# Degenerations of the Veronese and Applications

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## Introduction

The classical *Hermite interpolation* technique approximates a given differentiable function defined on an open subset of $\mathbb{R}^r$ with a polynomial of a given degree which takes at an assigned set of points $p_1, \ldots, p_n$, usually assumed to be sufficiently general, the same values as the function and as its derivatives up to order $m_1, \ldots, m_n$ respectively.

A geometric counterpart of Hermite interpolation is the following. Fix general points $p_1, \ldots, p_n$ in the complex projective space $\mathbb{P}^r$, and multiplicities $m_1, \ldots, m_n$. We will denote by $\mathcal{L} = \mathcal{L}_{r,d}(m_1, \ldots, m_n)$ the linear system of hypersurfaces of degree $d$ in $\mathbb{P}^r$ having multiplicity at least $m_i$ at $p_i$ for each $i = 1, \ldots, n$, and we will employ exponential notation $\mathcal{L}_{r,d}(m_1^{e_1}, \ldots, m_h^{e_h})$ for repeated multiplicities. A basic question is to determine the dimension of the system $\mathcal{L}$. In the 1–dimensional case $r = 1$ this is easy. Ruffini's theorem says that

$$\dim(\mathcal{L}) = \max\{-1, d - \sum_i m_i\}.$$

As soon as $r \geq 2$ this problem becomes very complicated and so far it is still unsolved in this generality. In the present paper will mainly deal with the planar case $r = 2$, and we will write $\mathcal{L}_d(m_1, \ldots, m_n)$ rather than $\mathcal{L}_{2,d}(m_1, \ldots, m_n)$ if there is no ambiguity. Then, the *virtual dimension* of $\mathcal{L}$ is

$$v(\mathcal{L}) = d(d+3)/2 - \sum_i m_i(m_i+1)/2$$

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and the *expected dimension* is

$$e(\mathcal{L}) = \max\{-1, v\}.$$

One has $\dim(\mathcal{L}) \geq e(\mathcal{L})$ and the system $\mathcal{L}$ is called *special* if $\dim(\mathcal{L}) > e(\mathcal{L})$. A special system is not empty.

Consider the blow–up $X$ of the plane at the points $p_1, \dots, p_n$. We abuse notation and denote by $\mathcal{L}$ also the proper transform of $\mathcal{L}$ to $X$. Suppose $\mathcal{L}$ is non empty, and assume that there is a $(-1)$–curve $C$ on $X$ such that $C \cdot \mathcal{L} \leq -2$. This forces $C$ to be a multiple fixed curve in the system and it is easy to see that $\mathcal{L}$ is special in this case. We will then say that $\mathcal{L}$ is $(-1)$–*special*. The Harbourne-Hirshowitz Conjecture (see also Gimigliano [26]) says that a system is special if and only if it is $(-1)$–special (see [15],[27],[28], etc.) Related to this conjecture, but weaker, is Nagata's Conjecture: if $n > 9$ and $d^2 < nm^2$ then $\mathcal{L}_d(m^n)$ is empty (see [30]).

There has been a substantial amount of partial progress on these conjectures; let us briefly recall some of the results. The Harbourne-Hirschowitz Conjecture is true for $n \leq 9$ (Castelnuovo, 1891 [6]; Nagata, 1960 [33]; Gimigliano [26]; Harbourne 1986 [27]). The Harbourne-Hirschowitz Conjecture is true for $m_i \leq 7$ (S. Yang, 2004 [42]; results have been announced for $m_i \leq 11$ by M. Dumnicki and W. Jarnicki, 2005 [20].) The Harbourne-Hirschowitz Conjecture is true for $\mathcal{L}_d(m^n)$ if $m \leq 20$ (see [10]; results have been announced for $m \leq 42$ by M. Dumnicki [21]). Nagata proved in [30] that his conjecture is true for $n = k^2$ points and he deduced from this a counterexample to the fourteenth problem of Hilbert. More specifically, the Harbourne-Hirschowitz Conjecture is true for $n = k^2$ points (see Evain 2005 [23]; also see [16] and [36]).

The results of Ciliberto and Miranda mentioned above (see [13], [14], [15], [16]; see also [10]) have systematically exploited a degeneration of the plane to the union of two surfaces, one of which is a plane and the other a ruled surface. In this work we describe some toric degenerations of the Veronese (see §§2 and 3) which we find useful in similar applications to interpolation results. In particular, using such degenerations, we give in §§4, 6 and 7 a new proof of the following:

**Theorem 0.1.** *The linear systems*

$$\mathcal{L}_d(2^n) \textit{ for } d \geq 5; \quad \mathcal{L}_{km}(m^{k^2}), \quad \mathcal{L}_{km+1}(m^{k^2})$$

*all have the expected dimension.*

These are all known results. In particular the first assertion is the planar instance of a famous theorem of Alexander–Hirschowitz (see [1]). However our purpose in this article is to introduce the technique of degenerations of the Veronese, and to illustrate its efficacy. This technique has interesting relations with tropical geometry (see [19]). It can be usefully applied to higher dimensional projective spaces and, more generally, to interpolation problems on toric varieties. In fact it has been recently applied to give a new combinatorial proof of Alexander–Hirschowitz Theorem in $\mathbb{P}^3$ (see [4]). The relation of double points interpolation problems with the study of secant varieties to the Veronese varieties is well known (see §1 and [9]). Section 5 contains some speculations on applications of degeneration techniques to the problem of computing the degree of secant varieties of the Veronese surfaces.

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## 1 Double Points, the Veronese, Secant Varieties

Consider $\mathcal{L}_d(2^n)$, i.e. the linear system of plane curves of degree $d$ with $n$ general double points. Recall the *Veronese embedding*

$$v_d : \mathbb{P}^2 \longrightarrow \mathbb{P}^{d(d+3)/2}$$

with image $V_d$, the *Veronese surface* of degree $d^2$. A plane curve of degree $d$ corresponds via $v_d$ to a hyperplane section of $V_d$; and such a plane curve has a double point at $p$ if and only if the corresponding hyperplane is tangent to $V_d$ at $v_d(p)$. Therefore the linear system $\mathcal{L}_d(2^n)$ corresponds to the linear system $\mathcal{H}$ of hyperplanes in $\mathbb{P}^{d(d+3)/2}$ which are tangent to $V_d$ at $n$ fixed (but general) points. The famous Terracini's Lemma (see [9]) relates this linear system to the tangent space to the a secant variety of $V_d$: the base locus of $\mathcal{H}$ is the general tangent space to $\mathrm{Sec}_{n-1}(V_d)$, the $(n-1)$–*secant variety* to $V_d$, i.e. the variety described by all linear spaces of dimension $n-1$ which are $n$–secant to $V_d$. One thus concludes that $\mathcal{L}_d(2^n)$ is special if and only if $\mathrm{Sec}_{n-1}(V_d)$ has smaller dimension than expected, namely $V_d$ is $(n-1)$–*defective*.

Next we will use toric degenerations of the Veronese surfaces to study interpolation problems and secant varieties.

## 2 Toric degenerations

In this section we quickly recall a few basic facts about toric degenerations of projective toric varieties that will be useful later. The interested reader is referred to [29], [34] for more information on the subject and to [25] for relations with tropical geometry.

