# On secant spaces to Enriques surfaces

Andreas Leopold Knutsen

**Abstract**

We relate the minimal gonality of smooth curves in a complete, ample, base point free linear system $|L|$ on an Enriques surface to the existence of certain secant spaces on the image of the surface mapped by the adjoint system. We also explicitly compute the minimal gonality in terms of invariants of the line bundle $L$. In particular, we obtain a precise criterion for the variation of the gonality of the curves.

## 1 Introduction

The purpose of this note is to study the relation between the minimal gonality of smooth curves in a complete linear system on an Enriques surface and the embedding properties of the adjoint linear system, as well as study the variation of the gonality in the linear system.

A line bundle $\mathcal{L}$ on a smooth, irreducible projective variety $X$ is said to be $k$-very ample, for an integer $k \geq 0$, if its sections separate subschemes of length $k+1$, i.e. if the natural restriction map $H^0(\mathcal{L}) \longrightarrow H^0(\mathcal{L} \otimes \mathcal{O}_Z)$ is surjective for any 0-dimensional subscheme $Z$ of $X$ of length $h^0(\mathcal{O}_Z) = k+1$. Note that $\mathcal{L}$ is 0-very ample if and only if it is generated by its global sections, and $\mathcal{L}$ is 1-very ample if and only if it is very ample. In general, if $\mathcal{L}$ is very ample and embeds $X$ in $\mathbb{P}^{h^0(\mathcal{L})-1}$, then $\mathcal{L}$ is $k$-very ample if and only if (the image of) $X$ has no $(k+1)$-secant $(k-1)$-planes. We refer to [BFS, BS1, BS2, BS3, BS4, Ba-So, C-G, Te] for some of the results developed on the subject on surfaces.

In [Kn3] we introduced the notion of birational $k$-very ampleness: A line bundle $\mathcal{L}$ is said to be *birationally $k$-very ample* if there exists a Zariski-open, dense

2000 *Mathematics Subject Classification :* Primary 14H51, 14C20, 14J28. Secondary 14J05, 14F17.

*Key words and phrases :* Enriques surfaces, line bundles, curves, gonality, higher order embeddings.

subset $\mathcal{U}$ of $X$ such that the restriction map $H^0(\mathcal{L}) \longrightarrow H^0(\mathcal{L} \otimes \mathcal{O}_Z)$ is surjective for any 0-dimensional subscheme $Z$ of $X$ of length $h^0(\mathcal{O}_Z) = k+1$ with $\mathrm{Supp}(Z) \subset \mathcal{U}$. If $\mathcal{L}$ is base point free, then $\mathcal{L}$ is birationally 1-very ample if and only if the morphism $\varphi_{\mathcal{L}}$ associated to $\mathcal{L}$ is birational. Furthermore, we showed in [Kn3, Kn1] that if $L$ is a globally generated line bundle and $X = S$ is a $K3$ or del Pezzo surface, then $L + K_S$ is birationally $k$-very ample if and only if all the smooth curves in $|L|$ have gonalities $\geq k+2$. Recall that the gonality of a smooth curve $C$, $\mathrm{gon}\, C$, is defined as the minimal integer $k$ such that $C$ carries a $g^1_k$. Since on a smooth curve $C$ it follows from Riemann-Roch and Serre duality that $\omega_C$ is $k$-very ample if and only if $\mathrm{gon}\, C \geq k+2$, the results in [Kn3, Kn1] can be seen as an attempt to "lift" this result to a surface.

In this note we show that the result above holds true on Enriques surfaces as well, under the additional assumption that $L$ is ample and $L^2 \geq 10$. In fact we prove a more precise result, relating the existence of curves with certain gonality to the existence of secant spaces on the adjointly embedded surface lying outside of curves of low degree.

To state the result, recall that a *nodal curve* $R$ on an Enriques surface is a smooth irreducible rational curve (whence with $R^2 = -2$) and a *halfpencil* is a reduced curve $E$ (not necessarily irreducible) such that $|2E|$ is an elliptic pencil (whence with $E^2 = 0$). Now we define, for any integer $s > 0$, and any big and nef line bundle $L$ on $S$,

$$\begin{aligned}\Theta_s(L) \;:=\; \{x \in S \mid\; & x \in R, \text{ with } R \text{ a nodal curve such that } R.L \leq s-2 \\ & \text{or } x \in E, \text{ with } E \text{ a halfpencil such that } E.L \leq s.\}\end{aligned} \tag{1}$$

Since the curves satisfying the conditions above are finitely many, $\Theta_s(L)$ is a proper, closed subset of $S$.

**Theorem 1.1.** *Let $L$ be an ample, globally generated line bundle on an Enriques surface $S$ such that $L^2 \geq 10$ and $k \geq 1$ an integer. Then the following conditions are equivalent:*

*(i)* *$L + K_S$ is birationally $k$-very ample.*

*(ii)* *The natural restriction map*

$$H^0(L+K_S) \longrightarrow H^0((L+K_S) \otimes \mathcal{O}_Z). \tag{2}$$

*is surjective for all 0-dimensional subschemes $Z \subset S$ of length $\leq k+1$ satisfying $Z \cap \Theta_{k+1}(L) = \emptyset$.*

*(iii)* *$\varphi_{L+K_S}$ is birational and (if $k \geq 2$) $\varphi_{L+K_S}(S)$ has no $(k+1)$-secant $(k-1)$-plane $\Pi$ such that $\Pi \cap \varphi_{L+K_S}(\Theta_{k+1}(L)) = \emptyset$.*

*(iv)* *All the smooth curves in $|L|$ have gonalities $\geq k+2$.*

Note that by the ampleness assumption on $L$ we have that $\varphi_{L+K_S}(S)$ is smooth if and only if $\phi(L) \geq 3$, and if $\phi(L) = 2$, then $\mathrm{Sing}\, \varphi_{L+K_S}(S) = \varphi_{L+K_S}(\Theta_2(L))$, cf. [Co1, Thm. 5.1] or [CD, Thm. 4.6.1, Lemma 4.6.1 and Thm. p. 281]. Recall from [CD] that the function $\phi : \mathrm{Pic}\, S \to \mathbb{Z}$ is defined as

$$\phi(L) = \inf\{|E.L| \;:\; E \in \mathrm{Pic}\, S,\; E^2 = 0\; E \not\equiv 0\}.$$

We also show that one can explicitly compute the minimal integer such that the equivalent conditions (i)-(iv) in Theorem 1.1 are not satisfied, or equivalently, the minimal gonality of smooth curves in $|L|$. Since we computed the gonality of the general curve in a complete linear system on an Enriques surface in [KL2], the results in this note complete the picture, at least for ample line bundles.

The particular question of which gonalities can occur for curves on special surfaces has also been studied widely throughout the years, cf. [SD, R, DM, GL, CP] for $K3$ surfaces, [Ma] for Hirzebruch surfaces, [Par, Kn2] for del Pezzo surfaces and [Ha] for elliptic ruled surfaces. It seems difficult to find results on other surfaces. Interestingly enough, not many examples are known where the gonality varies in a complete linear system.

To state our result, define, for any $L \in \operatorname{Pic} S$ with $L^2 > 0$ and $h^0(L) > 0$,

$$\beta(L) := \min\{B.L - 2 \mid B \in \operatorname{Pic} S \text{ with } B > 0, \text{ and } B^2 = 2\}$$

and

$$\mu(L) = \min\{B.L - 2 \mid B \in \operatorname{Pic} S \text{ with } B > 0,\ B^2 = 4,\ \phi(B) = 2 \text{ and } B \not\equiv L\},$$

the function defined in [KL2, Def. 1.2].

**Proposition 1.2.** *Let $L$ be an ample, globally generated line bundle on an Enriques surface $S$ such that $L^2 \geq 10$.*

*Let $d_{min}$ be the minimal gonality of a smooth curve in $|L|$.*

*If $(L^2, \phi(L)) = (10, 3)$, then $d_{min} = 3$ if $L \sim 2E + \Delta + K_S$, with $E$ a halfpencil and $\Delta$ a nodal cycle such that $\Delta.E = 3$, and $d_{min} = 4$ otherwise.*

*If $(L^2, \phi(L)) = (12, 2)$, then $d_{min} = 4$.*

*In all other cases,*

$$d_{min} = \min\{2\phi(L), \beta(L), \mu(L) - 2\}. \tag{3}$$

In particular, except for the one exceptional case $(L^2, \phi(L)) = (10, 3)$ above, the minimal gonality of smooth curves in $|L|$ and $|L + K_S|$ is the same.

We will also show that the right hand side in (3) can be computed in the following, more explicit way: Pick any $E$ such that $E^2 = 0$ and $E.L = \phi(L)$ and define, for any integer $i \geq 1$,

$$\phi_i(L, E) := \min\{F.L \mid F^2 = 0,\ F.E = i\}. \tag{4}$$

Then, for any $L \in \operatorname{Pic} S$ with $L^2 \geq 8$ and $h^0(L) > 0$, we have

$$\min\{2\phi(L), \beta(L), \mu(L) - 2\} = \phi(L) + \min\{\phi(L), \phi_1(L, E) - 2, \phi_2(L, E) - 4\}. \tag{5}$$

(In particular, this means - a posteriori - that the right hand side of (5) is independent of the choice of $E$.)

In [KL2] we computed the gonality $d_{gen}$ of a *general* curve in $|L|$. One can also state the results therein with the functions $\phi_i(L, E)$ so that we now have a precise description of both the general and minimal gonality of the smooth curves in $|L|$ (under the assumptions that $L$ is ample with $L^2 \geq 10$). This is given in Proposition

6.2 below. In particular, we obtain a precise criterion for the constancy of the gonality of smooth curves in $|L|$ in Corollary 6.4.

The note is organized as follows: In Section 2 we first gather some well-known results on Enriques surfaces that will be needed throughout and then we give a couple of results concerning particular decompositions of line bundles on Enriques surfaces. In Section 3 we show how these decompositions can be used to prove the existence of curves in $|L|$ of "low" gonalities. The principle, given in Lemma 3.1, is valid for any surface. In Section 4 we go through the well-known vector bundle methods that are used to treat problems of this kind and formulate a couple of results in the setting we need. Then, in Section 5, we prove Theorem 1.1 and Proposition 1.2, as well as (5). Finally, in Section 6, we first prove Proposition 6.2, that explicitly shows how the general and minimal gonality of the smooth curves in a complete linear system can be computed, and then a criterion for the constancy of the gonality in Corollary 6.4. We also give some examples showing how the gonality behaves.

**Remark 1.3.** We do not know if the assumptions that $L$ be ample and $L^2 \geq 10$ in the results above are necessary. But for sure, removing these assumptions would force us to treat very many special cases in the various proofs. The cases $(L^2, \phi(L)) = (10, 3)$ and $(12, 2)$ for instance already have proofs of their own. To keep the note of a reasonable length and to stay within the scope of it, we have decided not to try to weaken the hypotheses.

Many results are however true without the ampleness assumptions: Lemmas 2.3, 2.4(a)-(c), 3.1 and 3.2 are stated in general. In particular, the latter says that (iii) $\Leftrightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iv) in Theorem 1.1 hold even without the ampleness assumption on $L$ and only assuming $L^2 > 0$. It is also possible to obtain a version of Lemma 2.4(d) without the ampleness assumption, but with several exceptional cases.

The ampleness assumption first enters the picture in a crucial way in the proof of Proposition 3.3.

**Acknowledgements.** I thank the referee for several useful suggestions and remarks.

## 2 Decompositions of line bundles on Enriques surfaces

Let $S$ be a smooth, projective Enriques surface, that is, a smooth projective surface satisfying $h^1(\mathcal{O}_S) = 0$, $K_S \neq 0$ and $2K_S = 0$.

Note that if $D \geq 0$ is an effective divisor on $S$, then $h^2(D) = h^0(K_S - D) = 0$ by Serre duality. This will be used without further mentioning, as well as the consequence $h^0(D) = \frac{1}{2}D^2 + 1 + h^1(D)$ from Riemann-Roch.

