{G}_{n,k}$. For $\tilde{G}_{2^t-1,3}$ $(t \geq 3)$ our upper and lower bounds coincide, giving that the $\mathbb{Z}_2$-cup-length is $2^t - 3$ and the characteristic rank equals $2^t - 5$. Some applications to the Lyusternik-Shnirel'man category are also presented.

1 Introduction and statement of results

The $\mathbb{Z}_2$-cup-length, $\mathrm{cup}(X)$, of a compact path connected topological space X is defined to be the maximum of all numbers c such that there exist, in positive degrees, cohomology classes $a_1, \ldots, a_c \in H^*(X;\mathbb{Z}_2)$ such that their cup product $a_1 \cup \cdots \cup a_c$ is nonzero. Instead of the usual notation $a \cup b$, we shall mostly write

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$a \cdot b$ or just ab. For applications, the well known Elsholz inequality

$$\mathrm{cat}(X) \geq 1 + \mathrm{cup}(X), \tag{1}$$

is often important; here $\mathrm{cat}(X)$ is the Lyusternik-Shnirel'man category, that is, the least positive integer k such that X can be covered by k open subsets each of which is contractible in X (see, e.g., [4]).

If a closed smooth connected d-dimensional manifold M is (unorientedly) cobordant to zero, then it is not true that each element in the $\mathbb{Z}_2$-cohomology algebra $H^*(M;\mathbb{Z}_2)$ can be expressed as a polynomial in the Stiefel-Whitney classes $w_i(M)$. Indeed, if we admit that the contrary is true, then some of the Stiefel-Whitney classes, say $w_\kappa(M)$, must be nonzero. By Poincaré duality, there exists a cohomology class $x \in H^{d-\kappa}(M;\mathbb{Z}_2)$ such that $w_\kappa(M)x \neq 0$. The possibility of expressing x as a polynomial in the classes $w_i(M)$ yields then a contradiction to the fact that all the Stiefel-Whitney numbers of M vanish. Nevertheless, for any M of this type, there exists an integer z such that each element of $H^j(M;\mathbb{Z}_2)$ with $j \leq z$ can be expressed as a polynomial in the Stiefel-Whitney classes $w_i(M)$ (we observe that the same z can be taken for all smooth closed manifolds homotopy equivalent to M; we recall (see, e.g., [11, 11.13]) that the Stiefel-Whitney classes of homotopy equivalent manifolds are the same up to the obvious isomorphism). For example, since $w_0(M) = 1$, we can take $z = 0$ for any M.

For the cup-length of manifolds unorientedly cobordant to zero, we shall prove the following new inequality.

Theorem 1.1. *Let M be a closed smooth connected d-dimensional manifold unorientedly cobordant to zero. Let $\tilde{H}^r(M;\mathbb{Z}_2)$, $r < d$, be the first nonzero reduced cohomology group of M. Let z $(z < d-1)$ be an integer such that for $j \leq z$ each element of $H^j(M;\mathbb{Z}_2)$ can be expressed as a polynomial in the Stiefel-Whitney classes of the manifold M. Then we have that*

$$\mathrm{cup}(M) \leq 1 + \frac{d-z-1}{r}. \tag{2}$$

This theorem and some other results (see, e.g., [8, Theorem C]) motivate us to introduce a new numerical invariant of a closed smooth connected d-dimensional manifold M called the *characteristic rank*, denoted $\mathrm{charrank}(M)$, as the largest integer k, $0 \leq k \leq d$, such that each element of $H^j(M;\mathbb{Z}_2)$ with $j \leq k$ can be expressed as a polynomial in the Stiefel-Whitney classes of M. Of course, $\mathrm{charrank}(M)$ depends only on the homotopy type of M.

It is clear that to use Theorem 1.1 optimally means to use it with z equal to the characteristic rank, and so we are confronted with the following natural problem.

Problem 1.1. *Find the characteristic rank for all closed smooth connected manifolds.*

This task in its generality certainly represents a hard problem, requiring further research. Now let us look – in the context of the new estimate (2) and Problem 1.1 – at the oriented Grassmann manifolds.

We recall that the oriented Grassmann manifold $\tilde{G}_{n,k} \cong SO(n)/SO(k) \times SO(n-k)$ consists of oriented k-dimensional vector subspaces in Euclidean n-space $\mathbb{R}^n$. The spheres $\tilde{G}_{n,1} \cong S^{n-1}$ and the complex quadrics $\tilde{G}_{n,2}$ are well understood special cases, and so we shall suppose that $k \geq 3$ throughout the paper. In addition to this, due to the obvious diffeomorphism $\tilde{G}_{n,k} \cong \tilde{G}_{n,n-k}$, we shall suppose that $2k \leq n$.

The manifold $\tilde{G}_{n,k}$ is the universal double covering space for the Grassmann manifold $G_{n,k} \cong O(n)/O(k) \times O(n-k)$ of unoriented k-dimensional vector subspaces in $\mathbb{R}^n$. We denote by $p : \tilde{G}_{n,k} \to G_{n,k}$ the obvious covering projection. For the tangent bundles we have $T\tilde{G}_{n,k} \cong p^*(TG_{n,k})$.