Recall that the datum of a pair $(X, \mathcal{L})$, where $X$ is a projective, $n$–dimensional toric variety and $\mathcal{L}$ is a base point free, ample line bundle on $X$, is equivalent to the datum of an $n$ dimensional integral compact convex polytope $P$ in $\mathbb{R}^n$, which is determined up to translation. Thus we may assume all points of $P$ have non–negative coordinates (see [24], p. 72). If $m_i = (m_{i1}, \ldots, m_{in})$, $0 \leq i \leq r$, are the integral points of $P$, we can consider the monomial map

$$\phi_P : x \in (\mathbb{C}^*)^n \to [x^{m_0} : \ldots : x^{m_r}] \in \mathbb{P}^r$$

where $x = (x_1, \ldots, x_n)$ and

$$x^{m_i} := x_1^{m_{i1}} \cdots x_n^{m_{in}}.$$

The closure of the image of $\phi_P$ is the image $X_P$ of $X$ via the morphism $\phi_{\mathcal{L}}$ determined by the line bundle $\mathcal{L}$.

For example, if $P$ is the triangle

$$\Delta_d := \{(x,y) : x \geq 0, y \geq 0, x+y \leq d\}$$

then $X_{\Delta_d}$ is the Veronese surface $V_d$.

If $P$ is the rectangle

$$R_{a,b} := \{(x,y) : 0 \leq x \leq a, 0 \leq y \leq b\}$$

with $a,b$ positive integers, then $X_{R_{a,b}}$ is $\mathbb{F}_0 = \mathbb{P}^1 \times \mathbb{P}^1$ embedded in $\mathbb{P}^{ab+a+b}$ via the complete linear system $\mathcal{L}_{(a,b)}$ of curves of bidegree $(a,b)$.

If $P$ is the trapezoid

$$T_{a,b} := \{(x,y) : x \geq 0, 0 \leq y \leq b, x+y \leq a\}$$

with $a > b$ positive integers, then $X_{T_{a,b}}$ is $\mathbb{F}_1$, i.e. the plane blown up at a point $p$, embedded in $\mathbb{P}^r$, $r = ab + a - b(b+1)/2$, via the proper transform of the linear system of curves of degree $a$ with a point of multiplicity $a-b$ in $p$.

Now, consider a *subdivision* $D$ of $P$ into convex subpolytopes. This is a finite family of $n$ dimensional convex polytopes whose union is $P$ and such that any two of them intersect only along a face (which may be empty). Such a subdivision is called *regular* if there is a piecewise linear, positive function $F$ defined on $P$ such that:

(i) the polytopes of $D$ are the orthogonal projections on the hyperplane $z = 0$ of $\mathbb{R}^{n+1}$ of the $n$–dimensional faces of the *graph polytope*

$$G(F) := \{(x,z) \in P \times \mathbb{R} : 0 \leq z \leq F(x)\}$$

which are neither vertical, nor equal to $P$;

(ii) the function $F$ is *strictly convex*, i.e., the hyperplanes determined by each of the faces of $G(F)$ intersect $G(F)$ only along that face.

Once one has a regular subdivision $D$ as above, one can construct a flat, projective degeneration of $X_P$ parametrized by the affine line $\mathbb{C}$, to a reducible variety $X_0$ which is the union of the toric varieties $X_Q$, with $Q$ in $D$. The intersection of the components $X_Q$ of $X_0$ is dictated by the incidence relations of the corresponding polytopes: if $Q$ and $Q'$ have a common face $R$, then $X_Q$ intersects $X_{Q'}$ along the toric subvariety of both determined by the face $R$.

The degeneration can be described as follows. Consider the morphism

$$\phi_D : (x,t) \in (\mathbb{C}^*)^n \times \mathbb{C}^* \to [t^{F(m_0)}x^{m_0} : \ldots : t^{F(m_r)}x^{m_r}] \in \mathbb{P}^r.$$

The closure of the image of $(\mathbb{C}^*)^n \times \{t\}$, $t \neq 0$, is a variety $X_t$ which is projectively equivalent to $X_P$. The flat limit of $X_t$ when $t$ tends to 0 is the variety $X_0$. To see that $X_0$ is the union of the varieties $X_Q$, with $Q \in D$, one argues as follows. Suppose that $F|_Q$ is the linear function $a_1x_1 + \ldots + a_nx_n + b$. First act with the torus in the following way:

$$(x_1, \dots, x_n, t) \in (\mathbb{C}^*)^n \times \mathbb{C}^* \to (t^{-a_1}x_1, \dots, t^{-a_n}x_n, t) \in (\mathbb{C}^*)^n \times \mathbb{C}^*.$$

Then composing with $\phi_D$, and get

$$(x_1, \dots, x_n, t) \in (\mathbb{C}^*)^n \times \mathbb{C}^* \to [\dots : t^{F(m_i)}t^{-a_1 m_{i1} - \dots - a_n m_{in}} x^{m_i} : \dots] \in \mathbb{P}^r.$$

Note that the point

$$[\dots : t^{F(m_i)}t^{-a_1 m_{i1} - \dots - a_n m_{in}} x^{m_i} : \dots]$$

in $\mathbb{P}^r$ equals

$$t^{-b}[\dots : t^{F(m_i)}t^{-a_1 m_{i1} - \dots - a_n m_{in}} x^{m_i} : \dots]$$

i.e. the point

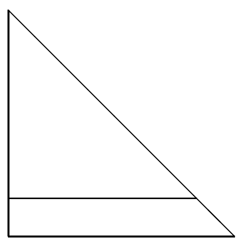
$$[\dots : t^{F(m_i) - F|_Q(m_i)} x^{m_i} : \dots].$$

Then by the definition of a regular subdivision, by letting $t \to 0$ in the above expression, we see that $X_Q$ sits in the flat limit $X_0$ of $X_t$.

## 3 Degenerations of the Veronese

Recall that a *rational normal scroll* $S(a, b)$, with $0 < a \le b$, is a smooth scroll surface of degree $d = a + b$ in $\mathbb{P}^{d+1}$, which is described by the lines joining corresponding points of two rational normal curves of degrees $a$ and $b$ lying in two linearly independent subspaces of dimensions $a$ and $b$ respectively. As an abstract surface, $S(a, b)$ is isomorphic to the *Hirzebruch surface* $\mathbb{F}_{b-a}$.

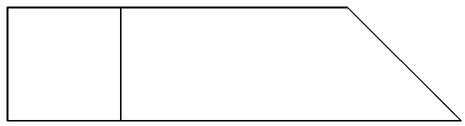
First of all, the Veronese $V_d$ degenerates to the union of $V_{d-1}$ and a scroll $S(d - 1, d)$:



This is a toric degeneration: the subdivision of the triangle $\Delta_d$ is clearly regular in this case. However, this degeneration can also be described directly as follows. In a trivial family $\mathcal{X}$ of $\mathbb{P}^2$'s parametrized by a disk $\Delta$, blow up the central fibre along a line $R$, thus getting a new family $\mathcal{X}'$. The general fibre of $\mathcal{X}'$ is still a plane, whereas the central fibre consists of the old plane $\mathbb{P}$ and of the exceptional divisor $\mathbb{F}$, which is an $\mathbb{F}_1$, meeting $\mathbb{P}$ along the line $R$. Note that $R$ is also the $(-1)$–curve on $\mathbb{F}$, meeting the ruling $F$ in one point.