A nonzero, effective divisor $E$ on $S$ is called *isotropic* if $E^2 = 0$ and in addition *primitive* if $E$ is not divisible in $\mathrm{Num}(S)$.

Any Enriques surface carries elliptic pencils [CD, Cor. 3.21], and any such pencil $|P|$ has two multiple fibers, $2E$ and $2(E + K_S)$, and $E$ and $E + K_S$ are called *halfpencils*. They are necessarily nef and primitive (isotropic). Conversely, any nef, primitive isotropic $E$ is a halfpencil, that is $|P| = |2E|$ is an elliptic pencil

[CD, Prop. 3.1.2 and Chp. 5, §3-4]. A divisor that is nef, primitive and isotropic is also called *primitive of canonical type* [Co1, (1.6.2.1)].

A *nodal curve* on $S$ is a smooth rational curve, or, equivalently by the genus formula, an irreducible curve with $R^2 = -2$. By Riemann-Roch it satisfies $h^0(R) = h^1(R) = 1$ and $h^2(R) = 0$, and by connectedness $h^i(R + K_S) = 0$ for $i = 0, 1, 2$. A *nodal cycle* is an effective divisor $\Delta > 0$ such that $\Delta'^2 \leq -2$ for any $0 < \Delta' \leq \Delta$. If $\Delta^2 = -2$, then Riemann-Roch implies $h^0(\Delta) = h^1(\Delta) = 1$ and $h^2(\Delta) = 0$, and Ramanujam's theorem on 1-connectedness implies that $h^i(\Delta + K_S) = 0$ for $i = 0, 1, 2$.

Let $L$ be a line bundle on an Enriques surface $S$. We will use the notation $L \geq 0$ to mean $h^0(L) > 0$ and $L > 0$ if in addition $L \not\sim \mathcal{O}_S$.

The $\phi$-function mentioned in the introduction has the following important properties that we will use throughout often without further mentioning:

(I) $\phi(L)^2 \leq L^2$ ([CD, Cor.2.7.1]).

(II) If $L$ is nef, then there exists a genus one pencil $|2E|$ such that $E.L = \phi(L)$. In particular, such an $E$ is nef. ([Co2, 2.11] or [CD, Cor.2.7.1, Prop.2.7.1 and Thm.3.2.1]).

(III) If $L$ is ample, then any $E$ such that $E^2 = 0$ and $E.L = \phi(L)$ is necessarily nef (left to the reader).

(IV) If $L$ is nef with $L^2 > 0$, then $|L|$ is base point free if and only if $\phi(L) \geq 2$. Moreover, if $\phi(L) = 1$, then the base scheme of $|L|$ consists of two distinct points, unless $L \sim 2E + R$, with $E$ a halfpencil and $R$ a nodal curve such that $E.R = 1$, in which case $R$ is the base scheme of $|L|$ ([CD, Prop. 3.1.6 and Thm.4.4.1]).

We will also constantly use the following fact: If $L$ is nef with $L^2 > 0$, then $h^i(L) = h^i(L + K_S) = 0$ by Kawamata-Viehweg vanishing, so that $h^0(L) = h^0(L + K_S) = \frac{1}{2}L^2 + 1$ by Riemann-Roch. Moreover, if in addition $L^2 > 2$, then the general member of $|L|$ is smooth and irreducible (by (IV) and Bertini's theorem, or [CD, Prop. 3.1.6 and Thm.4.10.2]).

We will also need the following strengthening of (I):

**Proposition 2.1.** *[KL2, Prop. 1] Let $L$ be a line bundle on an Enriques surface with $L > 0$ and $L^2 > 0$. If $L^2 \leq \phi(L)^2 + \phi(L) - 2$, then there exist primitive divisors $E_i$ with $E_i > 0$, $E_i^2 = 0$, for $i = 1, 2, 3$, $E_1.E_2 = E_1.E_3 = 2$, $E_2.E_3 = 1$ and an integer $h \geq 1$ so that one of the two following occurs:*

*(i)* $L^2 = \phi(L)^2$. *In this case* $L \equiv h(E_1 + E_2)$.

*(ii)* $L^2 = \phi(L)^2 + \phi(L) - 2$. *In this case either*

*(ii-a)* $L \sim h(E_1 + E_2) + E_3$; *or*

*(ii-b)* $L \sim (h + 1)E_1 + hE_2 + E_3$; *or*

*(ii-c)* $L \equiv 2(E_1 + E_2 + E_3)$ *(whence* $L^2 = 40$ *and* $\phi(L) = 6$*).*

A central tool for us will be to find suitable decompositions of line bundles $L$ on $S$ into effective classes. In particular, we will repeatedly use the following elementary fact that is an immediate consequence of the signature theorem [BPV, VIII.1]:

**Lemma 2.2.** *[KL2, Lemma 2.4] Let $S$ be an Enriques surface and $L$ be a line bundle on $S$ such that $L > 0$ and $L^2 \geq 0$. Let $F > 0$ be a divisor on $S$ such that $F^2 = 0$ and $\phi(L) = |F.L|$. Then $F.L > 0$ and if $\alpha > 0$ is such that $(L - \alpha F)^2 \geq 0$, then $L - \alpha F > 0$.*

This lemma will be used together with (I) and Proposition 2.1 to write effective decompositions.

**Lemma 2.3.** *Let $L$ be a nef line bundle on an Enriques surface such that $\phi(L) \geq 2$, $L^2 \geq 10$ and $(L^2, \phi(L)) \neq (10, 3)$. Set $k := \lfloor \frac{L^2}{4} \rfloor$. Then there is a decomposition $L \sim M + N$ such that $h^0(M) \geq 2$, $h^0(N) \geq 2$ and $M.N \leq k + 1$.*

*Proof.* We have $L^2 = 4k$ or $4k + 2$ with $k \geq 2$. Choose a nef $E$ with $E^2 = 0$ and $E.L = \phi(L)$. If

$$2\phi(L) \leq k + 1, \tag{6}$$

then $(L - 2E)^2 \geq 4k - 2(k + 1) = 2k - 2 \geq 2$ so that $h^0(L - 2E) \geq 2$ by Lemma 2.2 and Riemann-Roch and $L \sim 2E + (L - 2E)$ is the desired decomposition. From the facts that either $L^2 = \phi(L)^2$ or $L^2 \geq \phi(L)^2 - 2\phi(L) + 2$ by Proposition 2.1, one easily sees that (6) is verified except for the cases we now treat.

For the rest of the proof, we let $E$ be such that $E^2 = 0$ and $E.L = \phi(L)$ and all $E_i$s will be nonzero, effective, isotropic divisors. We will use Lemma 2.2 repeatedly without further mentioning.

**The case** $(k, L^2, \phi(L)) = (10, 42, 6)$**:** We have $(L - 2E)^2 = 18$.

If $\phi(L - 2E) = 3$, choose any $F > 0$ with $F^2 = 0$ and $F.(L - 2E) = 3$. Then $(L - 2E - 3F)^2 = 0$ and we can write $L \sim 2E + 3F + F'$ for $F' > 0$ such that $(F')^2 = 0$ and $F.F' = 3$. Now $6 = \phi(L) \leq F.L = 2E.F + 3$ implies that $E.F \geq 2$ and $6 = E.L = 3E.F + E.F'$ implies that $E.F = 2$ and $E.F' = 0$. Therefore $F' \equiv qE$ for some $q \geq 1$, giving the contradiction $3 = F.F' = 2q$. Therefore we cannot have $\phi(L - 2E) = 3$. The case $\phi(L - 2E) = 2$ is ruled out similarly.

Therefore we have $\phi(L - 2E) = 4$. Pick any $E_1$ with $E_1.(L - 2E) = 4$. Then one easily finds that $L - 2E \sim 2E_1 + E_2 + E_3$, with $E_1.E_2 = E_1.E_3 = 2$ and $E_2.E_3 = 1$. From $6 = E.L = 2E.E_1 + E.E_2 + E.E_3$ we find $E.E_2 \leq 3$ and $E.E_3 \leq 3$, but if $E.E_i = 3$ for $i = 2$ or $3$, then $(E + E_i)^2 = 6$ yields the contradiction $3\phi(L) = 18 \leq (E + E_i).L = 17$. Hence $E.E_2 \leq 2$ and $E.E_3 \leq 2$, so that we only get the two options $(E.E_1, E.E_2, E.E_3) = (1, 2, 2)$ and $(2, 1, 1)$. We set $M = 2E + E_1 + E_2$ and $N = E_1 + E_3$. Then $M.N = 11$.

**The case** $(k, L^2, \phi(L)) = (10, 40, 6)$**:** By Proposition 2.1 we have that either $L \sim 3E + 2E_1 + E_2$ with $E.E_1 = E.E_2 = 2$, $E_1.E_2 = 1$ or $L \equiv 2D$ for a $D > 0$ with $D^2 = 10$. In the second case we are done with $M = D$. In the first case we set $M = 2E + 2E_1$ and $N = E + E_2$. Then $M.N = 10$.

**The case** $(k, L^2, \phi(L)) = (9, 36, 6)$**:** We have $L \sim 3B$ with $B^2 = 4$ by Proposition 2.1 and we set $M = B$.

**The case** $(k, L^2, \phi(L)) = (8, 34, 5)$**:** We have $(L - 2E)^2 = 14$ and we can easily see, exactly as above, that $\phi(L - 2E) = 3$. Repeating the process find that

$L-2E \sim 2E_1+E_2+E_3$, with $E_1.E_3 = 2$ and $E_1.E_2 = E_2.E_3 = 1$. Moreover $E.(L-2E) = 5$ and $E_1.(L-2E) = 3$, whence $E.E_1 \geq 1$.

If $E.E_1 \geq 2$, then $5 = E.L \geq 2E.E_1$ implies $E.E_1 = 2$, whence $E.E_2 = 0$ or $E.E_3 = 0$. But $5 = \phi(L) \leq E_2.L = 2E.E_2+3$ implies $E \equiv E_3$, whence $L \equiv 3E+2E_1+E_2$. We set $M = 2E+E_1$ and $N = L-M \equiv E+E_1+E_2$. Then $M.N = 9$.

If $E.E_1 = 1$, then $(E.E_2, E.E_3) = (1,2)$ or $(2,1)$. We set $M = E+E_1+E_2$ and $N = E+E_1+E_3$. Then $M.N = 9$.

**The case** $(k, L^2, \phi(L)) = (8,32,5)$**:** We have $(L-3E)^2 = 2$ whence we have $L-3E \sim E_1+E_2$, with $E_i > 0$, $E_i^2 = 0$, $i = 1,2$ and $E_1.E_2 = 1$. From $5 = \phi(L) \leq E_i.L = 3E.E_i+1$ we get $E.E_i \geq 2$, $i = 1,2$, whence by symmetry we get $(E.E_1, E.E_2) = (2,3)$. We set $M = 2E+E_2$ and $N = E+E_1$. Then $M.N = 8$ and we are done.

**The case** $(k, L^2, \phi(L)) = (7,30,5)$**:** We have $(L-2E)^2 = 10$ and we can easily show, exactly as above, that $\phi(L-2E) = 3$. Repeating, we find that $L-2E \sim E_1+E_2+E_3$, with $E_1.E_2 = 1$ and $E_1.E_3 = E_2.E_3 = 2$. Since $E.(L-2E) = 5$ and $E_i.(L-2E) \leq 4$, we must have $E.E_i > 0$ for all $i$. At the same time, if $E.E_i = 3$ for $i = 1$ or 2, then $(E+E_i)^2 = 6$, so that $15 = 3\phi(L) \leq (E+E_i).L = 14$, a contradiction. Hence $E.E_i \leq 2$ for $i = 1,2$, and by symmetry we get the three possibilities $(E.E_1, E.E_2, E.E_3) = (1,1,3)$, $(1,2,2)$ and $(2,2,1)$. One easily sees that $M = E+E_1$ and $N = E+E_2+E_3$ yields the desired decomposition.