Let $\tilde{\gamma}_{n,k}$ (briefly $\tilde{\gamma}$) be the canonical oriented k-plane bundle over $\tilde{G}_{n,k}$, and let $\gamma_{n,k}$ (briefly γ) be the canonical k-plane bundle over $G_{n,k}$. Of course, the pullback $p^*(\gamma)$ is isomorphic to $\tilde{\gamma}$. By the ith *canonical* Stiefel-Whitney class of $\tilde{G}_{n,k}$ or $G_{n,k}$ we shall mean the ith Stiefel-Whitney class of the corresponding canonical k-plane bundle, that is, the class $w_i(\tilde{\gamma}_{n,k}) \in H^i(\tilde{G}_{n,k};\mathbb{Z}_2)$, briefly denoted by $\tilde{w}_i$, or $w_i(\gamma_{n,k}) \in H^i(G_{n,k};\mathbb{Z}_2)$, briefly denoted by w_i.

We now examine the cup-length of the oriented Grassmann manifolds; we first present some results on the height of cohomology classes. We recall that for a topological space X, the $\mathbb{Z}_2$-height (briefly height), $\mathrm{ht}(y)$, of a cohomology class $y \in H^*(X;\mathbb{Z}_2)$ is defined to be $\sup\{t; y^t \neq 0 \in H^*(X;\mathbb{Z}_2)\}$. For instance, $\mathrm{ht}(w_1(\gamma_{n,k}))$ is known due to R. Stong [13]; for $6 \leq 2k \leq n$ we cite the following: If s is such that $2^s < n \leq 2^{s+1}$, then

$$\mathrm{ht}(w_1(\gamma_{n,k})) = \begin{cases} 2^{s+1}-2 & \text{if } k=3 \text{ and } n = 2^s+1, \\ 2^{s+1}-1 & \text{otherwise.} \end{cases} \tag{3}$$

For $\mathrm{ht}(w_2(\tilde{\gamma}_{n,k}))$ with $k \geq 3$, no general formula is available up to now. Of course, we have

$$\mathrm{ht}(w_2(\tilde{\gamma}_{n,k})) \leq \mathrm{ht}(w_2(\gamma_{n,k})),$$

and so a source supplying some upper bounds for the height of $w_2(\tilde{\gamma}_{n,k})$ is S. Dutta and S. Khare's paper [5], where they calculated the values of $\mathrm{ht}(w_2(\gamma_{n,k}))$. Recently, in [10], we derived another interesting upper bound for $\mathrm{ht}(w_i(\tilde{\gamma}_{n,k}))$: in particular, we have that (for $6 \leq 2k \leq n$) $\mathrm{ht}(w_2(\tilde{\gamma}_{n,k}))$ does not exceed

$$\tilde{\kappa}(n,k) := \begin{cases} \lfloor \frac{(k-1)(n-k)}{2} \rfloor & \text{if } n \text{ is odd,} \\ \lfloor \frac{(k-1)(n-k+1)}{2} \rfloor & \text{if } n \text{ is even.} \end{cases} \tag{4}$$

Here and elsewhere $\lfloor a \rfloor$ denotes the integer part of $a \in \mathbb{R}$. For $k = 3$ the upper bound (4) is mostly better and never worse than the upper bound $\mathrm{ht}(w_2(\gamma_{n,k}))$, while for $k \geq 4$, (4) is sometimes better and sometimes worse.

In this paper, we shall prove that the bound (4) coincides with the actual value of the height for the manifolds $\tilde{G}_{2^t-1,3}$, $t \geq 3$. I am grateful to Tomohiro Fukaya who pointed out, after having read an earlier version of this paper, that my original proof for $\tilde{G}_{2^t-1,3}$ can readily be adjusted to also cover the manifolds $\tilde{G}_{2^t,3}$, $t \geq 3$. The bound (4) turns out to differ a little from the actual value of $\mathrm{ht}(w_2(\tilde{\gamma}_{2^t,3}))$.

Theorem 1.2. *For $t \geq 3$, we have*

$$\mathrm{ht}(w_2(\tilde{\gamma}_{2^t-1,3})) = \mathrm{ht}(w_2(\tilde{\gamma}_{2^t,3})) = 2^t - 4.$$