Consider the line bundle which is the pull–back of $\mathcal{O}_{\mathbb{P}^2}(d)$ to $\mathcal{X}$ . Call $\mathcal{O}(d)$ the pull–back of this line bundle to $\mathcal{X}'$ and consider the bundle $\mathcal{O}(d) \otimes \mathcal{O}(-\mathbb{F})$. The restriction of it to the general fibre is still $\mathcal{O}_{\mathbb{P}^2}(d)$, whereas its restriction to $\mathbb{P}$ is $\mathcal{O}_{\mathbb{P}^2}(d-1)$ and to $\mathbb{F}$ is $\mathcal{O}_{\mathbb{F}}(dF+R))$, which embeds $\mathbb{F}$ as $S(d-1,d)$. The projective degeneration in question can be obtained by using the sections of $\mathcal{O}(d) \otimes \mathcal{O}(-\mathbb{F})$.

Similarly, $S(d-1,d)$ degenerates to a quadric and a scroll $S(d-2,d-1)$:



Again this is a toric degeneration. On the other hand one can easily figure out how to imitate the previous construction in order to perform this degeneration in a direct way: in the central fibre of a trivial family of $\mathbb{F}_1$'s one has to blow up a ruling, thus creating an exceptional divisor which is an $\mathbb{F}_0$. We leave to the reader the task of computing which *twisting operation* of line bundles has to be made in order to finish the job.

The first construction can be iterated, and we thus see that $V_d$ degenerates to a union of a plane $V_1$ and a sequence of scrolls $S(1,2), S(2,3), \ldots, S(d-1,d)$ (see Figure 1). Also the second construction can be iterated, and we have that $S(a-1,a)$ degenerates to $a-1$ quadrics and a plane (see Figure 2).

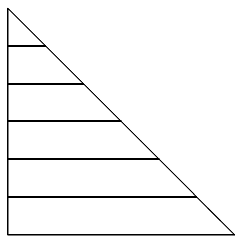


Figure 1:

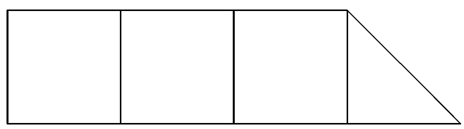
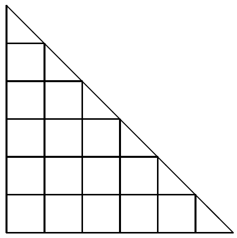


Figure 2:

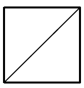
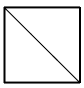
These degenerations can be combined in order to give rise to a degeneration of $V_d$ to a union of $d$ planes and $\binom{d}{2}$ quadrics, which we illustrate below for $d = 6$:

$d = 6$: 

This is again a toric degeneration: one has to prove that the subdivision of $\Delta_d$ into $d$ triangles and $\binom{d}{2}$ squares is regular, which is not difficult to see: a suitable strictly convex function $F$ determining this subdivision is the function $F(i, j) = i^2 + j^2$. Equivalently, one may translate the problem into the existence of the corresponding tropical curve (see [25]). Another way of proceeding is noticing that the *horizontal segments* in Figure 1 correspond to rational normal curves of degrees $1, 2, \ldots, d-1, d$ lying in linearly independent subspaces. The rulings of the scroll determine correspondences between these curves. The degeneration can be performed by letting each of these curves simultaneously degenerate to chains of lines (meanwhile also the correspondences between the curves degenerate). The existence of the degeneration can be verified also directly, but more painfully, verified by performing blow–up, blow-down and twist operations as illustrated above. Since we are now interested in drawing geometrical consequences form the existence of the degeneration, we skip these technical details and we leave them to the interested reader.

Summing up, the *vertices* of the last configuration of planes and quadrics are independent and therefore can be taken as the coordinate points of the ambient $\mathbb{P}^{d(d+3)/2}$. Each double curve in the above degeneration, along which two adjacent surfaces meet, is a line in the ambient space. We will refer to this degeneration as the *quadrics degeneration* of the Veronese.

Finally, each quadric can independently degenerate in its own space $\mathbb{P}^3$, to a union of two planes, in two ways:

 or  for each quadric.

Such a degeneration, which again can be understood from a toric, or tropical, viewpoint, can be performed directly by moving the quadric in a pencil in its embedding $\mathbb{P}^3$ and leaving the corresponding quadrilateral of double lines fixed.

If we degenerate each quadric to two planes, we obtain degenerations of $V_d$ to $d^2$ planes; we will refer to these as *planar degenerations* of the Veronese. Again, each double curve is a line in the ambient projective space, and the union of the planes spans this space. Each plane contains exactly three of the coordinate points of the projective space.

Since the vertices of these configurations are independent in the ambient projective space, any subset of $n$ of the planes which are pairwise disjoint will span a maximal dimensional space, of dimension $3n - 1$. Any such subset of a given planar degeneration $D$ of $V_d$ will be called a *skew n-set* of planes of $D$, and we will denote by $\nu_n(D)$, or simply by $\nu_n$ the number of such skew $n$–sets.

For the degenerations considered in this section and applications to secant varieties as in §5, see also [39].

## 4 Double point interpolation problems

Consider a double point interpolation system $\mathcal{L}_d(2^n)$, the linear system of plane curves of degree $d$ with $n$ general double points. As noted above, this system corresponds to the system of hyperplanes in $\mathbb{P}^{d(d+3)/2}$ which are tangent to the Veronese surface at $n$ general points. A famous theorem of Alexander–Hirschowitz (see [1], see also [28] and [2]) implies that $\mathcal{L}_d(2^n)$ is non–special, unless $d = 2, n = 2$ and $d = 4, n = 5$. The standard proof is based on an infinitesimal deformation analysis of singular hypersurfaces, implicit in Terracini's Lemma: it implies that speciality of $\mathcal{L}_{r,d}(2^n)$ forces the general element of this system to have non isolated singularities at the double base points $p_1, \ldots, p_n$ (see [2], [11], [7]). Employing a planar degeneration of the Veronese, we are able to reduce this result to a purely combinatorial property of the resulting configuration of planes.

**Lemma 4.1.** *Suppose that there exists a planar degeneration $D$ of $V_d$, and a skew $n$–set $S$ of planes of $D$. Then the linear system $\mathcal{L}_d(2^n)$ has the expected dimension $d(d+3)/2 - 3n$.*

*In particular, if there is a skew $n$–set of planes of $D$ whose planes contain all of the $(d+1)(d+2)/2$ coordinate points of the configuration, then $3n = (d+1)(d+2)/2$ and the linear system $\mathcal{L}_d(2^n)$ is empty.*

*Proof.* Consider the degeneration $D$, and let the general points $p_1, \ldots, p_n$ on $V_d$ degenerate in such a way that each point goes to a general point of the planes in the subset $S$. The limit of the system of hyperplanes tangent to $V_d$ at the points $p_1, \ldots, p_n$ is the system of hyperplanes tangent to the configuration $D$ at each limit point; but a hyperplane which is tangent to a plane at a point must contain that plane. Therefore the limiting system of hyperplanes is the system that contains the subset $S$ of $n$ planes in the configuration, which is the system of hyperplanes containing the span of $S$. Since $S$ consists of pairwise disjoint planes, it has maximal dimensional span, of dimension $3n - 1$; and therefore this limiting system of hyperplanes has codimension equal to $3n$.