**The case** $(k, L^2, \phi(L)) = (7,28,5)$**:** By Proposition 2.1 we have $L \sim 2E+2E_1+E_2$ with $E.E_1 = E_1.E_2 = 2$ and $E.E_2 = 1$. We set $M = E+E_1+E_2$ and $N = E+E_1$. Then $M.N = 7$.

**The case** $(k, L^2, \phi(L)) = (6,26,4)$**:** We have $(L-3E)^2 = 2$ , whence $L-3E \sim E_1+E_2$, with $E_1.E_2 = 1$. By symmetry we have the two possibilities $(E.E_1, E.E_2) = (2,2)$ and $(1,3)$. We set $M = 2E+E_2$ and $N = E+E_1$, and we get $M.N = 7$ and 6, respectively.

**The case** $(k, L^2, \phi(L)) = (6,24,4)$**:** We have $(L-2E)^2 = 8$. If $\phi(L-2E) = 1$, then we can write $L-2E \sim 4E_1+E_2$ with $E_1.E_2 = 1$, and $\phi(L) = 4 \leq E_1.L = 2E.E_1+1$ implies $E.E_1 \geq 2$, whence $E.L = 4E.E_1+E.E_2 \geq 8$, a contradiction. Hence $\phi(L-2E) = 2$, and we can write $L-2E \sim 2E_1+E_2$ with $E_1.E_2 = 2$.

We have $4 = 2E.E_1+E.E_2$. Hence we must have $E.E_1 = 1$ or 2. In the latter case we get $E.E_2 = 0$, then $E \equiv qE_2$ for some $q \geq 1$. From $E_1.E = E_1.E_2 = 2$ we get that $E \equiv E_2$ and we can set $M = 2E+E_1$ and $N = L-M \equiv E+E_1$. Then $M.N = 6$.

If $E.E_1 = 1$ and $E.E_2 = 2$, we set $M = E+E_1$ and $N = E+E_1+E_2$. Then $M.N = 6$.

**The case** $(k, L^2, \phi(L)) = (5,22,4)$**:** One easily sees that one can write $L \sim 2E+E_1+E_2+E_3$ with $E.E_1 = 2$, $E.E_j = E_i.E_j = 1$ for $1 \leq i < j \leq 3$ and one sets $M = 2E+E_1$ and $N = E_2+E_3$.

**The case** $(k, L^2, \phi(L)) = (5,20,4)$**:** Again one easily sees that $L \sim E+E_1+E_2+E_3+E_4$ with $E.E_i = E_i.E_j = 1$ for $i \neq j$ one sets and $M = E+E_1$ and $N = E_2+E_3+E_4$.

**The case** $(k, L^2, \phi(L)) = (4,18,4)$**:** One has $L \sim 2E+E_1+E_2$ with $E_1.E_2 = 1$, $E.E_1 = E.E_2 = 2$ and one sets and $M = E+E_1$ and $N = E+E_2$.

**The case** $(k, L^2, \phi(L)) = (4,18,3)$**:** We have $(L-2E)^2 = 6$.

If $\phi(L-2E)=1$, then we can write $L-2E \sim 3E_1+E_2$, with $E_1.E_2=1$. Since $E.(3E_1+E_2)=3$, we must have $E_2 \equiv E$, whence $L \equiv 3(E+E_1)$, and after possibly substituting $E_1$ with $E_1+K_S$ we get $L \sim 3(E+E_1)$. We set $M=2E+E_1$ and $N=E+2E_1$. Then $M.N=5$ and we are done.

If $\phi(L-2E)=2$, then $L-2E \sim E_1+E_2+E_3$, with $E_i.E_j=1$, for all $i \neq j$, and we easily see that $E.E_i=1$ for all $i$. We set $M=E+E_1+E_2$ and $N=E+E_3$. Then $M.N=5$.

**The case** $(k, L^2, \phi(L))=(4,16,4)$**:** We have $L \sim 2B$ with $B^2=4$ by Proposition 2.1 and one sets $M=B$.

**The case** $(k, L^2, \phi(L))=(4,16,3)$**:** One can write $L \sim 2E+E_1+E_2$ with $E.E_2=1$ and $E.E_1=E_1.E_2=2$ and one sets $M=E+E_1$ and $N=E+E_2$.

**The case** $(k, L^2, \phi(L))=(3,14,3)$**:** One can write $L \sim 2E+E_1+E_2$ with $E.E_1=E_1.E_2=1$, $E.E_2=2$ and one sets $M=E+E_1$ and $N=E+E_2$.

**The case** $(k, L^2, \phi(L))=(3,12,3)$**:** One can write $L \sim E+E_1+E_2+E_3$ with $E_i.E_j=1$ for $i \neq j$ and one sets $M=E+E_1$ and $N=E_2+E_3$.

**The case** $(k, L^2, \phi(L))=(3,10,2)$**:** One can write $L \sim 2E+E_1+E_2$ with $E.E_1=E_1.E_2=E.E_2=1$ and one sets $M=E+E_1$ and $N=E+E_2$. ■

**Lemma 2.4.** *Let L be as in Lemma 2.3 and $l>0$ the minimal integer such that there is a decomposition $L \sim M+N$ with $h^0(M) \geq 2$, $h^0(N) \geq 2$ and $M.N=l$.*

*Then*

$$3 \leq l \leq \lfloor \frac{L^2}{4} \rfloor + 1. \tag{7}$$

*and there is a decomposition $L \sim M+N$ with $M.N=l$, $h^0(M) \geq 2$ and $h^0(N) \geq 2$, and satisfying the following properties:*

*(a)* $N^2 \geq M^2$.

*(b)* $N^2>0$ *and* $h^1(N)=h^1(N+K_S)=0$.

*(c) M is one of the following:*

*(c-i)* $M \sim 2E$*, with* $|2E|$ *an elliptic pencil;*

*(c-ii)* $M^2=2$ *and* $|M|$ *has only two base points (which are distinct);*

*(c-iii)* $M^2=4$ *and* $|M|$ *is base point free (whence* $\phi(M)=2$*).*

*In particular, the general $D \in |M|$ is smooth and irreducible.*

*(d) If $M^2>0$ and L is ample, then $|N|$ is base component free. If in addition $(M^2, N^2, M.N) \neq (2,4,3)$, then $|\mathcal{O}_D(N)|$ is base point free for general $D \in |M|$.*

*Proof.* The right hand side inequality of (7) follows from Lemma 2.3. For the left hand side inequality note that as $h^0(M) \geq 2$, we must have $4 \leq 2\phi(L) \leq M.L = M^2+l$ by [KLM, Lemma 4.14], whence $M^2 \geq 2$ if $l \leq 2$, and likewise $N^2 \geq 2$. But then the Hodge index theorem implies $M \equiv N$ and $M^2=N^2=2$, so that $L^2=8$, a contradiction.

Now pick any decomposition $L \sim M+N$ with $h^0(M) \geq 2$, $h^0(N) \geq 2$ and $M.N=l$. By symmetry, we can assume (a).

If $M$ is not nef, then let $R$ be a nodal curve with $R.M < 0$. Then $R.N \geq -R.M$ by the nefness of $L$, whence

$$(M-R).(N+R) = l + 2 - R.N + R.M \leq l, \tag{8}$$

and as $h^0(M-R) = h^0(M)$ and $h^0(N+R) \geq h^0(N) \geq 2$, we get $R.N = -R.M = 1$, and we can substitute $M$ and $N$ with $M-R$ and $N+R$, respectively. Note that $(M-R)^2 = M^2$ and $(N+R)^2 = N^2$. Therefore, we can assume that $M$ is nef, in particular that $M^2 \geq 0$. It follows from (a) that also $N^2 \geq 0$, and if equality holds, then $N^2 = M^2 = 0$. Therefore $L^2 = 2M.N = 2l$, contradicting (7). Therefore $N^2 > 0$, and the same argument as above, with $M$ and $N$ interchanged, shows that any $\Delta > 0$ with $\Delta^2 = -2$ and $\Delta.N < 0$, must satisfy $\Delta.N = -1$ and $\Delta.M = 1$. This implies (b) by [KL1, Thm. 1], and, as $\Delta.L = 0$, also that

$$N \text{ is nef if } L \text{ is ample,} \tag{9}$$

a fact we will use later, in the proof of (d).

Now assume that $M^2 = 0$ and let $|M| = |M_0| + \Sigma_0$ be the decomposition into the moving and fixed part, respectively. The nefness of $M$ implies $M_0^2 = \Sigma_0^2 = M_0.\Sigma_0 = 0$, whence $|M_0| = |2lE|$ with $E$ nef such that $E^2 = 0$ and $E$ is not divisible in $\mathrm{Num}(S)$ and $l \geq 1$ an integer, by [CD, Prop. 3.1.4(ii) and Prop. 3.1.3], and $\Sigma_0 = 0$ or $\Sigma_0 \equiv E$, as a consequence of the signature theorem [BPV, VIII.1] (see also [KL1, Lemma 2.1]). Possibly adding $K_S$ to $E$, we can write $M \sim kE$, for an integer $k \geq 2$. If $k \geq 3$, then $h^0(M-E) \geq 2$, $h^0(N+E) \geq 2$ and

$$(M-E).(N+E) = M.N - E.N < M.N,$$

a contradiction. Hence $k = 2$ and we are in case (c-i).

We will now treat the case $M^2 > 0$ for the rest of the proof.

First of all we note that there is always a nef $E$ with $E.L = \phi(L)$ by [Co2, 2.11] or [CD, Cor. 2.7.1, Prop. 2.7.1 and Thm. 3.2.1]. If $2\phi(L) \leq l$, then $2\phi(L) \leq \min\{\lfloor \frac{L^2}{4} \rfloor + 1, 2\lfloor \sqrt{L^2} \rfloor\}$ by (7), and one easily checks that this implies $(L-2E)^2 > 0$, whence also $h^0(L-2E) \geq 2$ by Lemma 2.2 and Riemann-Roch. We would therefore be done with the proof. We will therefore henceforth assume that

$$2\phi(L) \geq l + 1. \tag{10}$$

We first show that we can assume $(M^2, \phi(M)) = (2,1)$ or $(4,2)$.

If $\phi(N) < \phi(M)$, then let $E$ be such that $E^2 = 0$ and $E.N = \phi(N)$. Then $h^0(M+E) \geq 2$ and

$$(N-E).(M+E) = M.N - E.M + E.N \leq M.N - \phi(M) + \phi(N) < M.N = l,$$

a contradiction unless $h^0(N-E) \leq 1$. The latter implies $(N^2, \phi(N)) = (4,2)$ or $(2,1)$ by Riemann-Roch and Lemma 2.2. But this is impossible, as $\phi(N) < \phi(M)$ and $N^2 \geq M^2$.

Therefore $\phi(M) \leq \phi(N)$, and the same argument as above, with $M$ and $N$ interchanged, shows that we can reduce to the cases $(M^2, \phi(M)) = (4,2)$ or $(2,1)$, as claimed.

In the first case $|M|$ is base point free by [CD, Prop. 3.1.6 and Thm. 4.4.1] as $M$ is nef with $\phi(M) \geq 2$, and we are in case (c-iii). In the second case $|M|$ has precisely two base points, necessarily distinct, unless $M \sim 2E + R$, where $|2E|$ is an elliptic pencil and $R$ is nodal with $R.E = 1$, by [CD, Prop. 3.1.6 and Thm. 4.4.1], in which case $R$ is the base component of $|2E + R|$. In this case, we add $K_S$ to both $M$ and $N$, and we are in case (c-ii).

