Bound on Seshadri constants on $\mathbb{P}^1 \times \mathbb{P}^1$

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Abstract

In the note we give an uniform bound for the multiple point Seshadri constants on $\mathbb{P}^1 \times \mathbb{P}^1$

1 Introduction

Let X be a projective algebraic surface (over $\mathbb{C}$) with an ample line bundle L. Let $P_1, ..., P_r$ be r different points on X. Let us recall the definition introduced by Demailly in [4].

Definition 1. *The Seshadri constant of L in $P_1, ..., P_r$ is defined as the number*

$$\varepsilon(L, P_1, ..., P_r) := inf\,\{\frac{LC}{mult_{P_1}C + ... + mult_{P_r}C} \mid C \text{ is a curve on } X\},$$

or, equivalently

$$\varepsilon(L, P_1, ..., P_r) := \ sup\ \{\varepsilon \mid \pi^* L - \varepsilon(E_1 + ... + E_r) \text{ is numerically effective}\},$$

where $\pi : \tilde{X} \longrightarrow X$ is the blow-up of X in $P_1, ..., P_r$.

Remark 2. *It follows from the definition that for an ample line bundle L on X*

$$0 < \varepsilon(L, P_1, ..., P_r) \leq \sqrt{\frac{L^2}{r}}.$$

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Notation:
For $P_1, ..., P_r$ generic on X we will write $\varepsilon(L, r)$ instead of $\varepsilon(L, P_1, ..., P_r)$.

Finding the exact value of these constants is in general a difficult problem. For example, for $\mathbb{P}^2$ with $L = \mathcal{O}_{\mathbb{P}^2}(1)$ the exact values of $\varepsilon(L, r)$ are known only if $r \leq 9$ or $r = k^2, k \in \mathbb{N}$. The famous Nagata Conjecture (cf [11]) states that $\varepsilon(\mathcal{O}_{\mathbb{P}^2}(1), r) = \sqrt{\frac{1}{r}}$ for $r \geq 10$. This problem is still open (see [7] for more about the subject). Moreover, all known values of Seshadri constants on algebraic surfaces are rational. In general, it is hard to find the value of a Seshadri constant even in one point. The interested reader may look for example in [1], [2], [5], [10], [14] and the references therein.

In his recent paper, [6], Fuentes Garcia investigated the Seshadri constants in one point on geometrically ruled surfaces, in case of ruled surfaces with the invariant $e > 0$ he computes $\varepsilon(A, x)$ explicitly, whereas for surfaces with $e \leq 0$ he either gives the exact value of ε or bounds for its value, depending on the position of the point on the surface.
On the other hand Syzdek in [13] studied the existence of so called Seshadri submaximal curves on $\mathbb{P}^1 \times \mathbb{P}^1$ with different polarizations L.

Definition 3. *1. A Seshadri submaximal curve is a curve C on X, such that*

$$\frac{LC}{mult_{P_1}C + ... + mult_{P_r}C} < \sqrt{\frac{L^2}{r}}.$$

2. A curve C in a linear system $|L|$ on a surface, passing through r points with multiplicities $m_1, ..., m_r$ is Riemann-Roch expected if

$$h^0(L) - \sum_{i=1}^{r} \binom{m_i + 1}{2} \geq 1.$$

Of course, when the Riemann-Roch theorem implies the existence of a submaximal curve, then the Seshadri constant is rational and less then $\sqrt{\frac{L^2}{r}}$. This follows from the fact that there is a finite number of Seshadri submaximal curves on a surface, see for example [12].

Syzdek in [13] gave a list of the Riemann-Roch expected submaximal curves on $\mathbb{P}^1 \times \mathbb{P}^1$. She also proved that there exists a number R_0 (depending on the type of the polarization), such that for $r \geq R_0$, there are no Riemann-Roch expected submaximal curves on $\mathbb{P}^1 \times \mathbb{P}^1$. In particular, she shows that for $r \leq 8$, a Riemann-Roch expected submaximal curve always exists.

In this note we give a uniform lower bound for the Seshadri constant on $\mathbb{P}^1 \times \mathbb{P}^1$, in case r is such, that there are no Riemann-Roch expected submaximal curves on $\mathbb{P}^1 \times \mathbb{P}^1$, so $\varepsilon(L, r)$ is "expected" to be maximal (ie equal to $\sqrt{\frac{L^2}{r}}$). We prove the following theorem.