By semicontinuity, we conclude that the system $\mathcal{L}_d(2^n)$ has codimension at least $3n$ in $\mathcal{L}_d$; but this is also the maximum possible codimension, since we are imposing $3n$ linear conditions on the plane curves.

In particular, if one can find a skew $n$–set $S$ of planes in $D$ that contains all of the coordinate points of the configuration, then $S$ will span the ambient space. Hence there can be no hyperplane that contains all of the planes of $S$, and we conclude, using the same argument as above, that the corresponding linear system must be empty. ∎

As a first application, we have:

**Lemma 4.2.** $\mathcal{L}_5(2^7)$ *is empty, and* $\dim(\mathcal{L}_6(2^9)) = 0$*, i.e. these systems have the expected dimensions.*

*Proof.* We illustrate below a skew 7-subset (respectively 9-subset) for a planar degeneration $D_5$ of $V_5$ (respectively $D_6$ of $V_6$):

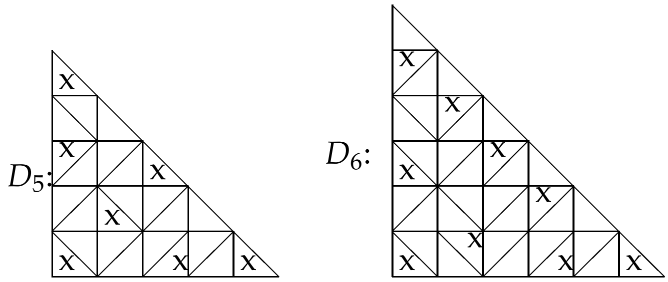
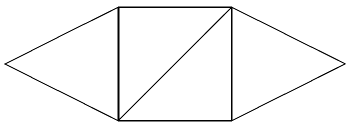



Figure 3:

Note that in the $d = 5$ case, the planes indicated by an 'x' form both a spanning 7-subset and a skew 7-subset of the indicated total planar degeneration. In the $d = 6$ example, the only vertex not covered by the 9-subset is the one at the upper left. ∎

**Remark 4.3.** In the above lemmas, the existence of a certain degeneration of the Veronese was used to prove that certain linear systems have the expected dimension. This argument can be reversed, if one knows that linear systems do not have the expected dimension. For example, the following union of four planes in $\mathbb{P}^5$ is *not* a degeneration of the Veronese $V_2$:



If it were, then the pair of planes on the two ends forms a spanning 2-subset. Therefore one could degenerate two points of $V_2$ so that one point goes on each end, and this would prove by the above Lemma that $\mathcal{L}_2(2^2)$ is empty, which it is not (for another, perhaps more conceptual explanation, see [5]).

On the other hand, it is clear that the above configuration of planes is a degeneration of the scroll $S(2,2)$.

As announced, the lemmas above enable us to reduce the problem of determining the dimension of $\mathcal{L}_d(2^n)$ to a tractable combinatorial one.

**Theorem 4.4.** *The linear system* $\mathcal{L}_d(2^n)$ *has the expected dimension whenever* $d \geq 5$.

*Proof.* The proof will be by induction on the degree $d$. Fix $n_0 = \lfloor (d+1)(d+2)/6 \rfloor$; with this number of points, we see that the virtual dimension of $\mathcal{L}_d(2^{n_0})$ is

$$v = d(d+3)/2 - 3n_0 = \begin{cases} -1 & \text{if } d \equiv 1,2 \mod 3 \\ 0 & \text{if } d \equiv 0 \mod 3. \end{cases}$$

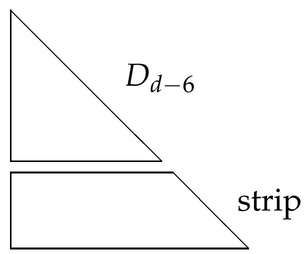
Suppose that the theorem is true for this $n = n_0$. Since the virtual dimension of $\mathcal{L}_d(2^{n_0})$ is at least $-1$, we conclude that the $3n_0$ conditions imposed by the $n_0$ double points are independent. Hence any fewer number of points will also impose independent conditions, and so $\mathcal{L}_d(2^k)$ will have the expected dimension for any $k < n_0$.

If $k > n_0$, then we must show that the system $\mathcal{L}_d(2^k)$ is empty; this is obvious if $d$ is not divisible by three, since by the computation above we have that $\mathcal{L}_d(2^{n_0})$ is already empty. If $3|d$, then we have that $\mathcal{L}_d(2^{n_0})$ has dimension zero, and so consists of a unique divisor; imposing any additional double points off this divisor will make the system empty.

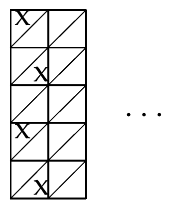
Therefore it will be sufficient to prove the theorem for $\mathcal{L}_d(2^{n_0})$; and this we will do by induction on $d$, employing Lemma (4.1). What we will show is that there is a skew $n_0$–subset $S$ of planes for a certain planar degeneration $D$ of $V_d$, if there is one for $V_{d-6}$.

To get the induction started, we must illustrate such degenerations and subsets for $5 \leq d \leq 10$; the $d = 5$ and $d = 6$ cases have been done above in Lemma (4.2). We leave to the reader the task of finding these for $7 \leq d \leq 10$, by drawing similar pictures to the ones in Lemma (4.2); it is not difficult at all, there are many ways to do each one, and it is an amusing exercise (see also [4]).

By induction, we assume that a total planar degeneration $D_{d-6}$ and a maximal skew subset $S'$ of it are available. The required total planar degeneration $D_d$ and the skew $n_0$–subset $S$ of it will be formed by grafting an appropriate strip onto $D_{d-6}$, as the following diagram indicates:

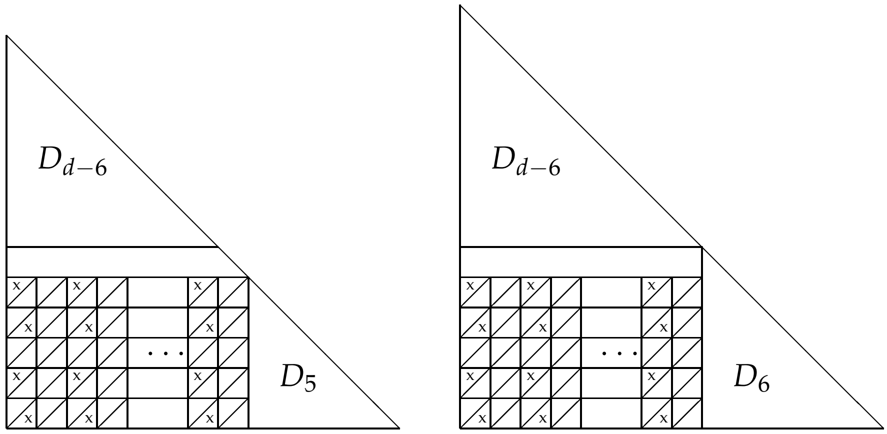



The strip is exactly 6 vertices high, and the configuration on the strip consists of $\lfloor (d+1)/2 \rfloor - 3$ copies of the rectangle




organized side by side; then on the far right of the strip, one has either the triangle with 6 vertices on a side, or a $5 \times 1$ rectangle adjacent to such a triangle, depending on the parity of $d$. These two pieces may be constructed using the configurations $D_5$ and $D_6$ presented in Lemma (4.2): the $D_5$ triangle configuration is used as is; the $D_6$ configuration is used after deleting the top vertex, creating the $5 \times 1$ rectangle adjacent to the triangle.