Finally, we prove (d). So assume that $L$ is ample. Then $N$ is nef by (9). If $|N|$ is not base component free, then $|N| = |2E| + R$, where $|2E|$ is an elliptic pencil and $R$ is nodal with $E.R = 1$, by [CD, Prop. 3.1.6]. In particular $N^2 = 2$, so that $M^2 = 2$ by (a). By (7) we have

$$4l - 4 \leq L^2 = 4 + 2M.N = 4 + 2l,$$

whence $(l, L^2) = (3, 10)$ or $(4, 12)$. Moreover from (10) we have

$$l + 1 \leq 2\phi(L) \leq 2E.L = N.L - R.L = 2 + l - R.L,$$

so that the only possibility is $(l, L^2, \phi(L), R.L) = (3, 10, 2, 1)$ by the ampleness of $L$. We are now done by adding $K_S$ to both $N$ and $M$, unless $M \sim 2E' + R' + K_S$, with $|2E'|$ an elliptic pencil and $R'$ is nodal with $E'.R' = 1$, by [CD, Prop. 3.1.6]. As $E.M = 1$, we get $E \equiv E'$. But then $1 = R.L = R.M = R.(2E + R') = 2 + R.R'$ yields the absurdity $R.R' = -1$.

It follows that $|N|$ is base component free.

Consequently, if $\mathcal{O}_D(N)$ is not base point free for a general, smooth irreducible curve $D \in |M|$, we must have that $|N|$ has base points, whence $\phi(N) = 1$, and moreover that $l = M.N = \deg \mathcal{O}_D(N) \leq 2g(D) - 1 = M^2 + 1$. Since we have proved that $\phi(M) \leq \phi(N)$, we obtain by (c) that $M^2 = 2$ and $l = 3$. Then $N^2 = 2$ or 4 by the Hodge index theorem. We now rule out the case $N^2 = 2$, finishing the proof of (d).

We have $h^0(\mathcal{O}_D(N)) = h^0(N) - \chi(\mathcal{O}_S(N - M)) = 2$, using (b), so that if $x$ is a base point of $|\mathcal{O}_D(N)|$, we must have $\mathcal{O}_D(N) \sim \omega_D(x)$, by the uniqueness of the $g^1_2$ on $D$. In particular $x$ is the only base point of $|\mathcal{O}_D(N)|$.

If both $M \sim 2E + R + K_S$ and $N \sim 2E' + R' + K_S$ with $|2E|$ and $|2E'|$ elliptic pencils and $R$ and $R'$ nodal curves with $R.E = R'.E' = 1$, then we get the same absurdity $R.R' = -1$ as above. By symmetry between $M$ and $N$, and using Lemma 2.2, we can therefore assume that $M \sim E + E_1$, with $E$ and $E_1$ nef, such that $E^2 = E_1^2 = 0$ and $E.E_1 = 1$. As $M.N = 3$, one easily sees that one can write $N \sim E + E_2$, with $E_2^2 = 0$ and $E.E_2 = E_1.E_2 = 1$.

Consider

$$0 \longrightarrow \mathcal{O}_S(N - 2M + K_S) \longrightarrow \mathcal{O}_S(N - M + K_S) \longrightarrow \mathcal{O}_D(x) \longrightarrow 0. \tag{11}$$

We have $h^0(N - 2M + K_S) = 0$ as $M.(N - 2M + K_S) = -1$.

We now claim that also

$$h^0(2M - N) = 0. \tag{12}$$

Indeed, if this were not the case, we would have $\Delta := E + 2E_1 - E_2 > 0$, with $\Delta^2 = -2$, $\Delta.E = 1$, $\Delta.E_1 = 0$ and $\Delta.E_2 = 3$. Since $|2E_1|$ is an elliptic pencil, as $E_1$ is nef, one easily sees that $\Delta$ must be contained entirely in one fiber of the elliptic fibration given by $|2E_1|$. Hence $\Delta' := 2E_1 - \Delta > 0$ and $E_2 \sim E + \Delta'$. As $N \sim E + E_2$ is nef, this implies that $\Delta'$ is a nodal curve with $\Delta'.E = 1$ and we have $N \sim 2E + \Delta'$. But then $\Delta'$ is a base component of $|N|$, a contradiction. This proves (12).

From (12) and Riemann-Roch we get $h^1(\mathcal{O}_S(N - 2M + K_S)) = 0$, so that (11) implies $h^0(N - M + K_S) > 0$. But this contradicts the ampleness of $L$, as $(N - M).L = 0$. ∎

**Remark 2.5.** Assume that $L > 0$ is a line bundle with $L^2 = 12$ and $\phi(L) = 2$. Let $E > 0$ be such that $E^2 = 0$ and $E.L = 2$. Then $(L - 2E)^2 = 4$. Using Lemma 2.2 one easily sees that the two cases $\phi(L - 2E) = 1$ and 2 yield, respectively,

(i) $L \equiv 3E + 2F$, $F > 0$ primitive, isotropic with $E.F = 1$;

(ii) $L \equiv 3E + F$, $F > 0$ primitive, isotropic with $E.F = 2$.

One easily verifies that the exceptional case in Lemma 2.4(d) yields case (i), with $M \equiv E + F$ and $N \equiv 2E + F$.

## 3 Zero-cycles in special position and minimal gonality of curves in a linear system

We first give a simple criterion to find zero-dimensional schemes on a surface such that (2) is not surjective.

**Lemma 3.1.** *Let $L$ be a line bundle on a surface $S$ with a decomposition $L \sim M + N$ such that $h^0(M) > 0$, $h^0(N) > 0$ and such that there is a smooth, irreducible curve $D \in |M|$ with $h^0(\mathcal{O}_D(N)) > 0$ and $\mathcal{O}_D(N)$ is nontrivial. Then, for any $Z \in |\mathcal{O}_D(N)|$, the natural restriction map* (2) *is not surjective.*

*Furthermore, if $L$ is big and nef, then this is equivalent to $H^1((L + K_S) \otimes \mathcal{J}_Z) \neq 0$.*

*Proof.* Pick a nonzero section $s \in H^0(\mathcal{O}_D(N))$, defining $0 \to \mathcal{O}_D \to \mathcal{O}_D(N)$. Tensoring with $M + K_S$, we obtain

$$0 \longrightarrow \omega_D \longrightarrow \mathcal{O}_D(L + K_S) \longrightarrow \mathcal{O}_{Z(s)}(L + K_S) \longrightarrow 0.$$

where $Z(s)$ is the scheme of zeroes of $s$. Since $h^1(\mathcal{O}_D(L + K_S)) = h^1(\omega_D(N)) = h^0(\mathcal{O}_D(-N)) = 0$ and $h^1(\omega_D) = 1$, the map $H^0(\mathcal{O}_D(L + K_S)) \to H^0(\mathcal{O}_{Z(s)}(L + K_S))$ is not surjective. If $L$ is big and nef, then $H^1(L + K_S) = 0$ and the last assertion is immediate. ∎

**Lemma 3.2.** *The implications (iii) $\Leftrightarrow$ (ii) $\Rightarrow$ (i) $\Rightarrow$ (iv) in Theorem 1.1 hold, even without the ampleness assumption on $L$ and assuming only $L^2 > 0$.*

*Proof.* Conditions (ii) and (iii) are obviously equivalent. Moreover, (ii) implies that (2) is surjective for all $Z \subset \mathcal{U} := S - \Theta_{k+1}(L)$, so that (ii) $\Rightarrow$ (i).

To see that (i) $\Rightarrow$ (iv), observe first that it is enough to show that the family of smooth curves in $|L|$ of gonality $\leq k + 1$ is positive-dimensional. Indeed, if this holds, then for any open $\mathcal{U} \subseteq S$, we can always find a smooth curve

$C \in |L|$ of gonality $\leq k+1$ such that $C \cap \mathcal{U} \neq \emptyset$ and a zero-dimensional scheme $Z$ in the gonality pencil (which is necessarily base point free) lying inside $\mathcal{U}$. As $h^1(\omega_C - Z) = h^0(\mathcal{O}_C(Z)) = 2$ and from

$$0 \longrightarrow K_S \longrightarrow (L+K_S) \otimes \mathcal{J}_Z \longrightarrow \omega_C - Z \longrightarrow 0 \tag{13}$$

we see that $h^1((L+K_S) \otimes \mathcal{J}_Z) = 1$, showing that the restriction map (2) cannot be surjective (by the last line of Lemma 3.1).

Now assume that $C \in |L|$ is a smooth curve of minimal gonality $l \leq k+1$, among the smooth curves in $|L|$. Let $Z \in |A_C|$ be any element in a pencil $|A_C|$ computing the gonality. As $h^1(K_S) = 0$, then (13) actually shows that $h^0(\mathcal{O}_{C'}(Z')) = 2$ for any smooth $C' \in |L \otimes \mathcal{J}_{Z'}|$ and any $Z' \in |A_C|$.

By Brill-Noether theory, $l \leq \lfloor \frac{g(C)+3}{2} \rfloor = \lfloor \frac{L^2}{4} \rfloor + 2$, and if equality holds, then all the smooth curves in $|L|$ would have gonality $l$ and we would be done.

Assume therefore that $l < \lfloor \frac{L^2}{4} \rfloor + 2$. Then $L^2 \geq 4l - 4$. The dimension of the family of curves in $|L|$ passing though some element $Z' \in |A_C|$, is, by the obvious incidence correspondence, at least

$$\begin{aligned} \dim |L \otimes \mathcal{J}_Z| + \dim |A_C| &\geq \dim |L| + 1 - l \\ &= \frac{1}{2}L^2 + 1 - l \geq 2l - 2 + 1 - l = l - 1 \geq 1, \end{aligned}$$

where $Z \in |A_C|$ is general. Therefore, we are done again. ∎

**Proposition 3.3.** *Let $S$ be an Enriques surface and $L$ be as in Lemma 2.3 and such that $(L^2, \phi(L)) \neq (12, 2)$. Let $l > 0$ the minimal integer such that there is a decomposition $L \sim M + N$ with $h^0(M) \geq 2$, $h^0(N) \geq 2$ and $M.N = l$.*

*If $L$ is ample, then there exists a positive-dimensional family of smooth curves in $|L|$ having gonality $\leq l$.*

*Proof.* Choose the decomposition $L \sim M + N$ satisfying the properties (a)-(d) in Lemma 2.4. In particular, as $(L^2, \phi(L)) \neq (12, 2)$ by assumption, the exceptional case in (d) does not occur (cf. Remark 2.5), so that $|N|$ is base component free and $\mathcal{O}_D(N)$ is base point free for the general smooth curve $D \in |M|$.

If $M^2 = 0$, there is nothing to prove.

If $M^2 = 2$, then $|M|$ has two distinct base points $x$ and $y$, by Lemma 2.4(c), so that if $C \in |L \otimes \mathcal{J}_x \otimes \mathcal{J}_y|$ is a smooth curve, then $|\mathcal{O}_C(M)(-x-y)|$ is a $g^1_l$ on $C$. As $\dim |L \otimes \mathcal{J}_x \otimes \mathcal{J}_y| \geq \dim |L| - 2 = \frac{1}{2}L^2 - 2 \geq 3$, we only need to show the existence of a smooth curve in $|L \otimes \mathcal{J}_x \otimes \mathcal{J}_y|$. From the short exact sequence

$$0 \longrightarrow N \longrightarrow L \otimes \mathcal{J}_x \otimes \mathcal{J}_y \longrightarrow \mathcal{O}_D(N) \longrightarrow 0, \tag{14}$$

the fact that $h^1(N) = 0$ by Lemma 2.4(b) and the base point freeness of $\mathcal{O}_D(N)$, we see that the base locus of $|L \otimes \mathcal{J}_x \otimes \mathcal{J}_y|$, off $x$ and $y$, is contained in $\mathrm{BS}\,|N|$ and does not intersect $D$. As $|N|$ is base component free, the general element of $|L \otimes \mathcal{J}_x \otimes \mathcal{J}_y|$ is smooth by Bertini's theorem, unless possibly if $|M|$ and $|N|$ share some base points, and this can only happen if $\phi(N) = 1$ and $|N|$ has $x$ or $y$ as one of its two base points, by [CD, Thm. 4.4.1]. But if the general element of $|L \otimes \mathcal{J}_x \otimes \mathcal{J}_y|$ were singular at $x$ (resp. $y$), then by (14) $x$ (resp. $y$) would be contained in every element of $|\mathcal{O}_D(N)|$, a contradiction on the base point freeness.