As regards the cup-length itself, our Theorem 1.1 applies, in particular, to the oriented Grassmann manifolds $\tilde{G}_{n,k}$ with n odd. Indeed, we recall (see the remark after the proof of [3, 24.2]) that each oriented Grassmann manifold is, thanks to the fixed point free involution $D \mapsto -D$ (where the oriented vector subspace $-D$ is obtained from D by reversing its orientation), unorientedly cobordant to zero. If n is odd, then the algebra $H^*(\tilde{G}_{n,k};\mathbb{Z}_2)$ is in dimensions less than or equal to $n-k-1$ generated by the canonical Stiefel-Whitney classes $\tilde{w}_2,\dots,\tilde{w}_k$ or, alternatively, by the classes $w_2(\tilde{G}_{n,k}),\dots,w_k(\tilde{G}_{n,k})$. [By an obvious adjustment of [1, 3.6.2], the class $w_i(\tilde{G}_{n,k})$ $(i \le k < n-k)$ can then be expressed as $w_i(\tilde{G}_{n,k}) = \tilde{w}_i + Q_i(\tilde{w}_2,\dots,\tilde{w}_{i-1})$, where Q_i is a $\mathbb{Z}_2$-polynomial. By induction, $\tilde{w}_i = w_i(\tilde{G}_{n,k}) + P_i(w_2(\tilde{G}_{n,k}),\dots,w_{i-1}(\tilde{G}_{n,k}))$, where P_i is a polynomial, for $i = 2,\dots,k$: we have $\tilde{w}_2 = w_2(\tilde{G}_{n,k})$ (it suffices to use the formula for $w_2(G_{n,k})$ from [1] and the fact that $p^*(\gamma) = \tilde{\gamma}$, $T\tilde{G}_{n,k} \cong p^*(TG_{n,k})$). By what we have said above, we have $\tilde{w}_j = w_j(\tilde{G}_{n,k}) +$ a polynomial in $\tilde{w}_2,\dots,\tilde{w}_{j-1}$ for $j \ge 2$. The induction hypothesis then implies that $\tilde{w}_j = w_j(\tilde{G}_{n,k}) + P_j(w_2(\tilde{G}_{n,k}),\dots,w_{j-1}(\tilde{G}_{n,k}))$ for some polynomial P_j.] As a consequence, if n is odd, then we have $r = 2$ and can take $z = n-k-1$ for $M = \tilde{G}_{n,k}$ in Theorem 1.1 (of course, in relation to Problem 1.1, we have that $\operatorname{charrank}(\tilde{G}_{n,k}) \ge n-k-1$ in this situation), which yields an interesting estimate:

$$\operatorname{cup}(\tilde{G}_{n,k}) \le 1 + \frac{(n-k)(k-1)}{2}.$$

To show its strength, we shall prove, using also the information coming from Theorem 1.2, that this estimate really gives the value of $\operatorname{cup}(\tilde{G}_{2^t-1,3})$. For n even, we obtain a general upper bound for $\operatorname{cup}(\tilde{G}_{n,k})$ implied by [9, Theorem A(b)] using the upper bound (4). This bound is sometimes better and sometimes worse than the upper bound given by [9, Proposition D]. More precisely, we shall prove the following theorem; we exclude $\tilde{G}_{6,3}$, because it is already known that $\operatorname{cup}(\tilde{G}_{6,3}) = 3$ by [9, Proposition D(a)].

Theorem 1.3. (a) *For the oriented Grassmann manifolds $\tilde{G}_{n,k}$ with n odd such that $6 \le 2k \le n$ we have that*

$$\operatorname{cup}(\tilde{G}_{n,3}) \le n-2;$$

in particular, for $t \ge 3$ we have

$$\operatorname{cup}(\tilde{G}_{2^t-1,3}) = 2^t - 3,$$

and for $k \ge 4$ we have that

$$\operatorname{cup}(\tilde{G}_{n,k}) \le \min\{u(n,k), \lfloor 1 + \frac{(n-k)(k-1)}{2} \rfloor\},$$

where

$$u(n,k)=\begin{cases}\lfloor\frac{5\cdot 2^s-13}{3}\rfloor & if\ n=2^s+1, s\geq 3, k=4,\\ 2^{s+1}-4 & if\ n=2^s+2, s\geq 3, k=4,\\ 2^{s+1}-3 & if\ n=2^s+3, s\geq 3, k=4,\\ \lfloor\frac{2^{s+1}+4n-17}{3}\rfloor & if\ 2^s+4\leq n\leq 2^{s+1}, k=4,\\ \lfloor\frac{(k+1)\cdot 2^s+k-k^2-1}{3}\rfloor & if\ n=2^s+1, s\geq 3, k\geq 5,\\ \lfloor\frac{2^{s+1}+kn-k^2-1}{3}\rfloor & if\ 2^s+2\leq n\leq 2^{s+1}, k\geq 5.\end{cases}$$

(b) *For the oriented Grassmann manifolds $\tilde{G}_{n,k}$ with n even such that $6\leq 2k\leq n$, $n\neq 6$, we have that*

$$\operatorname{cup}(\tilde{G}_{n,3})\leq\tilde{\kappa}(n,3)+\lfloor\frac{3(n-3)-2\tilde{\kappa}(n,3)}{3}\rfloor$$

and for $k\geq 4$

$$\operatorname{cup}(\tilde{G}_{n,k})\leq\min\{u(n,k),\tilde{\kappa}(n,k)+\lfloor\frac{k(n-k)-2\tilde{\kappa}(n,k)}{3}\rfloor\}.$$

We note that if n is odd, then the upper bound $\tilde{\kappa}(n,k)+\lfloor\frac{k(n-k)-2\tilde{\kappa}(n,k)}{3}\rfloor$ "mechanically" obtained from [9, Theorem A] is not better than the upper bound $\lfloor 1+\frac{(n-k)(k-1)}{2}\rfloor$ given in Theorem 1.3(a).