To be specific, we have the configurations




$d$ odd $d$ even

which show that if we have a solution for degree $d-6$, then we can construct one for $d$; this finishes the proof. ∎

The above proof is similar to a recent one by J. Draisma in [19], which uses tropical geometry rather than degenerations. These ideas have been exploited by S. Brannetti in [4] in order to give a similar combinatorial proof of the Alexander–Hirschowitz theorem in $\mathbb{P}^3$: the system $\mathcal{L}_{3,d}(2^n)$ has the expected dimension as soon as $d \geq 5$ (see also [40] and [37] for the classical approach of Terracini to this result). A more general result in the planar case has been proved by J. Roé in [35]. It would be very nice to have a full proof of the Alexander–Hirschowitz theorem using this technique.

## 5 Degree of secant varieties

In this section we want to indicate some enumerative applications of planar degenerations of the Veronese surfaces. Our considerations here are close to the ones in [39], where the Gröbner bases viewpoint is taken.

The general question we want to address is: given a projective, irreducible variety $X$ of dimension $n$ in $\mathbb{P}^r$, such that $\mathrm{Sec}_k(X)$ and has the expected dimension $nk+n+k$, what is the degree $s_k(X)$ of $\mathrm{Sec}_k(X)$ and what is the number $\mu_k(X)$ of $(k+1)$–secant $\mathbb{P}^k$ to $X$ passing through a general point of $\mathrm{Sec}_k(X)$?

Note that $\nu_k(X) := s_k(X) \cdot \mu_k(X)$ is the number of $(k+1)$–secant $\mathbb{P}^k$ to $X$ intersecting a general subspace of codimension $nk+n+k$ in $\mathbb{P}^r$. By the results in [7] and [8], one has $\mu_k(X) = 1$, unless $X$ is *k-weakly defective*, which is never the case if $X$ is a curve. The classification of weakly defective surfaces can be found in [7]. In [8] one can find the classification of $k$–weakly defective, but not $k$–defective, surfaces with $\mu_k(X) > 1$.

Recent contributions to the problem of computing the degree of secant varieties can be found in [17].

If $X$ is smooth, $\nu_1(X)$ coincides with the number of *apparent double points* of a general projection of $X$ to $\mathbb{P}^{2n}$ (see [12], [17]), and one has the so–called *double point formula*

$$2\nu_1(X) = d^2 - c_n(N_f)$$

where $d$ is the degree of $X$ and $N_f$ is the normal bundle to the projection morphism $f : X \to \mathbb{P}^{2n}$, defined by the exact sequence of vector bundles on $X$

$$0 \to T_X \to f^*(T_{\mathbb{P}^{2n}}) \to N_f \to 0;$$

hence $c_n(N)$ can be computed in terms of projective and birational invariants of $X$.

In the curve case, this is nothing but *Hurwitz formula*, i.e.

$$d_1(X) = \binom{d-1}{2} - g$$

where $d$ is the degree and $g$ the genus of $X$. In the surface case one has

$$\nu_1(X) = \frac{d(d-5)}{2} + 6\chi(\mathcal{O}_X) - 5(g-1) - K_X^2 \tag{5.1}$$

where $d$ is the degree and $g$ the sectional genus of $X$. This is also the degree of the secant variety, unless $X$ is weakly defective.

Similar, but more complicated, formulas are available for $\nu_k(X)$ in the case of curves (see [3], chapter VIII) and in the case of surfaces for $k \leq 5$ (see [31], [41], [32]). The question of computing $\nu_k(X)$ is open in general, even in the surface case and for Veronese surfaces. We devote this section to some speculations showing how planar degenerations of Veronese surfaces may be used to attack this problem.

Consider the Veronese surface $V_d$ and set $k_0 = n_0 - 1$, where $n_0$ was defined at the beginning of the proof of Theorem 4.4. Note that Theorem 4.4 implies that all secant varieties $\mathrm{Sec}_k(V_d)$ have the expected dimension unless $(d,k) = (2,1),(4,4)$. In particular, $\mathrm{Sec}_{k_0}(V_d)$ is a hypersurface if $d \equiv 0$ modulo 3, whereas it fills up the ambient space if $d \equiv 1,2$ modulo 3. Moreover, according to the results of [7] (see also [2]), $V_d$ is $k$–weakly defective but not defective only for $d = 6$ and $k = 8$, and one has $\mu_8(V_6) = 2$.

We wish to compute the numbers $s_{d,k} := s_k(V_d)$, or equivalently the numbers $\nu_{d,k} := \nu_{d,k}(V_d)$, which make sense for $1 \leq k \leq k_0$. Note that if $(d,k) \neq (2,1),(4,4)$ one has $\nu_{d,k} = s_{d,k}$ unless $(d,k) = (6,8)$ in which case $\nu_{6,8} = 2d_{6,8}$. Moreover it is known that $s_{2,1} = 3$ and $s_{4,4} = 6$ (see [17], Example 5.12).

**Proposition 5.2.** *Let $D$ be a planar degeneration of $V_d$, for $(d,k) \neq (2,1),(4,4)$. Then*

$$\nu_{d,k} \geq \nu_{k+1}(D). \tag{5.3}$$

*Proof.* First, let us look at the cases $d \equiv 0$ modulo 3 and $d \equiv 1,2$ modulo 3 and $k < k_0$, in which $\mathrm{Sec}_k(V_d)$ is a proper subvariety of the ambient space. The proof of Theorem 4.4 implies that for any skew $(k+1)$–subset $S$ of planes of $D$, the span $\Pi$ of these planes sits in the flat limit of $\mathrm{Sec}_k(V_d)$. Moreover for the general point of $\Pi$ there is a unique subspace of dimension $k$ meeting the $k+1$ planes of $S$ each in one point. This proves the assertion in this case.

Similarly, if $d \equiv 1,2$ modulo 3 and $k = k_0$, given a planar degeneration $D$ of $V_d$, any skew $n_0$–subset $S$ of planes of $D$ spans the whole space, and each such subset contributes to the number $\nu_{d,k_0}$. ■

If one can find degenerations $D$ for which equality holds in (5.3), the computation of $\nu_{d,k}$ would turn into a purely combinatorial problem. Following [39] one may call these degeneration *k–delightful*. A degeneration which is $k$–delightful for all $k \leq k_0$ can be simply called a *delightful degeneration*.

Next we discuss a few interesting examples concerning the computation of $\nu_{d,k_0}$.