Finally, let $M^2 = 4$.

The hypotheses of Lemma 3.1 are satisfied, and letting $D$ and $Z$ be as in that lemma, using $h^1(\mathcal{O}_S) = 0$, we have for any smooth curve $C \in |L \otimes I_Z|$, that

$$h^1(\omega_C - Z) = h^1((L + K_S) \otimes \mathcal{J}_Z) + 1 \geq 2.$$

Therefore $h^0(\mathcal{O}_C(Z)) \geq 2$. Moreover, as

$$\dim |L \otimes \mathcal{J}_Z| \geq \dim |L| - l = \frac{1}{2}L^2 - l \geq l - 2 \geq 1$$

by (7), we only have left to show that, for sufficiently general $D \in |M|$ and $Z \in |\mathcal{O}_D(N)|$, there is a smooth curve in $|L \otimes \mathcal{J}_Z|$.

Since $h^1(N) = 0$ by Lemma 2.4(b), the short exact sequence

$$0 \longrightarrow N \longrightarrow L \otimes \mathcal{J}_Z \longrightarrow \mathcal{O}_D(M) \longrightarrow 0,$$

is exact on global sections. Now $N^2 \geq M^2 = 4$, by Lemma 2.4(a), so that $M$ and $N$ are base point free by [CD, Prop. 3.1.6 and Thm. 4.4.1]. Therefore, also $\mathcal{O}_D(M)$ is base point free, so that the base scheme of $|L \otimes \mathcal{J}_Z|$ is precisely $Z$, which we can choose to consist of $l$ distinct points, as $|\mathcal{O}_D(N)|$ is base point free. Thus, there is a smooth curve in $|L \otimes \mathcal{J}_Z|$ by Bertini's theorem. ∎

## 4 Vector bundles methods

In this section we recall some well-known vector bundle methods already used by Tyurin, Reider, Beltrametti-Francia-Sommese, Lazarsfeld and others [Re, Ty, La, BFS, BS4]. We formulate some results in the language of our setting. These are well-known to the experts and this section is only included for completeness and to ease the reading.

Assume that $Z$ is a zero-dimensional subscheme of length $l \geq 1$ on an Enriques surface $S$ and $L$ a big and nef line bundle on $S$ such that the natural restriction map in (2) is not surjective on $Z$, but is surjective for any proper subscheme $Z' \subsetneq Z$. In other words, $Z$ is a *minimal* zero-dimensional subscheme for which the surjectivity of (2) fails. For instance (cf. the proof of Lemma 3.2), any zero-scheme $Z$ in the linear system of a pencil computing the gonality $l$ of a smooth curve $C \in |L|$, satisfies this condition, because $h^0(\mathcal{O}_C(Z')) = 1$ for any proper subscheme $Z' \subsetneq Z$, by the base point freeness of any pencil computing the gonality of a curve.

Then (see e.g. [Ty, (1.12)] and [BS4, Thm. 2.1]) there is a rank-two vector bundle on $\mathcal{E}$ on $S$ fitting into a short exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{E} \longrightarrow L \otimes \mathcal{J}_Z \longrightarrow 0 \tag{15}$$

and satisfying

$$\det \mathcal{E} = L; \tag{16}$$

$$c_2(\mathcal{E}) = \operatorname{length} Z = l; \tag{17}$$

$$h^i(\mathcal{E} \otimes \omega_S) = 0, \quad i = 1, 2. \tag{18}$$

We will make use of the following two lemmas, which are variants of well-known results (see e.g. [DM, Kn3, GLM1, KL2]):

**Lemma 4.1.** *Assume there are effective, nontrivial line bundles $M$ and $N$ on $S$, and a zero-dimensional subscheme $X \subset S$, fitting into a short exact sequence*

$$0 \longrightarrow N \longrightarrow \mathcal{E} \longrightarrow M \otimes \mathcal{J}_X \longrightarrow 0. \tag{19}$$

*Then*

*(a)* $L \sim M + N$.

*(b)* $M.N = \text{length}\, Z - \text{length}\, X \leq \text{length}\, Z = l$.

*(c)* $|M|$ *contains an effective divisor $D$ passing through $Z$.*

*(d)* $h^1(M + K_S) = h^2(M + K_S) = 0$.

*If furthermore $Z \cap \Theta_l(L) = \emptyset$, then $h^0(M) \geq 2$ and $M^2 \geq 0$.*

*Proof.* Taking $c_1$ and $c_2$ of (19) and using (16) and (17) we obtain (a) and (b). Tensoring (15) and (19) with $\mathcal{O}_S(-N)$ and using the fact that $h^0(-N) = 0$ as $N$ is effective and nontrivial, we obtain $h^0(M \otimes \mathcal{J}_Z) > 0$, proving (c). Finally, (d) is an immediate consequence of (18) and (19).

Now assume that $Z \cap \Theta_l(L) = \emptyset$.

If $M^2 \leq -2$, then by (d) and Riemann-Roch, we must have $M^2 = -2$ and $h^0(M + K_S) = 0$, so that $M$ is a nodal cycle, and as such, only supported on nodal curves. As $M.L = M.N + M^2 \leq l - 2$, this and (c) contradict our assumptions on $Z$. Therefore $M^2 \geq 0$.

If $M^2 = 0$ and $h^0(M) = 1$, then, as $h^0(M + K_S) = 1$ by (d), we have that $M$ is only supported on precisely one halfpencil, and possibly some nodal curves in addition. As $M.L = M.N + M^2 \leq l$, we again obtain a contradiction on our assumptions on $Z$.

If $M^2 \geq 2$ then $h^0(M) \geq 2$ by Riemann-Roch. ∎

**Lemma 4.2.** *In the above situation, assume furthermore that $L^2 \geq 4l - 2$ and that $Z \cap \Theta_l(L) = \emptyset$. Then, either*

*(i) there are line bundles $M$ and $N$ on $S$, and a zero-dimensional subscheme $X \subset S$, fitting into a short exact sequence like* (19), *and such that $h^0(N) \geq 2$, $h^0(M) \geq 2$, $M^2 \geq 0$, $M.N \leq l$ and $N^2 \geq M^2 \geq 0$; or*

*(ii) $L^2 = 4l - 2$ and for any $\Sigma \geq 0$ such that $h^0(\mathcal{E}(-\Sigma)) > 0$, we can find a line bundle $N \geq \Sigma$ and a nodal cycle $\Delta$ such that* (19) *holds with $M \sim N + \Delta + K_S$, $X = \emptyset$, $h^0(M) \geq 2$, $M^2 \geq 0$, $M.N = l$, $\Delta^2 = -2$, $h^0(\Delta) = 1$ and $h^0(\Delta + K_S) = 0$.*

*Proof.* We first consider the case where either $h^0(\mathcal{E} \otimes \mathcal{E}^*) \geq 2$ or $h^2(\mathcal{E} \otimes \mathcal{E}^*) \geq 2$ and we will show that we end up in case (i).

For any ample divisor $H$ on $S$, we have that $\mathcal{E}$ is not $H$-stable, because if it were, we would have had $h^0(\mathcal{E} \otimes \mathcal{E}^*) = 1$ by [F, Cor. 4.8] and $h^2(\mathcal{E} \otimes \mathcal{E}^*) \leq 1$ by

[F, Prop. 4.7]. Bidualizing and saturating (if necessary) we find two line bundles $N, M$ on $S$ and a zero-dimensional subscheme $X \subset S$ such that $\mathcal{E}$ fits into an exact sequence like (19). By Lemma 4.1, we have $M.N \leq l$, $h^0(M) \geq 2$ and $M^2 \geq 0$. Furthermore, by construction, $N$ destabilizes $\mathcal{E}$, that is $\mu_H(N) \geq \mu_H(\mathcal{E})$ (where $\mu_H$ denotes the $H$-slope), or, equivalently, $H.(N-M) \geq 0$.

If $h^0(N-M) > 0$, then $h^0(N) \geq 2$ and $N.L \geq M.L$ as $L$ is nef, whence $N^2 \geq M^2$, and we are in case (i).

Assume now that $h^0(N-M) = 0$.

If $h^2(N-M) > 0$ then $N - M \sim K_S$. It follows that $0 = (N-M)^2 \geq -2 + 4\,\mathrm{length}(X)$, whence $X = \emptyset$ and (19) splits since $\mathrm{Ext}^1(M,N) \cong H^1(N-M) = 0$, so that $h^0(N) \geq 2$ by Lemma 4.1 applied with $M$ and $N$ interchanged.

If $h^2(N-M) = 0$ then by Riemann-Roch we get

$$0 = h^1(N-M) + 1 + \frac{1}{2}(N-M)^2 = h^1(N-M) + \frac{1}{2}L^2 - 2l + 1 + 2\,\mathrm{length}(X) \geq 0$$

therefore $L^2 = 4l - 2$, $X = \emptyset$ and $h^1(N-M) = 0$, whence (19) splits and again $h^0(N) \geq 2$.

Furthermore, in both the last cases $L.N \geq L.M$, whence $N^2 \geq M^2$ and we have proved that we are in case (i).

Finally we treat the case where $h^0(\mathcal{E} \otimes \mathcal{E}^*) = 1$ and $h^2(\mathcal{E} \otimes \mathcal{E}^*) \leq 1$. We will see that we are in case (ii).

By Riemann-Roch we get that

$$2 \geq h^1(\mathcal{E} \otimes \mathcal{E}^*) + \chi(\mathcal{E} \otimes \mathcal{E}^*) = h^1(\mathcal{E} \otimes \mathcal{E}^*) + L^2 - 4l + 4 \geq 2,$$

whence $h^1(\mathcal{E} \otimes \mathcal{E}^*) = 0$, $h^2(\mathcal{E} \otimes \mathcal{E}^*) = 1$ and $L^2 = 4l - 2$. Such a vector bundle $\mathcal{E}$ is called *exceptional* [Ki] and for any $\Sigma \geq 0$ such that $h^0(\mathcal{E}(-\Sigma)) > 0$ we can find an $N \geq \Sigma$ such that $h^0(\mathcal{E}(-N)) > 0$ but $h^0(\mathcal{E}(-N-B)) = 0$ for any $B > 0$. Applying [Ki, Thm. 3.4] we get that $\mathcal{E}(-N)$ fits into an exact sequence

$$0 \longrightarrow \mathcal{O}_S \longrightarrow \mathcal{E}(-N) \longrightarrow \Delta + K_S \longrightarrow 0,$$

with $\Delta^2 = -2$, $h^0(\Delta) = 1$, $h^0(\Delta + K_S) = 0$ and $\Delta$ is a nodal cycle. Setting $X = \emptyset$ and $M \sim N + \Delta + K_S$, we have an exact sequence like (19) and the rest follows from Lemma 4.1. ∎

## 5 Proofs of Theorem 1.1 and Proposition 1.2

In this section we first prove Theorem 1.1 and Proposition 1.2, dividing the treatment into three cases. Then we prove (5) from the introduction.

### 5.1 Proofs of Theorem 1.1 and Proposition 1.2 in the cases where $(L^2, \phi(L)) \neq (10,3), (12,2)$

By Lemma 3.2 the part that remains to be proved of Theorem 1.1 is that (iv) implies (ii).

Arguing by contradiction, assume that there is a zero-dimensional scheme $Z \subset S$ of length $l \leq k+1$, such that $Z \cap \Theta_l(L) = \emptyset$ and $Z$ is a minimal subscheme such that (2) is not surjective.

By Lemma 2.3 and Proposition 3.3 there is a smooth curve in $|L|$ of gonality $\leq \lfloor \frac{L^2}{4} \rfloor + 1$, so that we can assume that $L^2 \geq 4k+4 \geq 4 \operatorname{length} Z = 4l$. By Lemma 4.2 there is a decomposition $L \sim M+N$ with $h^0(M) \geq 2$, $h^0(N) \geq 2$ and $M.N \leq l$. Therefore, by Proposition 3.3, there is a smooth curve in $|C|$ of gonality $\leq l \leq k+1$, as desired.