Let us come back for a while to our Problem 1.1. On the one hand, we have $\operatorname{cup}(\tilde{G}_{2^t-1,3})=2^t-3$ ($t\geq 3$) by Theorem 1.3, and this of course implies that z in Theorem 1.1 cannot exceed 2^t-5 if $M=\tilde{G}_{2^t-1,3}$. On the other hand, we know that $\operatorname{charrank}(\tilde{G}_{n,k})\geq n-k-1$ if n is odd, and so we see (for $t\geq 3$) that

$$\operatorname{charrank}(\tilde{G}_{2^t-1,3})=2^t-5.$$

The question of what is $\operatorname{charrank}(\tilde{G}_{n,3})$ (more generally: $\operatorname{charrank}(\tilde{G}_{n,k})$) for an arbitrary n odd seems to remain open. The situation is different for $\tilde{G}_{n,k}$ with n even. Indeed, for $\tilde{G}_{n,k}$ (recall that $6\leq 2k\leq n$) we have $H^1(\tilde{G}_{n,k};\mathbb{Z}_2)=0$ (of course, also $w_1(\tilde{G}_{n,k})=0$) and $H^2(\tilde{G}_{n,k};\mathbb{Z}_2)\cong\mathbb{Z}_2$; this together with the fact that (apply [1, Theorem 1.1]) $w_2(\tilde{G}_{n,k})=0$ if n is even implies that $\operatorname{charrank}(\tilde{G}_{n,k})=1$ if n is even.

We close the presentation of our results on the cup-length by the following.

Theorem 1.4. *Given an oriented Grassmann manifold $\tilde{G}_{n,k}$ such that $6\leq 2k\leq n$ and $(n,k)\neq(6,3)$, let t be the largest integer such that $2^t-4\leq n-k$. At the same time, let s be the largest integer and p the least integer such that, under the conditions $p\geq 1$ and $0\leq z\leq 2^{p-1}-2$, we have that $2^s-2^p+z-1\leq n-k$. Then*

$$\operatorname{cup}(\tilde{G}_{n,k})\geq\max\{2^t-3,2^s-3\cdot 2^{p-1},l(n,k)\},$$

where

$$l(n,k) = \begin{cases} \frac{n+3}{2} & if\ n \geq 7\ is\ odd,\ n \notin \{9,11\}, k=3, \\ \frac{n+2}{2} & if\ n \geq 8\ is\ even,\ n \notin \{10,12\}, k=3, \\ 5 & if\ (n,k) \in \{(9,3),(10,3),(11,3),(12,3)\}, \\ \frac{n-k+6}{2} & if\ n-k+3\ is\ odd,\ n-k+3 \notin \{9,11\}, k \geq 4, \\ \frac{n-k+5}{2} & if\ n-k+3\ is\ even,\ n-k+3 \notin \{10,12\}, k \geq 4, \\ 5 & if\ n-k+3 \in \{9,10,11,12\}, k \geq 4. \end{cases}$$

Finally, we add a corollary on the Lyusternik-Shnirel'man category. By Grossman ([6], [7, Proposition 5.1]), we have $\mathrm{cat}(X) \leq 1 + \frac{\dim(X)}{r}$ if X has the homotopy type of an $(r-1)$-connected $(r \geq 1)$ finite CW-complex. For those oriented Grassmann manifolds $\tilde{G}_{n,k}$ $(6 \leq 2k \leq n)$ having even dimension, Grossman's upper bound can be improved by one, and we obtain a wider result.

Corollary 1.1. *For the oriented Grassmann manifold* $\tilde{G}_{n,k}$ *such that* $6 \leq 2k \leq n$ *we have that*

$$1 + \max\{2^t - 3, 2^s - 3 \cdot 2^{p-1}, l(n,k)\} \leq \mathrm{cat}(\tilde{G}_{n,k}) \leq \lceil \frac{k(n-k)}{2} \rceil;$$

the numbers t, s, p, *and* $l(n,k)$ *are specified in Theorem 1.4, and* $\lceil a \rceil$ *is the ceiling value of* $a \in \mathbb{R}$.

In particular, we have (for all $q \geq 3$*) that*

$$2^q - 2 \leq \mathrm{cat}(\tilde{G}_{2^q-1,3}) \leq 3 \cdot 2^{q-1} - 6. \tag{5}$$

Remark 1.1. Work on the cup-length of the oriented Grassmann manifolds has also been done independently by T. Fukaya. The author learned of this work thanks to Mamoru Mimura and Masaki Nakagawa. In a 2006 preprint (Cup-length of $\tilde{G}_{n,3}$), Fukaya presented the results of computer calculations of the cup-length of $\tilde{G}_{n,3}$ for $n = 7, \ldots, 200$. This established that $\mathrm{cat}(\tilde{G}_{7,3}) = 6$. In October 2007, he posted improved results; see arXiv:0710.4033v1 [math.AT] (Application of Gröbner bases to the cup-length of oriented Grassmann manifolds). There are two specific overlaps with this paper. We both calculated the $\mathbb{Z}_2$-cup-length of $\tilde{G}_{2^t-1,3}$ using different methods, and Fukaya's Corollary (op. cit., p. 2) coincides with estimate (5) of our Corollary 1.1.