**Example 5.4.** If $d = 3$, then $n_0 = 3$ and $s_{3,2} = \mu_{3,2} = 4$. In fact $\mathrm{Sec}_2(V_3)$ is the hypersurface in $\mathbb{P}^9$ consisting of points corresponding to cubic curves which, after a change of variables, can be written in the form $x_0^3 + x_1^3 + x_2^3 = 0$. These are the so–called *equianharmonic cubics*, i.e. the ones with $J$–invariant equal to 0. It is a classical result of invariant theory due to Aronhold that the degree of this hypersurface is 4 (see [22], p. 194). Note that this also follows by Le Barz formula in [31] (see also [17]).

Consider the planar degeneration $D_3$ of $V_3$ corresponding to the picture:

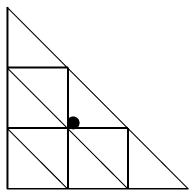
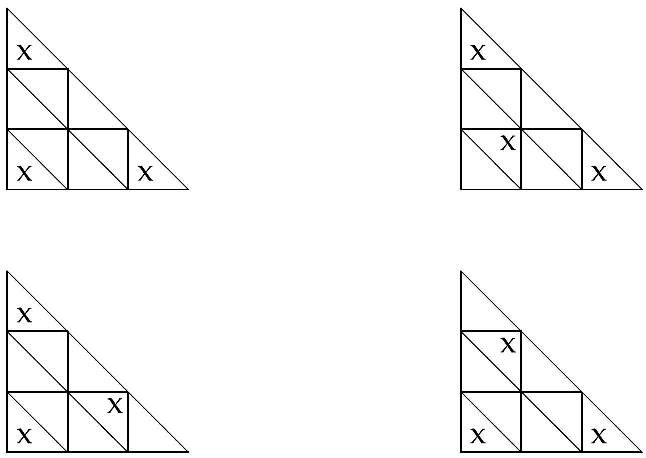


Figure 4:

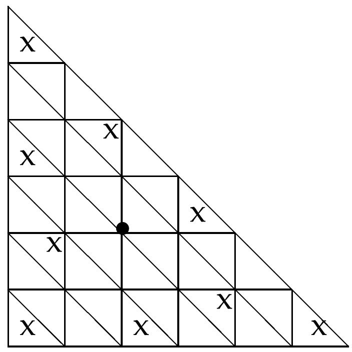
This is 3–delightful. In fact there are four skew 3–sets of planes in this picture, given by:



**Example 5.5.** If $d = 5$, then $n_0 = 7$ and $\mathrm{Sec}_6(V_5)$ fills up the whole space $\mathbb{P}^{20}$. It is a classical result which goes back to Hilbert that $\nu_{5,6} = 1$ (see also [38] and [17]).

Consider the planar degeneration $D_5$ of $V_5$ shown in Figure 3. This is 6–delightful. Indeed, it is not difficult to see that there is a unique skew 7–set of planes in $D_5$, i.e. the one indicated in Figure 3.

**Example 5.6.** If $d = 6$, then $n_0 = 9$. As we said $\mu_{6,8} = 2$. Consider the planar degeneration $D_6$ of $V_6$ shown in Figure 3. This is not 8–delightful. Indeed, the skew 9–set of planes indicated in $D_6$ indicated in the figure spans a linear space of the correct dimension 26. However in order to have equality in (5.3), the same subspace should also be spanned by another skew 9–set of planes in $D_6$. The reader can easily verify this is not the case. On the other hand the fact that $\mu_{6,8} \geq 2$, and therefore $V_6$ is 8–weakly defective, can be read off from the following degeneration:



In fact the indicated planes form a skew 9–set $D$ spanning a hyperplane in $\mathbb{P}^{27}$, since they contain all the vertices but the *central* one indicated by $\bullet$ . The same hyperplane is spanned by the 9–set of planes obtained from $D$ with a reflection with respect to the line $x + y = 0$. It would be interesting to see whether this degeneration is 8–delightful.

Let us turn now to the other extreme case, i.e. to $\nu_{d,1}$. For $d \geq 5$ and $d = 3$, the variety $\mathrm{Sec}(V_d)$ has dimension 5, one has $s_{d,1} = \nu_{d,1}$ and, by (5.1), its degree is

$$s_{d,1} = \frac{d^4 - 10d^2 + 15d}{2} - 3. \tag{5.7}$$

Is there some 1–delightful planar degeneration of $V_d$?

Surprisingly enough, the answer is no, in general. For instance, $s_{3,1} = 15$. However, if we look at the planar degeneration $D_3$ of $V_3$ described in Figure 4, one sees only 12 pairs of disjoint planes. Therefore the conclusion is that the union of the twelve $\mathbb{P}^5$'s generated by these skew 2–sets is only a part of the flat limit of the secant variety of $V_3$.

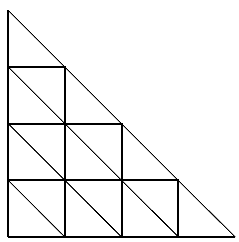
The reason for the discrepancy between (5.7) and the number of skew 2–sets of planes in a degeneration can be understood as follows. As we said, if $X$ a smooth surface, $\nu_1(X)$ computes the number of double points of a general projection $X'$ of $X$ to $\mathbb{P}^4$. These double points are the only non Cohen–Macaulay points of $X'$; thus $\nu_1(X)$ measures the length of the scheme of non Cohen–Macaulay points of $X'$, if this has finite length.

Now let us go back to $V_d$ and more specifically to $V_3$. When we project the union of planes depicted in Figure 4 down to $\mathbb{P}^4$, we find a reducible surface $D$ which has 12 nodes corresponding to the pairs of planes which are disjoint in $D_3$, but whose projections meet in $\mathbb{P}^4$. There is however one further non Cohen–Macaulay point, namely the *central* sextuple point $\bullet$. This used to be Cohen–Macaulay in $\mathbb{P}^9$ (its original embedding dimension is 5), but it is no longer Cohen–Macaulay in $\mathbb{P}^4$. We claim its *Cohen–Macauliness defect* is 3, i.e. *it counts for* 3 *more nodes of* $X'$. This restores the number 15 for $\nu_{3,1}$.

To give evidence to our claim, consider the following picture:

The left hand sides shows a polytope $P$ with horizontal and vertical sides of length 1. The corresponding projective toric variety $X_P$ is a del Pezzo surface of degree 6 in $\mathbb{P}^6$, i.e. the plane blown up at three distinct points anticanonically embedded in $\mathbb{P}^6$. By (5.1), the degree of the secant variety to $X_P$ is 3. The right hand side shows a regular subdivision of $P$, corresponding to a degeneration of $X_P$ to 6 planes, which forms a cone with vertex at $\bullet$ over a cycle of six lines spanning a $\mathbb{P}^5$. Since there are no pairs of disjoint planes here, we see that the singularity at $\bullet$ has to count with multiplicity 3 in the computation of the degree of the secant variety.

Similarly, $\nu_{4,1} = 75$. On the other hand in the planar degeneration of $V_4$ corresponding to the picture



we see only 66 pairs of disjoint planes. However the three sextuple points contribute each by three in this computation restoring the right degree $75 = 66 + 9$.

Similar computations should be available for $\nu_{d,k}$, with $1 \leq k \leq k_0$. However the situation may turn out to be very complicated, and, in general, the question is: how do the singularities of the configuration influence the lack of delightfulness and in particular the combinatorial computation of $\nu_{d,k}$?