This concludes the proof of Theorem 1.1.

We next prove Proposition 1.2.

Let $d_{min}$ be the minimal gonality of a smooth curve in $|L|$. We first prove the inequality

$$d_{min} \geq \min\{2\phi(L), \beta(L), \mu(L) - 2\}. \tag{20}$$

We first note that there is a decomposition $L \sim M+N$ with $h^0(M) \geq 2$, $h^0(N) \geq 2$ and $M.N \leq d_{min}$. Indeed, if $L^2 \leq 4d_{min} - 2$ this follows from Lemma 2.3, and if $L^2 \geq 4d_{min}$, then this follows from Theorem 1.1 and Lemma 4.2.

By Lemma 2.4(c) we therefore have that (20) holds. We now prove the opposite inequality.

By Lemma 2.3 and Proposition 3.3 we have $L^2 \geq 4d_{min} - 4$.

Now clearly $d_{min} \leq 2\phi(L)$.

Assume, to get a contradiction, that $d_{min} > \beta(L)$. Let $B$ be such that $B^2 = 2$ and $\beta(L) = B.L - 2$. By the Hodge index theorem we have $(B.L)^2 \geq B^2L^2 \geq 20$, whence $B.L \geq 5$, so that $\beta(L) \geq 3$ and $d_{min} \geq 4$. Hence

$$\begin{aligned}
(L-B)^2 &= L^2 + B^2 - 2L.B = L^2 - 2\beta(L) - 2 \geq (4d_{min} - 4) - 2(d_{min} - 1) - 2 \\
&= 2d_{min} - 4 \geq 4.
\end{aligned}$$

Since $(L-B).B = L^2 - 2 > 0$, we have $h^2(L-B) = 0$ by Serre duality, as a consequence of the signature theorem [BPV, VIII.1] (cf. also [KL1, Lemma 2.1]). Hence $h^0(L-B) \geq 3$ by Riemann-Roch. Proposition 3.3 then implies that $d_{min} \leq B.(L-B) = \beta(L)$, a contradiction.

Finally, assume, to get a contradiction, that $d_{min} > \mu(L) - 2$. Let $B$ be such that $B^2 = 4$, $\phi(B) = 2$ and $\mu(L) = B.L - 2$. By the Hodge index theorem we have $(B.L)^2 \geq B^2L^2 \geq 40$, whence $B.L \geq 7$, so that $\mu(L) \geq 5$ and $d_{min} \geq 4$. As above, we obtain that $(L-B)^2 \geq 2d_{min} - 6 \geq 2$ and that $h^0(L-B) \geq 2$. Proposition 3.3 then implies that $d_{min} \leq B.(L-B) = \mu(L) - 2$, a contradiction.

### 5.2 Proofs of Theorem 1.1 and Proposition 1.2 in the case $(L^2, \phi(L)) = (12, 2)$

Since $\phi(L) = 2$, we have that $\operatorname{gon} C \leq 4$ for all smooth curves $C \in |L|$, so that (i)-(iv) in Theorem 1.1 do not hold for $k = 3$, by Lemma 3.2.

As above, by Lemma 3.2, the part that remains to be proved of Theorem 1.1 is that (iv) implies (ii). In fact we will now prove that (ii) holds for $k \leq 2$. This will also prove that $\operatorname{gon} C = 4$ for all smooth curves $C \in |L|$, whence Proposition 1.2.

Assume, to get a contradiction, that (ii) does not hold for some $k \leq 2$ and let $Z \subset S$ be a zero-dimensional subscheme of length $l = 2$ or 3, such that $Z \cap \Theta_l(L) = \emptyset$ and $Z$ is a minimal subscheme such that (2) is not surjective. Let $\mathcal{E}$ be the associated vector bundle, as in Section 4. Since $L^2 = 12 \geq 4l$ we can apply Lemma 4.2(i). Thus $L \sim M + N$ with $N^2 \geq M^2$, $h^0(M) \geq 2$, $h^0(N) \geq 2$ and $M.N \leq l \leq 3$. The Hodge index theorem yields the only possibility $l = 3$, $N^2 = 4$ and $M^2 = 2$. (It follows that (ii) holds for $k = 1$, whence also (i), (iii) and (iv) by Lemma 3.2.) Using Lemma 2.2 successively, one easily verifies that we must be in case (i) of Remark 2.5, with $N \equiv 2E + F$ and $M \equiv E + F$. Recall that $E > 0$ and $F > 0$ are primitive, isotropic with $E.F = 1$. Moreover, $E$ is nef as $E.L = 2$ and $L$ is ample. Moreover, if $R$ is nodal with $R.F < 0$, then $R.F = -1$, otherwise $(F - R)^2 \geq 2$ and $(F - R).L \leq 2 = \phi(L)$, a contradiction. As $R.E > 0$ by the ampleness of $L$, we get that both $N$ and $M$ are nef. In particular, $|N|$ is base component free by [CD, Prop. 3.1.6]. It also follows that $h^0(N - M) = 1$, $h^1(N - M) = 0$ and $h^1(-M) = 0$, so that tensoring (21) and (15) by $\mathcal{O}_S(-M)$, we therefore find that $h^0(N \otimes \mathcal{J}_Z) = h^0(\mathcal{E}(-M)) = 2$.

Assume now that the pencil $|N \otimes \mathcal{J}_Z|$ has a fixed part $\Sigma > 0$. Then $(N - \Sigma).L \geq 2\phi(L) = 4$, by Lemma 2.2, as $N - \Sigma$ moves, so that $\Sigma.L \leq 3$. If $Z \cap \Sigma = \emptyset$, then $h^0(N \otimes \mathcal{J}_Z) = h^0((N - \Sigma) \otimes \mathcal{J}_Z)$. However, tensoring (21) and (15) by $\mathcal{O}_S(-M - \Sigma)$ as above, we obtain $h^0((N - \Sigma) \otimes \mathcal{J}_Z) = h^0(\mathcal{E}(-M - \Sigma)) = h^0(N - M - \Sigma) \leq 1$, a contradiction. Therefore, $Z \cap \Sigma \neq \emptyset$. Hence $\Sigma^2 \leq -2$ and $\Sigma.L \geq 2$, as $Z \cap \Theta_3(L) = \emptyset$ by assumption. Moreover, as $(N - \Sigma).M \geq 2\phi(M) = 2$, we must have $\Sigma.M \leq 1$, whence $\Sigma.N \geq 1$, so that $(N - \Sigma)^2 = N^2 + \Sigma^2 - 2N.\Sigma \leq 0$. Therefore, $(N - \Sigma)^2 = 0$, so that $\Sigma^2 = -2$, $N.\Sigma = 1$ and $|N - \Sigma|$ must be an elliptic pencil. But then $1 = N.\Sigma = N.(N - \Sigma) + \Sigma^2$ is even, a contradiction.

Therefore, the pencil $|N \otimes \mathcal{J}_Z|$ is base component free. Since $N^2 = 4$, $Z$ must contain at least one of the two base points of $|N|$ (recall that $\phi(N) = 1$). But these are contained in $F$, as $N \sim 2E + F$ and $|2E|$ is base point free. Hence $Z \cap \Theta_3(L) \neq \emptyset$, the desired contradiction.

### 5.3 Proofs of Theorem 1.1 and Proposition 1.2 in the case $(L^2, \phi(L)) = (10, 3)$

Since $g(C) = 6$ for all smooth curves $C \in |L|$, we have $\operatorname{gon} C \leq 4$, so that (i)-(iv) in Theorem 1.1 do not hold for $k = 3$, by Lemma 3.2.

As above, by Lemma 3.2, the part that remains to be proved of Theorem 1.1 is that (iv) implies (ii).

Assume now that (ii) does not hold for some $k \leq 2$ and let $Z \subset S$ be a zero-dimensional subscheme of length $l = 2$ or 3, such that $Z \cap \Theta_l(L) = \emptyset$ and $Z$ is a minimal subscheme such that (2) is not surjective. Let $\mathcal{E}$ be the associated vector bundle, as in Section 4. Since $L^2 = 10 \geq 4l - 2$ we can apply Lemma 4.2.

If we are in (i) of that lemma, then $L \sim M + N$ with $N^2 \geq M^2$, $h^0(M) \geq 2$, $h^0(N) \geq 2$ and $M.N \leq l \leq 3$. But then, by the Hodge index theorem, $M^2 \leq 2$, so that $M.L \leq 5$, whence $\phi(L) \leq 2$, by [KLM, Lemma 4.14], a contradiction.

Therefore, we are in (ii) of Lemma 4.2, so that $l = 3$. It follows that (ii) in Theorem 1.1 holds for $k = 1$, whence also (i), (iii) and (iv) by Lemma 3.2. We have a short exact sequence

$$0 \longrightarrow N \longrightarrow \mathcal{E} \longrightarrow M \longrightarrow 0, \tag{21}$$

with $M \sim N + \Delta + K_S$, $h^0(M) \geq 2$, $M.N = 3$ and $\Delta$ a nodal cycle. The ampleness of $L$ implies $0 < N.L = 3 + N^2$, whence $N^2 \geq -2$.

If $N^2 = -2$, then $M^2 = 6$ with $M.L = 9 = 3\phi(L)$. Therefore, using Lemma 2.2 successively, $M \sim E_1 + E_2 + E_3$ with $E_i^2 = 0$, $E_i.L = 3$, $E_i.E_j = 1$ for $i \neq j$ and each $E_i$ is nef, as $L$ is ample. In particular $E_i.N = E_i.L - E_i.M = 1$ for all $i$. Twisting (21) with $\mathcal{O}_S(-E_1)$ we obtain $h^0(\mathcal{E} - E_1) \geq h^0(M - E_1) + \chi(N - E_1) = 2 + \frac{1}{2}(N - E_1)^2 + 1 = 1$. Hence, by Lemma 4.2(ii), we can in fact find new line bundles $N'$ and $M'$ as in (21) with $N' \geq E_1$, in particular with $N'.L \geq E_1.L = 3$, so that $N'^2 \geq 0$.

We can therefore assume that $N^2 \geq 0$. As $h^0(M) \geq 2$, we must by [KLM, Lemma 4.14] have $6 = 2\phi(L) \leq M.L = 3 + M^2$, whence $M^2 = 4$ and $N^2 = 0$. We have that $N$ is nef with $h^0(N) = 1$, as $N.L = \phi(L) = 3$.

If $M$ were not nef, then there would be a nodal curve $R$ with $R.M = R.(N + \Delta) < 0$, whence $R.N \geq 2$ by the ampleness of $L$, so that $R.\Delta \leq -3$. It follows that $(\Delta - R)^2 \geq 2$ with $\Delta - R > 0$ and $(\Delta - R).L \leq 3$, a contradiction. Therefore $M$ is nef.

As $M.L = 7$ we have $\phi(M) = 2$. Indeed, if $\phi(M) = 1$, we would have, using Lemma 2.2 successively, $M \sim 2E_1 + E_2$ with $E_1$ and $E_2$ isotropic such that $E_1.E_2 = 1$, so that $M.L \geq 3\phi(L) = 9$. It follows that $|M|$ is base point free.