2 Proofs of the results

2.1 Proof of Theorem 1.1

The cup-length of the manifold M is realized by a nonzero product of maximal degree,

$$\alpha_1 \cdots \alpha_m \cdot \beta_1 \cdots \beta_s \in H^d(M; \mathbb{Z}_2),$$

such that $r \leq \deg(\alpha_i) \leq z$ and $\deg(\beta_j) \geq z+1$. Set $\alpha = \alpha_1 \cdots \alpha_m$ and $\beta = \beta_1 \cdots \beta_s$. Then

$$\begin{aligned} \mathrm{cup}(M) &= m+s \\ &\leq \deg(\alpha)/r + \deg(\beta)/(z+1) \\ &= \deg(\alpha)/r + (d - \deg(\alpha)/(z+1)) \\ &= \frac{(z+1-r)\deg(\alpha) + d\cdot r}{r\cdot(z+1)}. \end{aligned} \tag{6}$$

By the definition of z, it is clear that α is in the subalgebra generated by the Stiefel-Whitney classes of M, and we also know that $\deg(\beta) \geq z+1$, because otherwise the product $\alpha \cdot \beta$ would be a nontrivial characteristic number, contrary to the hypothesis that M is cobordant to 0. Hence we have

$$\deg(\alpha) \leq d - (z+1).$$

This inequality combined with (6) implies that

$$\begin{aligned} \mathrm{cup}(M) &\leq \frac{(z+1-r)(d-(z+1)) + d\cdot r}{r\cdot(z+1)} \\ &= 1 + \frac{d-(z+1)}{r}. \end{aligned}$$

The proof of Theorem 1.1 is finished.

2.2 An auxiliary result on $H^*(\tilde{G}_{n,k};\mathbb{Z}_2)$

We identify (see, e.g., [2]) the $\mathbb{Z}_2$-cohomology ring $H^*(G_{n,k};\mathbb{Z}_2)$ with

$$\mathbb{Z}_2[w_1,\ldots,w_k]/I_{n,k},$$

where $I_{n,k}$ is the ideal generated by the dual Stiefel-Whitney classes

$$\bar{w}_{n-k+1}(\gamma),\ldots,\bar{w}_n(\gamma).$$

In other words, $I_{n,k}$ is generated by the homogeneous components of

$$\frac{1}{1+w_1+\cdots+w_k} = 1 + w_1 + \cdots + w_k + (w_1+\cdots+w_k)^2 + \ldots$$

in dimensions $n-k+1,\ldots,n$.

At the same time, we denote by $g_i(w_2(\gamma),\ldots,w_k(\gamma))$ or briefly g_i, for $i = n-k+1,\ldots,n$, the i-dimensional homogeneous component of

$$\frac{1}{1+w_2+\cdots+w_k} = 1 + w_2 + \cdots + w_k + (w_2+\cdots+w_k)^2 + \ldots.$$

So we can pass from $\bar{w}_i(\gamma)$ to g_i by reducing it modulo w_1. Let $J_{n,k}$ denote the ideal of $\mathbb{Z}_2[w_2,\ldots,w_k]$ generated by $g_{n-k+1},\ldots,g_n$.

The Gysin exact sequence associated with the double covering projection $p : \tilde{G}_{n,k} \to G_{n,k}$ (see, e.g., [11, 12.3]) implies that $\tilde{w}_2^{i_2} \cdots \tilde{w}_k^{i_k} = p^*(w_2^{i_2} \cdots w_k^{i_k})$ (where $p^* : H^*(G_{n,k};\mathbb{Z}_2) \to H^*(\tilde{G}_{n,k};\mathbb{Z}_2)$ is the induced cohomology homomorphism) does not vanish if and only if $w_2^{i_2} \cdots w_k^{i_k} \in H^{2i_2+\cdots+ki_k}(G_{n,k};\mathbb{Z}_2)$ cannot be represented as a multiple of w_1. By the definitions of the ideals $I_{n,k}$ and $J_{n,k}$, it is clear that $w_2^{i_2} \cdots w_k^{i_k} \in H^{2i_2+\cdots+ki_k}(G_{n,k};\mathbb{Z}_2)$ is a multiple of w_1 if and only if $w_2^{i_2} \cdots w_k^{i_k} \in J_{n,k}$. Thus we have verified the following.

Lemma 2.1. *The cohomology class $\tilde{w}_2^{i_2} \cdots \tilde{w}_k^{i_k} \in H^*(\tilde{G}_{n,k};\mathbb{Z}_2)$ does not vanish if and only if $w_2^{i_2} \cdots w_k^{i_k}$ is not a multiple of w_1, hence if and only if $w_2^{i_2} \cdots w_k^{i_k}$ does not belong to the ideal $J_{n,k}$.*

2.3 Proof of Theorem 1.2

Our upper bound (4), applied to $\tilde{G}_{2^t-1,3}$, immediately gives that the height of $\tilde{w}_2$ does not exceed $2^t - 4$ (note that, in particular, we have $\tilde{w}_2^{2^t-3} = 0$). We wish to prove that $2^t - 4$ is also a lower bound; for this it suffices to show that $\tilde{w}_2^{2^t-4} \neq 0$.

We know by [9, (3.3), p. 2982] that the ideal $J_{2^t,3}$ is generated by

$$g_\kappa(w_2(\gamma_{2^t,3}), w_3(\gamma_{2^t,3})) := \sum_{\frac{\kappa}{3} \le i \le \frac{\kappa}{2}} \binom{i}{3i-\kappa} w_2(\gamma_{2^t,3})^{3i-\kappa} w_3(\gamma_{2^t,3})^{\kappa-2i}, \tag{7}$$

$\kappa = 2^t - 2, 2^t - 1, 2^t$.