**Remark 5.8.** The considerations we made in the present section can be more generally applied to toric varieties. As an example, consider $S(2, 2)$. This is well known to be a OADP– variety, i.e. there is a unique secant line to $S(2,2)$ passing through the general point of $\mathbb{P}^5$ (see [12]).

To see this combinatorially, look at the planar degeneration of $S(2,2)$ displayed in Remark 4.3 and note that there is here a unique skew 2–set of planes, given by the two end planes of the configuration.

Similar considerations can be made for higher dimensional varieties. For instance, look at the Segre embedding $X$ of $\mathbb{P}^1 \times \mathbb{P}^1 \times \mathbb{P}^1$ in $\mathbb{P}^7$. Again this is an OADP– variety (see [12]). The reader will be amused to rediscover this by looking at the well known decomposition of the 3–dimensional cube in six tetrahedra. This decomposition is regular and gives rise to a degeneration of $X$ to a union of six 3–spaces. This degeneration is delightful: indeed the decomposition of the cube possesses a unique pair of disjoint tetrahedra, corresponding to two independent 3-spaces in the limit of $X$.

The problem of studying the secant variety to a toric variety has been considered in [18].

## 6 Points of higher multiplicity

In this section we go back to interpolation and we want to use the planar degenerations of the Veronese surfaces in order to study multiple points interpolation problems. We recall a basic fact.

**Lemma 6.1.** *Let $X$ be a variety, $M$ a line bundle on $X$, and $D$ an effective Cartier divisor on $X$. Set $M(D) = M \otimes \mathcal{O}_X(D)$. Suppose that $H^0(X,M) = H^1(X,M) = 0$. Then the restriction map from $H^0(X,M(D))$ to $H^0(X,M(D)|_D)$ is an isomorphism.*

This lemma applies in the following two cases:

**Lemma 6.2.**

*(a) Let $X$ be the blow up of $\mathbb{P}^2$ at a point, with exceptional divisor $E$ and line class $H$. Let $M = \mathcal{O}_X((m-1)H - mE)$. Then $H^0(M) = H^1(M) = 0$.*

*(b) Let $X$ be the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at two general points, with exceptional divisors $E_1$ and $E_2$, and denote by $V$ the vertical fiber class and by $H$ the horizontal fiber class. Let $M = \mathcal{O}_X((m-1)H + mV - mE_1 - mE_2)$ (or, symmetrically, $M = \mathcal{O}_X(mH + (m-1)V - mE_1 - mE_2)$ ). Then $H^0(M) = H^1(M) = 0$.*

*Proof.* In both cases, the systems are empty, so that $H^0 = 0$, and by using Serre Duality, $H^2$ is also zero. Riemann-Roch gives $\chi = 0$ as well, which implies $H^1 = 0$. ∎

We can apply the previous Lemmas to different divisors $D$. It is useful for our applications, given the degeneration constructions we have introduced, that the divisors $D$ be subdivisors of the double curves of the planes or quadrics in the degeneration.

In the planar case, this means that we will be applying the lemmas for a divisor $D$ consisting of a subdivisor of a triangle of lines $L_1 + L_2 + L_3$. We have the following list of $M(D)$'s in this case.

**Lemma 6.3.** *Let $X$ be the blow up of $\mathbb{P}^2$ at a point, with exceptional divisor $E$ and line class $H$. Let $L_1$, $L_2$, and $L_3$ be a triangle $T$ of lines not passing through the point. Let $M = \mathcal{O}_X((m-1)H - mE)$. Then the restriction maps*

$$(1):\ H^0(X, \mathcal{O}_X(mH - mE)) \to H^0(L_i, \mathcal{O}_X(mH)|_{L_i})$$
$$(2):\ H^0(X, \mathcal{O}_X((m+1)H - mE)) \to H^0(L_i + L_j, \mathcal{O}_X((m+1)H - mE)|_{L_i+L_j})$$
$$(3):\ H^0(X, \mathcal{O}_X((m+2)H - mE)) \to H^0(T, \mathcal{O}_X((m+2)H - mE)|_T)$$

*are isomorphisms.*

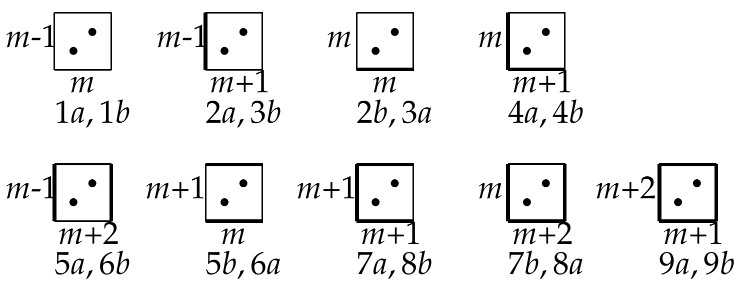
We note that these three spaces have dimensions $m+1$, $2m+3$, and $3m+6$, respectively.

In the quadric case, we present the information in a table. Let $X$ be the blow up of $\mathbb{P}^1 \times \mathbb{P}^1$ at two general points, with exceptional divisors $E_1$ and $E_2$, and denote by $V$ the vertical fiber class and by $H$ the horizontal fiber class. We fix two vertical fibers $V_1$ and $V_2$, and two horizontal fibers $H_1$ and $H_2$, not passing through the two points; our divisor $D$ will be a subdivisor of $H_1 + H_2 + V_1 + V_2$. By Lemma 6.2, there are two possibilities for the line bundle $M$ for each divisor $D$; these are presented in the last two columns of the table.

**Lemma 6.4.** *Using the above notation, the restriction map from $H^0(X, M(D))$ to $H^0(D, M(D)|_D)$ is an isomorphism, for all $D$ and $M(D)$ in the following table:*

| | *Divisor D* | *M(D)-a* | *M(D)-b* |
|---|---|---|---|
| 1. | $0$ | $mV + (m-1)H$ | $(m-1)V + mH$ |
| 2. | $V_1$ *or* $V_2$ | $(m+1)V + (m-1)H$ | $mV + mH$ |
| 3. | $H_1$ *or* $H_2$ | $mV + mH$ | $(m-1)V + (m+1)H$ |
| 4. | $V_i + H_j$ | $(m+1)V + mH$ | $mV + (m+1)H$ |
| 5. | $V_1 + V_2$ | $(m+2)V + (m-1)H$ | $(m+1)V + mH$ |
| 6. | $H_1 + H_2$ | $mV + (m+1)H$ | $(m-1)V + (m+2)H$ |
| 7. | $H_1 + H_2 + V_i$ | $(m+1)V + (m+1)H$ | $mV + (m+2)H$ |
| 8. | $V_1 + V_2 + H_j$ | $(m+2)V + mH$ | $(m+1)V + (m+1)H$ |
| 9. | $V_1 + V_2 + H_1 + H_2$ | $(m+2)V + (m+1)H$ | $(m+1)V + (m+2)H$ |

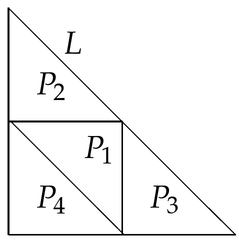
*(We abused notation and denoted the $M(D)$'s using the divisor classes only.)*




We will apply these lemmas by constructing a degeneration of the $d$-fold Veronese $V_d$ to a union of planes and quadrics as described above. We will degenerate the bundle $\mathcal{O}(d)$ to a bundle on the degenerate configuration which will have certain degrees on the planes and bidegrees on the quadrics. The general multiple points will degenerate either to one point on a plane or to two general points on a quadric. Our goal is to show that we may determine the dimension of the limit system by analysing, via the above lemmas, the dimension on each surface of the degeneration.