Pick a general, smooth $D \in |M|$ and consider

$$0 \longrightarrow -\Delta - K_S \longrightarrow N \longrightarrow \mathcal{O}_D(N) \longrightarrow 0.$$

Then, as $h^1(N) = 0$, since $h^0(N) = 1$, and $h^1(-\Delta + K_S) = h^1(\Delta) = 1$, we have that $|\mathcal{O}_D(N)|$ is a $g^1_3$, which is necessarily base point free, as $D$ is nonhyperelliptic by [Ve]. Let now $Z \in |\mathcal{O}_D(N)|$ be a general element, lying outside $N \cap D$. From

$$0 \longrightarrow N \longrightarrow L \otimes \mathcal{J}_Z \longrightarrow \mathcal{O}_D(M) \longrightarrow 0$$

and the fact that $h^1(N) = 0$ and $\mathcal{O}_D(M)$ is base point free, we see that the base locus of $|L \otimes \mathcal{J}_Z|$, off $Z$, is contained in $N$ and does not intersect $D$. Since $Z \cap N = \emptyset$, we have that $|L \otimes \mathcal{J}_Z|$ is base component free, whence its general member is irreducible. As $N \cup D \in |L \otimes \mathcal{J}_Z|$, the singularities of the general member of $|L \otimes \mathcal{J}_Z|$ can only lie in $\operatorname{Sing}(N \cup D) - (N \cap D)$, but this is equal to $\operatorname{Sing} N$. If now $x$ is a singular point of some general $C \in |L \otimes \mathcal{J}_Z|$, we must therefore have

$$N.L \geq \operatorname{mult}_x N \cdot \operatorname{mult}_x C \geq 4,$$

a contradiction.

This shows that the general element of $|L \otimes \mathcal{J}_Z|$ is smooth. By Lemma 3.1, we have $h^1(L \otimes \mathcal{J}_Z) \neq 0$, whence by (13) we have $h^0(\mathcal{O}_C(Z)) = h^1(\omega_C - Z) \geq 2$ and $C$ has gonality $\leq 3$, in fact gonality 3, as (i)-(iv) in Theorem 1.1 hold for $k = 1$.

Now both Theorem 1.1 and Proposition 1.2 have been proved.

**Remark 5.1.** The particular polarization given by $L = 2E + \Delta + K_S$ appearing above is called a *Reye polarization* [CV, DR].

### 5.4 Proof of (5)

We first need:

**Lemma 5.2.** *Let $L > 0$ be a line bundle on an Enriques surface with $L^2 \geq 8$ and $E$ such that $E^2 = 0$ and $E.L = \phi(L)$. Let $\beta(L)$, $\mu(L)$ and $\phi_i(L, E)$ be as defined in the introduction.*

*(a) If $\mu(L) - 2 < 2\phi(L)$, then $\mu(L) + 2 = \phi(L) + \phi_2(L, E)$.*

*(b) If $\beta(L) < 2\phi(L)$, then $\beta(L) + 2 \geq \phi(L) + \phi_i(L, E)$ for $i = 1$ or $2$.*

*Proof.* (a) Let $B$ satisfy $B^2 = 4$, $\phi(B) = 2$, $B \not\equiv L$ and $B.L = \mu(L) + 2$. By assumption, $B.L \leq 2\phi(L) + 3$ and by using Lemma 2.2, we can write $B \sim E_1 + E_2$ with $E_1^2 = E_2^2 = 0$ and $E_1.E_2 = 2$. We can assume that $E_1.L \leq \phi(L) + 1$ and $E_2.L \leq \phi(L) + 2$.

We want to find an $F$ such that $F.E = 2$ and $F.L \leq \mu(L) + 2 - \phi(L)$.

If $E.E_i = 2$ for $i = 1$ or $2$, then we are done with $F = E_i$.

If $E.E_i \geq 3$ for $i = 1$ or $2$, then $(E + E_i)^2 \geq 6$. Using Lemma 2.2 successively we can write an effective decomposition of $E + E_i$ containing at least three components of square zero. Hence $3\phi(L) \leq (E + E_i).L \leq 2\phi(L) + 2$, so that $\phi(L) \leq 2$ and $(E + E_i).L \leq 6$. But then $(E + E_i)^2 L^2 \geq 48 > 36 \geq ((E + E_i).L)^2$, contradicting the Hodge index theorem.

The remaining possibility is that $E.B = 2$, and in this case we are done with $F := B - E$.

(b) Let $B$ satisfy $B^2 = 2$ and $B.L = \beta(L) + 2$. By assumption, $B.L \leq 2\phi(L) + 1$ and by using Lemma 2.2, we can write $B \sim E_1 + E_2$ with $E_1^2 = E_2^2 = 0$ and $E_1.E_2 = 1$. We can assume that $E_1.L = \phi(L)$ and $E_2.L \leq \phi(L) + 1$.

We want to find an $F$ such that $F.E = 1$ or $2$ and $F.E \leq \beta(L) + 2 - \phi(L)$. If $E.E_i \leq 2$ for $i = 1$ or $2$, then we are done with $F = E_i$.

If $E.E_i \geq 3$ for $i = 1$ or $2$, we reach the same contradiction as in (a). ∎

We now prove (5). So assume $L \in \operatorname{Pic} S$ with $L^2 \geq 8$ and $L > 0$ and pick any $E$ such that $E^2 = 0$ and $E.L = \phi(L)$.

If $F$ satisfies $F^2 = 0$ and $F.E = i$, for $i = 1$ or $2$, then $(E + F)^2 = 2i$, $\phi(E + F) = i$ and $(E + F).L = \phi(L) + F.L$. In particular we have that $\phi(L) + \phi_1(L, E) \geq \beta(L) + 2$ and $\phi(L) + \phi_2(L, E) \geq \mu(L) + 2$. Therefore

$$\min\{2\phi(L), \beta(L), \mu(L) - 2\} \leq \min\{2\phi(L), \phi(L) + \phi_1(L, E) - 2, \phi(L) + \phi_2(L, E) - 4\}$$

At the same time, Lemma 5.2 yields the opposite inequality. Thus, (5) is proved.

## 6 On the variation of the gonality

First we recall:

**Theorem 6.1.** *([KL2, Thm. 1]) Let $|L|$ be a base-component free complete linear system on an Enriques surface $S$ such that $L^2 > 0$. Then, for a general $C \in |L|$, we have*

$$\operatorname{gon}(C) = \min\{2\phi(L), \mu(L), \lfloor \frac{L^2}{4} \rfloor + 2\}.$$

We have

**Proposition 6.2.** *Let $L$ be an ample, globally generated line bundle on an Enriques surface $S$ such that $L^2 \geq 10$. Let $d_{gen}$ be the gonality of the general smooth curve in $|L|$ and $d_{min}$ be the minimal gonality of a smooth curve in $|L|$. Let $E$ be such that $E^2 = 0$ and $E.L = \phi(L)$.*

*If $(L^2, \phi(L)) \neq (10,3), (12,2)$, then*

$$d_{min} = \phi(L) + \min\{\phi(L), \phi_1(L,E) - 2, \phi_2(L,E) - 4\}$$

*and*

$$d_{gen} = \phi(L) + \min\{\phi(L), \phi_2(L,E) - 2\},$$

*except for the following cases, where $d_{gen} = \lfloor \frac{L^2}{4} \rfloor + 2 = d_{min} + 1$:*

$$(L^2, \phi(L)) \in \{(12,3), (14,3), (18,4), (20,4), (22,4), (30,5)\}. \tag{22}$$

*If $(L^2, \phi(L)) = (12,2)$, then $d_{gen} = d_{min} = 4$.*

*If $(L^2, \phi(L)) = (10,3)$, then $d_{gen} = 4$, and $d_{min} = 3$ if $L \sim 2E + \Delta + K_S$, with $E$ a halfpencil and $\Delta$ a nodal cycle such that $\Delta.E = 3$ and $d_{min} = 4$ otherwise.*

*Proof.* It is easy to check that $2\phi(L) \leq \lfloor \frac{L^2}{4} \rfloor + 2$ except for the special pairs in (22) and $(L^2, \phi(L)) = (10,3)$, where $d_{gen} = \lfloor \frac{L^2}{4} \rfloor + 2$, by [KL2, Cor. 1]. Using the decompositions in the proof of Lemma 2.3 one checks that the minimal gonality of a curve in $|L|$ is $d_{min} = d_{gen} - 1 = \min\{\beta(L), \mu(L) - 2\} < 2\phi(L)$ in all these cases, by Proposition 1.2, except in the case $(L^2, \phi(L)) = (10,3)$ and $L$ is not of the special form $L \sim 2E + \Delta + K_S$ in Proposition 1.2.

Therefore, the proposition is proved in all these cases.

In the remaining cases the result follows from Proposition 1.2, Lemma 5.2 and Theorem 6.1. ∎

In particular, as $\phi_1(L,E) \geq \phi(L)$, we obtain:

**Corollary 6.3.** *Let $L$ be an ample, globally generated line bundle on an Enriques surface $S$ such that $L^2 \geq 10$. Let $d_{gen}$ be the gonality of the general smooth curve in $|L|$ and $d_{min}$ be the minimal gonality of a smooth curve in $|L|$. Then*

$$d_{min} \geq d_{gen} - 2.$$

This result was also proved in [KL2, Cor. 2] without the ampleness assumption and assuming only $L^2 > 0$, although we did not find the exact value for the minimal gonality there. We will see in Examples 6.5-6.7 that all cases $d_{min} = d_{gen} - 2$, $d_{gen} - 1$ and $d_{gen}$ actually occur.

Another immediate corollary of Proposition 6.2 is the following criterion for the constancy of the gonality:

**Corollary 6.4.** *Let $L$ be an ample, globally generated line bundle on an Enriques surface $S$ such that $L^2 \geq 10$.*

*Then, all the smooth curves in $|L|$ have the same gonality if and only if*

$$\phi_1(L,E) \geq \phi(L) + 2 \;\text{ and }\; \phi_2(L,E) \geq \phi(L) + 3$$

*or $(L^2, \phi(L)) = (12,2)$, in which case the gonality is precisely $2\phi(L)$; or $(L^2, \phi(L)) = (10,3)$ and $L$ is not of the special form in Proposition 1.2, in which case the gonality is 4.*

In particular, we see that if the gonality is constant and $(L^2, \phi(L)) \neq (10, 3)$, then there is a unique numerical equivalence class $[E] \in \mathrm{Num}(S)$ such that $E.L = \phi(L)$, and $F.L \geq \phi(L) + 2$ for any $F > 0$ such that $F \not\equiv E$ and $F^2 = 0$.

We conclude this note with some examples. To this end, recall from [KL2, Lemma 2.14] that if $L > 0$ is any line bundle on an Enriques surface with $L^2 \geq 0$, then there is an integer $n$ such that $1 \leq n \leq 10$ and, for every $i = 1, \ldots, n$, there are primitive divisors $E_i > 0$ with $E_i^2 = 0$ and integers $a_i > 0$ such that

$$L \equiv a_1 E_1 + \cdots + a_n E_n$$

and one of the three following intersection sets occurs:

(i) $E_i.E_j = 1$ for $1 \leq i < j \leq n$.

(ii) $n \geq 2$, $E_1.E_2 = 2$ and $E_i.E_j = 1$ for $2 \leq i < j \leq n$ and for $i = 1, 3 \leq j \leq n$.

(iii) $n \geq 3$, $E_1.E_2 = E_1.E_3 = 2$ and $E_i.E_j = 1$ for $3 \leq i < j \leq n$, for $i = 1$, $4 \leq j \leq n$ and for $i = 2, 3 \leq j \leq n$.

This way of writing a decomposition of any $L$ is useful in order to explicitly compute the general gonality and minimal gonality of the smooth curves in $|L|$.

**Example 6.5.** Let $L$ be an ample, globally generated line bundle on an Enriques surface $S$ such that $L^2 \geq 10$ and such that $L$ is not as in (22) and $(L^2, \phi(L)) \neq (12, 2)$. Let $d_{gen}$ be the gonality of the general smooth curve in $|L|$ and $d_{min}$ be the minimal gonality of a smooth curve in $|L|$.

Assume that $L$ has a decomposition as in (i) above, that is,

$$L \equiv a_1 E_1 + \cdots + a_n E_n,$$

with $n \leq 10$ and all $E_i.E_j = 1$ for $i \neq j$.

We can assume that $a_1 \geq \cdots \geq a_n$. Set $a := a_1 + \cdots + a_n$.