By the same [9, (3.3)], the ideal $J_{2^t-1,3}$ is generated by

$$\begin{aligned} &g_\kappa(w_2(\gamma_{2^t-1,3}), w_3(\gamma_{2^t-1,3})) \\ &:= \sum_{\frac{\kappa}{3} \le i \le \frac{\kappa}{2}} \binom{i}{3i-\kappa} w_2(\gamma_{2^t-1,3})^{3i-\kappa} w_3(\gamma_{2^t-1,3})^{\kappa-2i}, \end{aligned} \tag{8}$$

$\kappa = 2^t - 3, 2^t - 2, 2^t - 1$.

We shall show, a little later, that

$$g_{2^t-3}(w_2(\gamma_{2^t-1,3}), w_3(\gamma_{2^t-1,3})) = 0. \tag{9}$$

Taking (9) for granted, we prove that $\tilde{w}_2(\gamma_{2^t-1,3})^{2^t-4}$ does not vanish by showing (see Lemma 2.1) that $w_2(\gamma_{2^t-1,3})^{2^t-4}$ is not in the ideal $J_{2^t-1,3}$. Indeed, suppose that $w_2(\gamma_{2^t-1,3})^{2^t-4}$ is in $J_{2^t-1,3}$. Then there are some $\mathbb{Z}_2$-polynomials $a(x,y)$ and $b(x,y)$ such that $w_2(\gamma_{2^t-1,3})^{2^t-4}$ is equal to

$$\begin{aligned} &a(w_2(\gamma_{2^t-1,3}), w_3(\gamma_{2^t-1,3})) \cdot g_{2^t-2}(w_2(\gamma_{2^t-1,3}), w_3(\gamma_{2^t-1,3})) \\ +\; &b(w_2(\gamma_{2^t-1,3}), w_3(\gamma_{2^t-1,3})) \cdot g_{2^t-1}(w_2(\gamma_{2^t-1,3}), w_3(\gamma_{2^t-1,3})). \end{aligned}$$

But (see (7) and (8)) since the generators

$$g_{2^t-2}(w_2(\gamma_{2^t-1,3}), w_3(\gamma_{2^t-1,3})) \text{ and } g_{2^t-2}(w_2(\gamma_{2^t,3}), w_3(\gamma_{2^t,3}))$$

are "the same", and also the generators

$$g_{2^t-1}(w_2(\gamma_{2^t-1,3}), w_3(\gamma_{2^t-1,3})) \text{ and } g_{2^t-1}(w_2(\gamma_{2^t,3}), w_3(\gamma_{2^t,3}))$$

are "the same", then we must also have

$$\begin{aligned} w_2(\gamma_{2^t,3})^{2^t-4} &= a(w_2(\gamma_{2^t,3}), w_3(\gamma_{2^t,3})) \cdot g_{2^t-2}(w_2(\gamma_{2^t,3}), w_3(\gamma_{2^t,3})) \\ &+ b(w_2(\gamma_{2^t,3}), w_3(\gamma_{2^t,3})) \cdot g_{2^t-1}(w_2(\gamma_{2^t,3}), w_3(\gamma_{2^t,3})). \end{aligned}$$

This means that $w_2(\gamma_{2^t,3})^{2^t-4}$ is in $J_{2^t,3}$, and so (by Lemma 2.1) $w_2(\gamma_{2^t,3})^{2^t-4}$ is a multiple of $w_1(\gamma_{2^t,3})$. But the latter is impossible, because it is known, thanks to Stong [13, p. 104], that $w_1(\gamma_{2^t,3})^{2^t-1} w_2(\gamma_{2^t,3})^{2^t-4}$ does not vanish, and by [13, p. 103] (see (3) in Sec. 1) the height of the class $w_1(\gamma_{2^t,3})$ is $2^t - 1$.

Of course, we also proved that $w_2(\tilde{\gamma}_{2^t,3})^{2^t-4} \neq 0$. Since we know (see the very beginning of this proof and Lemma 2.1) that $w_2(\gamma_{2^t-1,3})^{2^t-3}$ is in $J_{2^t-1,3}$, it is clear, by what we have seen above, that $w_2(\gamma_{2^t,3})^{2^t-3}$ is in $J_{2^t,3}$; therefore we have $w_2(\tilde{\gamma}_{2^t,3})^{2^t-3} = 0$.

It remains to show that, for $t \geq 3$, we have

$$g_{2^t-3}(w_2, w_3) = 0.$$

In the proof, we shall use, among other things, the well known Lucas' theorem saying that if a has the dyadic expansion $\sum_{i=0}^m a_i 2^i$ and b has the dyadic expansion $\sum_{i=0}^m b_i 2^i$, then

$$\binom{a}{b} \equiv \binom{a_0}{b_0}\binom{a_1}{b_1} \cdots \binom{a_m}{b_m} \pmod 2.$$

By this, e.g., one immediately sees that each binomial coefficient of the form $\binom{\text{even}}{\text{odd}}$ is even, and if a and b are even, then

$$\binom{a}{b} \equiv \binom{a \cdot 2^{-1}}{b \cdot 2^{-1}} \pmod 2. \tag{10}$$

We also recall Pascal's identity

$$\binom{a+1}{b+1} = \binom{a}{b} + \binom{a}{b+1}. \tag{11}$$

To prove that $g_{2^t-3}(w_2, w_3) = 0$ for $t \geq 3$, we first observe that, by the formula (8) cited above, we have

$$g_{2^t-3} = \begin{cases} \displaystyle\sum_{\frac{2^t-1}{3} \leq i \leq 2^{t-1}-2} \binom{i}{3i-2^t+3} w_2^{3i-2^t+3} w_3^{2^t-3-2i} & \text{if } t \text{ is even,} \\ \displaystyle\sum_{\frac{2^t-2}{3} \leq i \leq 2^{t-1}-2} \binom{i}{3i-2^t+3} w_2^{3i-2^t+3} w_3^{2^t-3-2i} & \text{if } t \text{ is odd.} \end{cases} \tag{12}$$