Theorem 4. *Let L be a line bundle in $\mathbb{P}^1 \times \mathbb{P}^1$, of type (α, β). Let r be such, that there exist no Riemann-Roch expected submaximal curves on $(\mathbb{P}^1 \times \mathbb{P}^1, L)$. Then*

$$\varepsilon(L, r) \geq \sqrt{\frac{2\alpha\beta}{r + \frac{1}{2}}}.$$

2 Useful facts

Lemma 5. (See [3]). *Let L be a line bundle on $\mathbb{P}^1 \times \mathbb{P}^1$, of type (a, b). Assume that there exists $C \in |L|$, a reduced and irreducible curve on $\mathbb{P}^1 \times \mathbb{P}^1$, passing through points $P_1, ..., P_r$ with multiplicities $m_1, ..., m_r$, where $m_j > 0, j = 1, ..., r$. Then, for any chosen m_j, there exists a reduced and irreducible curve on $\mathbb{P}^2$ of degree $d = a + b - m_j$, passing through $r + 1$ points on $\mathbb{P}^2$ with multiplicities $a - m_j, b - m_j, m_1, ..., m_{j-1}, m_{j+1}, ..., m_r$.*

Definition 6. *A curve D on a surface X, passing through points $P_1, ..., P_r$ with multiplicities $m_1, ..., m_r$, is called almost homogeneous if all but at most one m_j are equal.*

Lemma 7. (See [13], Proposition 2.10.). *Let (X, L) be a polarized surface with Picard number ϱ. Let $P_1, ..., P_r$ be general points on X. If $\varrho = 1$ or $\varrho = 2$, then any reduced and irreducible Seshadri submaximal curve on X is almost homogeneous.*

Lemma 8. (See [15], Lemma 1). *Let C be a reduced and irreducible curve on a surface X, passing through a general point $P \in X$ with multiplicity $m \geq 2$. Then*

$$C^2 \geq m^2 - m + 1.$$

Lemma 9. (See [16]). *Let C be a reduced and irreducible curve of degree d on $\mathbb{P}^2$, passing through the general points $P_1, ..., P_r$ with multiplicities $m_1, ..., m_r$. Then*

$$d^2 \geq \sum_{j=1}^{r} m_j^2 - m_q,$$

for any q such that $m_q > 0$.

3 Proof

To prove Theorem 4 we have to exclude the existence of (reduced and irreducible) Seshadri submaximal curves C on $\mathbb{P}^1 \times \mathbb{P}^1$, passing through r general points with multiplicities $m_1, ..., m_r$ and satisfying

$$\frac{LC}{m_1 + ... + m_r} < \sqrt{\frac{2\alpha\beta}{r + \frac{1}{2}}}. \tag{1}$$

Suppose, that such a curve C exists (and is not Riemann-Roch expected). Let C be of type (a, b).

First, observe that we may exclude the situation $a = 0$ or $b = 0$. Indeed, suppose for example that $b = 0$. Then, as the curve is reduced and irreducible, it must be $a = 1$, so $m = 1$, and such a curve is always Riemann-Roch expected.

Then, observe that Lemma 7 implies that it is enough to exclude the existence of almost homogeneous curves, satisfying inequality (1). So, let us assume that C is of type (a,b) on $\mathbb{P}^1 \times \mathbb{P}^1$, with $a > 0, b > 0$, passing through general points $P_1, ..., P_r$ with multiplicity m in $P_1, ..., P_{r-1}$ and with multiplicity μ in P_r. We will denote such a curve by $((a,b); m^{\times(r-1)}, \mu)$ and we will write $a + b =: c$.

From Lemma 5 it follows that we may move our considerations to $\mathbb{P}^2$, considering instead of C the curve on $\mathbb{P}^2$ (in what follows also denoted by C), of degree $c - m$ and with multiplicities $a - m, b - m, m^{\times(r-2)}, \mu$ in $r + 1$ general points. We will denote such a curve by $((c - m); a - m, b - m, m^{\times(r-2)}, \mu)$.