Now one easily computes:

$$\begin{aligned} \phi(L) &= E_1.L = a - a_1, \\ \phi_1(L, E_1) &= E_2.L = \phi(L) + a_1 - a_2. \end{aligned}$$

Moreover, we have $\phi_2(L, E_1) \geq 2a_1 + a_2 + \cdots + a_n = \phi(L) + 2a_1$, so that $\phi_2(L, E_1) \geq \phi(L) + 4$, unless possibly if $a_1 = 1$, in which case $a_2 = \cdots = a_n = 1$, so that $\phi_1(L, E_1) = \phi(L)$ and $\phi_2(L, E_1) \geq \phi(L) + 2$.

It follows from Proposition 6.2 that

$$d_{gen} = 2\phi(L) = 2(a - a_1),$$

and

$$d_{min} = \begin{cases} 2\phi(L) & \text{if } a_1 \geq a_2 + 2; \\ 2\phi(L) - 1 & \text{if } a_1 = a_2 + 1; \\ 2\phi(L) - 2 & \text{if } a_1 = a_2. \end{cases}$$

**Example 6.6.** Let $L$ be an ample, globally generated line bundle on an Enriques surface $S$ such that $L^2 > 10$ and such that $L$ is not as in (22). Let $d_{gen}$ be the gonality of the general smooth curve in $|L|$ and $d_{min}$ be the minimal gonality of a smooth curve in $|L|$.

Assume that $L$ has a decomposition as in (ii) above, that is,

$$L \equiv a_1E_1 + \cdots + a_nE_n,$$

with $n \leq 10$ and all $E_i.E_j = 1$ for $i \neq j$ except $E_1.E_2 = 2$.

We can assume that $a_1 \geq a_2$ and $a_3 \geq \cdots \geq a_n$. Set $a := a_1 + \cdots + a_n$.

One easily computes

$$\phi(L) = \min\{E_1.L = a + a_2 - a_1, E_3.L = a - a_3\}.$$

**Case I:** $a_1 - a_2 \geq a_3$**.**

We have $\phi(L) = E_1.L = a + a_2 - a_1$ and one easily finds:

$$\phi_1(L, E_1) = E_3.L = a - a_3 \text{ and } \phi_2(L, E_1) = E_2.L = a + a_1 - a_2.$$

Hence, by Proposition 6.2 we have

$$d_{gen} = 2a + \min\{2(a_2 - a_1), -2\}$$

and

$$d_{min} = 2a + \min\{2(a_2 - a_1), a_2 - a_1 - a_3 - 2, -4\}.$$

**Case II:** $a_1 - a_2 \leq a_3 - 1$**.**

We must have $a_3 > 0$. We have $\phi(L) = E_3.L = a - a_3$ and one easily finds:

$$\phi_1(L, E_3) = E_1.L = a + a_2 - a_1.$$

If $F > 0$ satisfies $F^2 = 0$, $F.E_3 = 2$ and $F.L \leq \phi(L) + 3$, then $F \not\equiv E_i$ for all $i = 1, \ldots, n$, whence

$$a + a_3 \leq F.L \leq a - a_3 + 3,$$

so that $a_3 = 1$. It follows that $a_4 = \cdots = a_n = 1$ and $a_1 = a_2$. In this case one finds $(E_1 + E_2 - E_3).L = a + 1$, $(E_1 + E_2 - E_3)^2 = 0$ and $(E_1 + E_2 - E_3).E_3 = 2$, so that $\phi_2(L, E_3) = a + 1$.

Therefore, we have proved that

$$\phi_2(L, E_3) \geq \phi(L) + 4 \text{ unless } a_1 = a_2,\ a_3 = \cdots = a_n = 1, \quad \text{in which case } \phi_2(L, E_3) = a + 1.$$

Hence, by Proposition 6.2 we have $d_{gen} = 2(a - a_3)$ and

$$d_{min} = 2a + \min\{-2a_3, a_2 - a_1 - a_3 - 2\},$$

unless $a_1 = a_2$ and $a_3 = \cdots = a_n = 1$, in which case $d_{min} = 2a - 4$.

We leave it to the interested reader to explicitly work out the case when $L$ has a decomposition as in (iii) above, in the same way as in the two last examples. We will restrict ourselves to the following particular case:

**Example 6.7.** Let $L$ be an ample, globally generated line bundle on an Enriques surface $S$ such that $L^2 \geq 10$. Let $d_{gen}$ be the gonality of the general smooth curve in $|L|$ and $d_{min}$ be the minimal gonality of a smooth curve in $|L|$.

By [KL2, Def. 2.9, Prop. 2.10], we have $\mu(L) < 2\phi(L)$ if and only if $L$ is of one of the following three types, where $E_1, E_2, E_3$ are primitive such that $E_i > 0$, $E_i^2 = 0, i = 1, 2, 3, E_1.E_2 = E_1.E_3 = 2$ and $E_2.E_3 = 1$:

($\mu_1$) $L \equiv h(E_1 + E_2), h \geq 3$;

($\mu_2$) $L \sim h(E_1 + E_2) + E_3, h \geq 1$;

($\mu_3$) $L \sim (h+1)E_1 + hE_2 + E_3, h \geq 1$.

In each of these cases one easily computes:

($\mu_1$) $\phi(L) = E_1.L = E_2.L = 2h, \phi_1(L, E_1) \geq 2h, \phi_2(L, E_1) = E_2.L = 2h$;

($\mu_2$) $\phi(L) = E_2.L = 2h+1, \phi_1(L, E_2) \geq 2h+1, \phi_2(L, E_2) = E_1.L = 2h+2$;

($\mu_3$) $\phi(L) = E_1.L = 2h+2, \phi_1(L, E_1) \geq 2h+2, \phi_2(L, E_1) = E_2.L = 2h+3$.

Hence, by Proposition 6.2 we have:

($\mu_1$) $d_{gen} = 4h - 2, d_{min} = 4h - 4$;

($\mu_2$) $d_{gen} = 4h + 1, d_{min} = 4h - 1$;

($\mu_3$) $d_{gen} = 4h + 3, d_{min} = 4h + 1$.

## References

[Ba-So] E. Ballico, A. J. Sommese, *Projective surfaces with k-very ample line bundles of degree* $\leq 4k+4$, Nagoya Math. J. **136** (1994), 57-79.

[BPV] W. Barth, C. Peters, A. van de Ven, *Compact complex surfaces*. Ergebnisse der Mathematik und ihrer Grenzgebiete **4**. Springer-Verlag, Berlin-New York, 1984.

[BFS] M. Beltrametti, P. Francia, A. J. Sommese, *On Reider's method and higher order embeddings*, Duke Math. J. **58** (1989), 425–439.

[BS1] M. Beltrametti, A. J. Sommese, *On k-spannedness for projective surfaces*. In: Sommese, A. J. et. al. (eds.). Hyperplane sections, L'Aquila Proceedings 1988, (Springer LNM **1417**, pp. 24–51), Berlin–Heidelberg–New York: Springer 1990

[BS2] M. Beltrametti, A. J. Sommese, *On the preservation of k-very ampleness under adjunction*, Math. Z. **212** (1993), 257–283.

[BS3] M. Beltrametti, A. J. Sommese, *The Adjunction Theory of Complex Projective Varieties*. De Gruyter Expositions in Mathematics **16**, Berlin: Walter de Gruyter, 1995.

[BS4] M. Beltrametti, A. J. Sommese, *Zero cycles and k-th order embeddings of smooth projective surfaces*. In: Projective surfaces and their classification, Cortona proceedings Symp. Math. INDAM **32**, pp. 33–48, New York, London: Academic press, 1988.

[C-G] F. Catanese, L. Göttsche, *d-very ample line bundles and embeddings of Hilbert schemes of* 0*-cycles*, Manuscr. Math. **78** (1990), 193-212.

[CV] A. Conte, A. Verra, *Reye constructions for nodal Enriques surfaces*, Trans. Amer. Math. Soc. **336** (1993), 79–100.

[CD] F. Cossec, I. Dolgachev, *Enriques Surfaces I*. Progress in Mathematics **76**. Birkhäuser Boston, MA, 1989.

[Co1] F. R. Cossec, *Projective models of Enriques surfaces*, Math. Ann. **265** (1983), 283–334.

[Co2] F. R. Cossec, *On the Picard group of Enriques surfaces*, Math. Ann. **271** (1985), 577–600.

[CP] C. Ciliberto, G. Pareschi, *Pencils of minimal degree on curves on a K*3 *surface*, J. Reine Angew. Math. **460** (1995), 15–36.

[DR] I. Dolgachev, I. Reider, *On rank* 2 *vector bundles with* $c_1^2 = 10$ *and* $c_2 = 3$ *on Enriques surfaces*, Algebraic geometry (Chicago, IL, 1989), 39–49, Lecture Notes in Math. **1479**, Springer, Berlin, 1991.

[DM] R. Donagi, D. R. Morrison, *Linear systems on K*3*-sections*, J. Diff. Geom. **29** (1989), 49–64.

[F] R. Friedman, *Algebraic surfaces and holomorphic vector bundles*. Universitext. Springer-Verlag, New York, 1998.

[GL] M. Green, R. Lazarsfeld, *Special divisors on curves on a K*3 *surface*, Invent. Math. **89** (1987), 357–370.

[GLM1] L. Giraldo, A. F. Lopez, R. Muñoz, *On the projective normality of Enriques surfaces*, Math. Ann. **324** (2002), 135–158.

[Ha] T. Harui, *The gonality and the Clifford index of curves on an elliptic ruled surface*, Arch. Math. **84** (2005), 131–147

[Ki] H. Kim, *Exceptional bundles on nodal Enriques surfaces*, Manuscr. Math. **82** (1994), 1–13.

[Kn1] A. L. Knutsen, *Higher order birational embeddings of Del Pezzo surfaces*, Math. Nachr. **254-255** (2003), 183–196.

[Kn2] A. L. Knutsen, *Exceptional curves on Del Pezzo surfaces*, Math. Nachr. **256** (2003), 58–81.

[Kn3] A. L. Knutsen, *On kth order embeddings of K*3 *surfaces and Enriques surfaces*, Manuscr. Math. **104** (2001), 211–237.

[KL1] A. L. Knutsen, A. F. Lopez, *A sharp vanishing theorem for line bundles on K3 or Enriques surfaces*, Proc. Am. Math. Soc. **135**, 3495-3498 (2007).

[KL2] A. L. Knutsen, A. F. Lopez, *Brill-Noether theory of curves on Enriques surfaces, I: the positive cone and gonality*, Math. Z. **261** (2009), 659-690.

[KLM] A. L. Knutsen, A. F. Lopez, R. Muñoz, *On the extendability of projective surfaces and a genus bound for Enriques-Fano threefolds*. Preprint 2006, available at math.AG/0605750.

[La] R. Lazarsfeld, *Brill-Noether-Petri without degenerations*, J. Diff. Geom. **23** (1986), 299–307.

[Ma] G. Martens, *The gonality of curves on a Hirzebruch surface*, Arch. Math. (Basel) **67** (1996), 349–352,

[Par] G. Pareschi, *Exceptional linear systems on curves on Del Pezzo surfaces*, Math. Ann. **291** (1991), 17–38.

[R] M. Reid. *Special linear systems on curves lying on a K3 surface*, J. London Math. Soc. **13** (1976), 454–458.

[Re] I. Reider, *Vector bundles of rank* 2 *and linear systems on algebraic surfaces*, Ann. of Math. **127** (1988), 309–316.

[SD] B. Saint-Donat, *Projective models of* $K-3$ *surfaces*, Amer. J. Math., **96** (1974), 602–639.

[Te] H. Terakawa, *Higher order embeddings of algebraic surfaces of Kodaira dimension zero*, Math. Z. **229** (1998), 417-433.

[Ty] A. N. Tyurin, *Cycles, curves and vector bundles on an algebraic surface*, Duke Math. J. **54** (1987), 1–26.

[Ve] A. Verra, *On Enriques surface as a fourfold cover of* $\mathbb{P}^2$, Math. Ann. **266** (1983), 241–250.

Matematisk institutt, Universitetet i Bergen,
Johannes Brunsgate 12, 5008 Bergen, Norway.
e-mail :`andreas.knutsen@math.uib.no`