One readily proves (e.g., by induction) that the integer $3^{-1} \cdot (2^t - 1)$ is of the form $4s + 1$ if t is even, and if t is odd, then the integer $3^{-1} \cdot (2^t - 2)$ is of course of the form $8z + 2$.

Let us first suppose that $t = 2r$, $r \geq 2$. The coefficients in the expression (12) for $g_{2^{2r}-3}$ are now of the form

$$\binom{3^{-1} \cdot (2^{2r} - 1) + y}{3y + 2}, \tag{13}$$

where $y \geq 0$ (as always, we have $\binom{a}{b} = 0$ if $a < b$). Writing $3^{-1} \cdot (2^{2r} - 1) = 4s + 1$, we show, by induction on r, that all the coefficients (13) are even.

First, one directly calculates that they are even for $r = 2$. It is also clear, for any r, that the coefficient $\binom{3^{-1}\cdot(2^{2r}-1)+y}{3y+2}$ is even if y is odd. If $y = 4k$ for some k, then the coefficient (13) is $\binom{4(s+k)+1}{12k+2}$, hence obviously even by Lucas' theorem. Finally, let us suppose that $x = 4k + 2$ for some k. So the coefficient (13) is $\binom{4(s+k)+3}{12k+8}$ and, using Pascal's identity (11), one readily sees that

$$\binom{4(s+k)+3}{12k+8} \equiv \binom{4(s+k)+2}{12k+8} \pmod 2.$$

Then (see (10))

$$\binom{4(s+k)+2}{12k+8} \equiv \binom{2(s+k)+1}{6k+4} \pmod 2.$$

Having this, using (11) and (10) again, we obtain that

$$\binom{2(s+k)+1}{6k+4} \equiv \binom{s+k}{3k+2} \pmod 2,$$

hence

$$\binom{2(s+k)+1}{6k+4} \equiv \binom{3^{-1} \cdot (2^{2r-2} - 1) + k}{3k+2} \pmod 2.$$

So we see that

$$\binom{3^{-1} \cdot (2^{2r} - 1) + y}{3y+2} \equiv \binom{3^{-1} \cdot (2^{2r-2} - 1) + k}{3k+2} \pmod 2.$$

Of course, the coefficient on the right is even by the induction hypothesis, and so we proved that the coefficient (13) is even for any even $t \geq 4$.

For t odd, the proof is similar. Indeed, let us write $t = 2r + 1$, $r \geq 1$. The binomial coefficients in the expression (12) for $g_{2^{2r+1}-3}$ are now of the form

$$\binom{3^{-1} \cdot (2^{2r+1} - 2) + x}{3x + 1}, \tag{14}$$

where $x \geq 0$, and we show, by induction on r, that all of them are even. Let us write $3^{-1} \cdot (2^{2r+1} - 2) = 8z + 2$. The claim is obviously true for $r = 1$. For any r, the coefficient (14) is clearly even if x is even. If $x = 4k + 3$ for some k, then the coefficient (14) is $\binom{4(2z+k)+5}{12k+10}$, hence an even number by Lucas' theorem. Finally, let us suppose that $y = 4k + 1$ for some k. So the coefficient (14) is now $\binom{4(2z+k)+3}{12k+4}$. Using (11) and (10) similarly to the case of t even, we obtain that

$$\binom{3^{-1} \cdot (2^{2r+1} - 2) + x}{3x + 1} \equiv \binom{3^{-1} \cdot (2^{2r-1} - 2) + k}{3k + 1} \pmod 2.$$

Again, the coefficient on the right hand side is even by the induction hypothesis, and so we proved that the coefficient under question is also even for any odd $t \geq 3$.

We have verified that

$$g_{2^t-3}(w_2, w_3) = 0$$

for any $t \geq 3$; this proves that $\mathrm{ht}(w_2(\tilde{\gamma}_{2^t-1,3})) = \mathrm{ht}(w_2(\tilde{\gamma}_{2^t,3})) = 2^t - 4$.

2.4 Proof of Theorem 1.3

Proof of Theorem 1.3(a). The upper bound $1 + \dfrac{(n-k)(k-1)}{2}$ in Theorem 1.3(a) is a special case of Theorem 1.1 (where we have $r = 2$ and $z = n - k - 1$). To complete the proof of Theorem 1.3(a), we first note that for $\tilde{G}_{n,3}$ the upper bound $n - 2$ implied by Theorem 1.1 is always better than the upper bound given in [9, Proposition D(b)]. In particular, our upper bound for $\mathrm{cup}(\tilde{G}_{2^t-1,3})$, which is $2^t - 3$, coincides with the lower bound implied by the fact that $\tilde{w}_2^{2^t-4}$ is nonzero (see Theorem 1.2) and Poincaré duality (by this, there is some y such that $w_2^{2^t-4}y \neq 0 \in H^{\mathrm{top}}(\tilde{G}_{2^t-1,3}; \mathbb{Z}_2)$).