Let us assume L is of type (α, β). We have to exclude the existence of an almost homogeneous submaximal curve on $\mathbb{P}^1 \times \mathbb{P}^1$, such that

$$C = ((a,b); m^{r-1}, \mu),$$

so we have to exclude the existence of a curve C, such that:
1. On $\mathbb{P}^2$: C is numerically equivalent to $((c - m); a - m, b - m, m^{\times(r-2)}, \mu)$
2. On $\mathbb{P}^1 \times \mathbb{P}^1$: $\dfrac{LC}{(r-1)m + \mu} < \sqrt{\dfrac{2\alpha\beta}{r + 0.5}}$.

Then, if

$$LC < \sqrt{\frac{2\alpha\beta}{r + 0.5}}((r-1)m + \mu), \tag{2}$$

we have

$$2\sqrt{ab\alpha\beta} \leq \alpha b + \beta a \leq \sqrt{\frac{2\alpha\beta}{r + 0.5}}((r-1)m + \mu) = 2\sqrt{\frac{\alpha\beta}{2r + 1}}((r-1)m + \mu), \tag{3}$$

which gives

$$\sqrt{ab} < \sqrt{\frac{1}{2r + 1}}\left((r-1)m + \mu\right). \tag{4}$$

Thus, our aim is to prove that there are no curves on $\mathbb{P}^2$, satisfying $C \equiv ((c - m); a - m, b - m, m^{\times(r-2)}, \mu)$ and $\sqrt{ab} < \frac{1}{\sqrt{2r+1}}\left((r-1)m + \mu\right)$.

Let us now assume that $m \geq 2$. Consider the curve

$$\tilde{C} = (c - m)H - (a - m)E_1 - (b - m)E_2 - mE_3 - ... - mE_{r-1} - \mu E_{r+1}$$

on the blow up of $\mathbb{P}^2$ in $P_1, ..., P_{r-1}, P_{r+1}$, passing with multiplicity m through P_r. To this curve we may apply Lemma 8. We get

$$(\tilde{C})^2 = (a + b - m)^2 - (a - m)^2 - (b - m)^2 - (r - 3)m^2 - \mu^2 \geq m^2 - m + 1,$$

which gives

$$2ab - (r-1)m^2 - \mu^2 + m - 1 \geq 0. \tag{5}$$

Together with the inequality (4), we get

$$\frac{2}{2r + 1}((r-1)m + \mu)^2 - (r-1)m^2 - \mu^2 + m - 1 \geq 0. \tag{6}$$

Assuming first, that $m = 2$, from the above inequality we obtain

$$-(2r-1)\mu^2 + 8(r-1)\mu - 10r + 13 \geq 0. \tag{7}$$

This is a quadratic inequality with respect to μ, with a negative leading term coefficient. It is easy to check that the discriminant of the quadratic function is negative for all $r > 2$, so there are no solutions for the above inequality.

Assume now, that $m \geq 3$, so $m \leq \frac{m^2}{3}$. From inequality (6) after multiplying by $(2r+1)$ we get

$$2(r-1)^2 m^2 + 4(r-1)m\mu + 2\mu^2 - (2r+1)(r-1)m^2 - \\ (2r+1)\mu^2 + (2r+1)(m-1) \geq 0, \tag{8}$$

so, writing $\frac{m^2}{3}$ instead of $m-1$ we get

$$\left(\frac{-7}{3}r + \frac{10}{3}\right) m^2 + 4(r-1)m\mu - (2r-1)\mu^2 \geq 0. \tag{9}$$

Treating this as a quadratic inequality with m as a variable and r, μ as parameters, we see that the m^2-coefficient is negative and the discriminant equals

$$\frac{4}{3}\mu^2(-2r^2 + 3r + 2), \tag{10}$$

so the inequality has no solutions for $r \geq 3$.

Now assume that $m = 1$ and $\mu \geq 2$. Then inequality (4) becomes

$$\sqrt{ab} < \frac{1}{\sqrt{2r+1}}(r - 1 + \mu). \tag{11}$$

Apply Lemma 8 to the curve $((c-1); a-1, b-1, 1^{\times(r-2)})$ passing through the last point with multiplicity μ. We get

$$(a+b-1)^2 - (a-1)^2 + (b-1)^2 - (r-2) - \mu^2 + \mu - 1 \geq 0, \tag{12}$$

equivalent to

$$2ab - r - \mu^2 + \mu \geq 0. \tag{13}$$

Using inequality (11) we get

$$\mu^2(1-2r) + \mu(4r-1) - 5r + 2 \geq 0, \tag{14}$$

this gives a contradiction.