For $\tilde{G}_{n,k}$ with $k \geq 4$, the upper bound $u(n,k)$ coming from [9, Proposition D(b)] is sometimes worse and sometimes better than the upper bound given by Theorem 1.1. That is why we take the minimum in Theorem 1.3(a). The proof of Theorem 1.3(a) is finished.

Proof of Theorem 1.3(b). Now we have n even. By [9, Theorem D(b)], we know that $u(n,k)$ is an upper bound. Using the formula (b1) of [9, Theorem A(b)], taking $r = 2$, $q = 3$, and $k_1 = \tilde{\kappa}(n,k)$ there, we also obtain

$$\tilde{\kappa}(n,k) + \left\lfloor \frac{k(n-k) - 2\tilde{\kappa}(n,k)}{3} \right\rfloor$$

as an upper bound; for $\tilde{G}_{n,3}$, one directly checks that this upper bound is always better than $u(n,k)$. This finishes the proof of Theorem 1.3.

2.5 Proof of Theorem 1.4

Let $j : \tilde{G}_{a,b} \to \tilde{G}_{a+1,b+1}$ be the standard inclusion such that $j^*(\tilde{\gamma}_{a+1,b+1}) = \tilde{\gamma}_{a,b} \oplus \varepsilon^1$, where ε^1 denotes the trivial line bundle; so we have $j^*(w_2(\tilde{\gamma}_{a+1,b+1})) = w_2(\tilde{\gamma}_{a,b})$. In addition to this, we have the standard inclusion $i : \tilde{G}_{n-1,q} \to \tilde{G}_{n,q}$ such that $i^*(\tilde{\gamma}_{n,q}) = \tilde{\gamma}_{n-1,q}$. Suitably composing such inclusions j or i repeatedly, we obtain, for t such that $2^t - 4 \leq n - k$, an inclusion $\iota : \tilde{G}_{2^t-1,3} \to \tilde{G}_{n,k}$ such that

$\iota^*(w_2(\tilde{\gamma}_{n,k})) = w_2(\tilde{\gamma}_{2^t-1,3})$. By Theorem 1.2, we have $w_2(\tilde{\gamma}_{2^t-1,3})^{2^t-4} \neq 0$, hence also $w_2(\tilde{\gamma}_{n,k})^{2^t-4} \neq 0$ and, using Poincaré duality, we obtain that

$$\mathrm{cup}(\tilde{G}_{n,k}) \geq 2^t - 3.$$

In addition to this, by Stong [13, p. 104] we know (cf. the proof of Theorem 1.2 for a similar case) that the power $w_2(\gamma_{2^s-2^p+2+z,3})^{2^s-3\cdot 2^{p-1}-1}$ is not a multiple of $w_1(\gamma)$, hence we have

$$w_2(\tilde{\gamma}_{2^s-2^p+2+z,3})^{2^s-3\cdot 2^{p-1}-1} \neq 0.$$

Using an inclusion argument as above, we conclude that

$$\mathrm{cup}(\tilde{G}_{n,k}) \geq 2^s - 3 \cdot 2^{p-1}$$

if s, z, and p are as in the statement of the theorem.

Finally, the lower bounds $l(n,k)$ come from [9, Proposition B]. This finishes the proof of Theorem 1.4.

2.6 Proof of Corollary 1.1

In view of the Elsholz inequality (1), the lower bound stated in the corollary is implied by Theorem 1.4. The upper bound is implied by the following theorem due to Berstein and Švarc (see [14, Theorem 20] or [7, Proposition 5.3]).

Theorem 2.1 (I. Berstein - A. S. Švarc)**.** *Let X be a finite $(r-1)$-connected CW-complex, where $r \geq 2$, and suppose that* $\dim(X) \leq rm$*, where $m \geq 1$. Let $\varphi_r(X) \in \tilde{H}^r(X;\pi_r(X))$ be the fundamental class of X. Then* $\mathrm{cat}(X) \leq m$ *if and only if the m-fold cup-product $\varphi_r(X) \cup \cdots \cup \varphi_r(X) = 0$.*

Applying this theorem to $\tilde{G}_{n,k}$ $(6 \leq 2k \leq n)$, we have $r = 2$, $m = \lceil \frac{k(n-k)}{2} \rceil$, $\pi_2(\tilde{G}_{n,k}) \cong \mathbb{Z}_2$ (see, e.g., Steenrod [12, 25.8]), and

$$\tilde{H}^2(\tilde{G}_{n,k};\pi_2(\tilde{G}_{n,k})) = \{0, \tilde{w}_2\} \cong \mathbb{Z}_2.$$

Hence the fundamental class $\varphi_2(\tilde{G}_{n,k})$ can be identified with the canonical Stiefel-Whitney class $\tilde{w}_2$. Our (4) of course implies that

$$\tilde{w}_2^{\lceil \frac{k(n-k)}{2} \rceil} = 0,$$

and we obtain that

$$\mathrm{cat}(\tilde{G}_{n,k})) \leq \lceil \frac{k(n-k)}{2} \rceil.$$

This finishes the proof of the corollary.

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