Assume now that $m = \mu = 1$. Consider the linear system of curves of degree $c-1$ passing through two points with multiplicities $a-1, b-1$. In [11], Nagata proved that the dimension of such a system is either expected or every curve in the system contains a line. However, our curve C is reduced and irreducible and passes through at least seven more points in general position. Thus, the dimension of the system $((c-1); a-1, b-1)$ is expected. Moreover, the conditions on the system to pass through points in general position with multiplicity one are independent, so each such point causes the dimension of the system to become one less. So, in our case we have a curve $((c-1); a-1, b-1)$ passing through $r-1$ points with multiplicity one. This means that

$$\frac{(a+b-1)(a+b+2)}{2} - \frac{(a-1)a}{2} - \frac{(b-1)b}{2} - (r-1) \geq 0. \tag{15}$$

This implies that

$$ab + a + b - r \geq 0, \tag{16}$$

and this would mean that our curve C on $\mathbb{P}^1 \times \mathbb{P}^1$ is Riemann-Roch expected, contrary to our assumptions.

Remark 10. *Harbourne and Roé showed us that from their Theorem I.2.1.(a) in [9], it follows easily that for $r \geq 3\frac{(\alpha+\beta)^2}{\alpha\beta}$*

$$\varepsilon(L, r) \geq \sqrt{\frac{2\alpha\beta}{r + \frac{r}{2r-5}}}.$$

They suggested as well, that using their method one might be able to improve the result, to get the bound asymptotically $\sqrt{\frac{2\alpha\beta}{r+\frac{1}{3}}}$. This will be the aim of our future project.

Harbourne also pointed out that from his paper [8] applied to our situation, it follows, that if $L^2 r = 2\alpha\beta r$ is a square of a natural number and $r \geq L^2$, then the Seshadri constant $\varepsilon(L, r)$ has maximal possible value, $\varepsilon(L, r) = \sqrt{\frac{L^2}{r}}$.

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References

[1] Bauer, Th.: Seshadri constants on algebraic surfaces. Math. Ann. **313**, (1999) 547-583

[2] Bauer, Th., Szemberg, T.: Seshadri constants on abelian surfaces. (An appendix to: Bauer, Th: Seshadri constants and periods of polarized abelian varieties). Math. Ann. **312**, (1998) 618-623

[3] De Volder, C., Laface, A. On linear systems on $\mathbb{P}^3$ through multiple points. J. Algebra **310** (2007) no. 1, 207-217

[4] Demailly, J.-P.: *Singular Hermitian metrics on positive line bundles.* Lect. Notes Math. 1507, Springer 1992, 87-104

[5] Ein, L., Küchle, O., Lazarsfeld, R.: Local positivity of ample line bundles. J. Diff. Geom. **42**, (1995) 193-219

[6] Garcia, Fuentes L.: Seshadri constants on ruled surfaces: the rational and elliptic cases. manuscripta math. **119**, (2006), 483-505

[7] Harbourne, B.: On Nagata's conjecture. J.Algebra **236** (2001) 692-702

[8] Harbourne, B.: Seshadri constants and very ample divisors on algebraic surfaces J. Reine Angew. Math. **559**, (2003), 115-122

[9] Harbourne, B., Roé, J.: Discrete behaviour of Seshadri constants on surfaces. http://mat.uab.es/~jroe/papers/SeshSurfaces.pdf

[10] Küchle, O.: Multiple point Seshadri constants and the dimension of adjoint linear series. Ann. Inst. Fourier **46**, (1996) 63-71

[11] Nagata, M., On the 14-th problem of Hilbert, Am. J. Math. **81** (1959), 766-772

[12] Syzdek, W.: Nagata submaximal curves on $\mathbb{P}^1 \times \mathbb{P}^1$, Ann. Polon. Math. **80** (2003), 223-230

[13] Syzdek, W.: Seshadri constants and geometry of surfaces, preprint, http://www.uni-essen.de/ mat903/alggeo.html

[14] Szemberg, T.: Global and local positivity of line bundles. Habilitationsshrift Essen 2001

[15] Xu, G.: Ample line bundles on smooth surfaces, J. Reine Angew. Math. **469**, (1995), 199209

[16] Xu, G.: Curves in $\mathbb{P}^2$ and symplectic packings, Math. Ann. **299**, (1994), 609-613

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