An example will illustrate the argument.

Consider the linear system of curves of degree $2m$ with 4 points of multiplicity $m$. It is well known that this is a non–special system of dimension $m$, composed with the pencil of conics through the four points; each element of the system consists of $m$ conics in the pencil. This is the expected dimension of the system $\mathcal{L}_{2m}(m^4)$: $v = 2m(2m+3)/2 - 4m(m+1)/2 = m$. We want to exploit the degeneration of the Veronese surface $V_2$ to four planes to give a combinatorial proof of non–speciality of $\mathcal{L}_{2m}(m^4)$. The degeneration to four planes $P_1, \dots, P_4$, may be illustrated as below:




**Lemma 6.5.** *The system $\mathcal{L}_{2m}(m^4)$ has the expected dimension $m$.*

*Proof.* Since this is a degeneration of $V_2$, the bundle $\mathcal{O}(m)$ (which has degree $m$ on each plane) corresponds to the bundle $\mathcal{O}(2m)$ on $\mathbb{P}^2$. We degenerate the four points to one in each plane.

We note that on each plane, we have curves of degree $m$ with one point of multiplicity $m$. Therefore we are in the situation of Lemma 6.3, and we see that the linear system on any one of the planes is determined by the linear system restricted to any one of the three lines on that plane indicated in the figure; indeed, the map on global sections is an isomorphism.

Denote by $L_j$ the line of intersection between the interior plane $P_1$ and one of the other three planes $P_j$.

To determine the space of sections on the entire configuration of four planes, we choose any one of the other non-double lines, say the line $L$ indicated in the figure, on the plane $P_2$. The restriction of the system to $L$ has degree $m$, and the vector space of sections has dimension $m+1$.

By Lemma 6.3, any section restricted to $L$ determines a section on the plane $P_2$. This then determines the section on the double line $L_2$, via the corresponding restriction isomorphism. Again by Lemma 6.3, the section on $L_2$ determines the section on the interior plane $P_1$. Similarly, this determines sections on the two other double lines $L_3$ and $L_4$. Finally Lemma 6.3 now implies that the sections on the other two planes are determined as well.

All of these determinations are made via isomorphisms, and the result is that the space of sections of $\mathcal{O}(m)$ on the configuration of four planes, with the four multiple points, is isomorphic to the space of sections on the restriction to the line $L$. We conclude that the limiting bundle has $h^0 = m+1$ as desired. ∎

Another phenomenon occurs when a configuration of surfaces meets in a cycle, with an interior vertex. To fix notation, suppose that we have $n$ surfaces $S_1, \dots, S_n$, meeting in a cycle with a common vertex. i.e., there is a smooth rational curve $L_i = S_i \cap S_{i+1}$ (with indices taken modulo $n$) and a point $v$ common to all the curves $L_i$. Therefore on the surface $S_i$ we have the two curves $L_i$ and $L_{i-1}$, meeting at $v$.

We assume that we have a linear system $M_i$ on each surface $S_i$, which restricts to linear systems on both $L_i$ and $L_{i-1}$. We assume that these are compatible with the configuration, in the sense that the two linear systems on each $L_i$ (one restricted from $M_i$ on $S_i$ and one restricted from $M_{i+1}$ on $S_{i+1}$) are the same, corresponding to a vector space $V_i$ of sections. We assume that not every section in $M_i$ vanishes at the interior vertex $v$.

We further assume that these restriction maps from the vector space of sections corresponding to $M_i$ to both $L_{i-1}$ and $L_i$ are isomorphisms. The restriction isomorphisms give a map $g_i : V_{i-1} \to V_i$, as the composition of the inverse of the restriction from $S_i$ to $L_{i-1}$, followed by the restriction to $L_i$. Since these restrictions are assumed to be isomorphisms, each $g_i$ is an isomorphism. The composition of these in sequence around the cycle of surfaces gives a map $F : V_1 \to V_1$.

We have the following result.

**Lemma 6.6.** *In the above settings, if the maps $g_i$ are general enough, then there is a unique non–zero section of $V_1$ (up to scalar) which does not vanish at $v$ and is fixed by $F$.*

*Proof.* The generality of the maps allows us to assume that the composition $F$, which is an automorphism of $V_1$, can be diagonalized. Consider the subspace $W \subset V_1$ of sections vanishing at the interior vertex $v$. Note that $W$ has codimension one in $V_1$, by the assumption that not every section in $M_1$ vanishes at $v$. Furthermore, $W$ is $F$-invariant: indeed, a section vanishing at $v$ is mapped to a section vanishing at $v$ by all restriction maps, hence by all maps $g_i$, and hence by $F$.

Therefore there is exactly one eigenspace $W_1 \subset V_1$ such that $V_1 = W \oplus W_1$.

We finish the proof by showing that the eigenvalue $\lambda$ for the one–dimensional eigenspace $W_1$ is $\lambda = 1$. Choose a section $s \in W_1$; note that $s(v) \neq 0$. We also have $F(s)(v) = s(v)$; that corresponding sections have the same value at $v$ is true for all restrictions, hence for all $g_i$, and hence for $F$. We have $F(s) = \lambda s$; evaluating at $v$, and noting that $s(v) \neq 0$, we see that $\lambda = 1$. ∎

We apply this to the following case.

**Lemma 6.7.** *The system $\mathcal{L}_{3m}(m^9)$ has the expected dimension* 0.

*Proof.* We make the total planar degeneration $D_3$ of $V_3$ to nine planes showed in Figure 4. The line bundle $\mathcal{O}(m)$ which has degree $m$ on each of the nine planes of the degeneration of $V_3$ is a degeneration of the bundle $\mathcal{O}(3m)$ on $\mathbb{P}^2$. Hence on each plane we have curves of degree $m$ with a general point of multiplicity $m$ that agree on the double lines; moreover there is exactly one interior vertex •.

In this configuration we see six planes adjacent to the interior vertex • and linear systems that satisfy the hypothesis of the Lemma 6.6, each being determined by the restriction to a line. Indeed, they each satisfy the conditions of the first case in Lemma 6.3.

By Lemma 6.6, there is a unique non–zero section, up to scalar, on the six middle planes, which does not vanish on the interior vertex $\bullet$ and agrees on all of the double curves around the vertex. This gives a unique section on each of the three double lines that meet the three corner planes in the configuration. Again by the first case in Lemma 6.3, these sections lift uniquely to sections of $\mathcal{O}(m)$ on the three corner planes.

On the other hand there is no non–zero section vanishing on the interior vertex $\bullet$. Any such section $s$, in fact, would vanish also on each of the 9 points of multiplicity $m$ located in each of the planes of $D_3$. In particular the divisor of $s$ around $v$ would consist of six general lines passing through $v$, one in each of the six planes surrounding $v$. This however is clearly not a Cartier divisor on the reducible surface $D_3$, which gives a contradiction.

Hence the dimension of the linear system is zero, which is the expected dimension. ∎

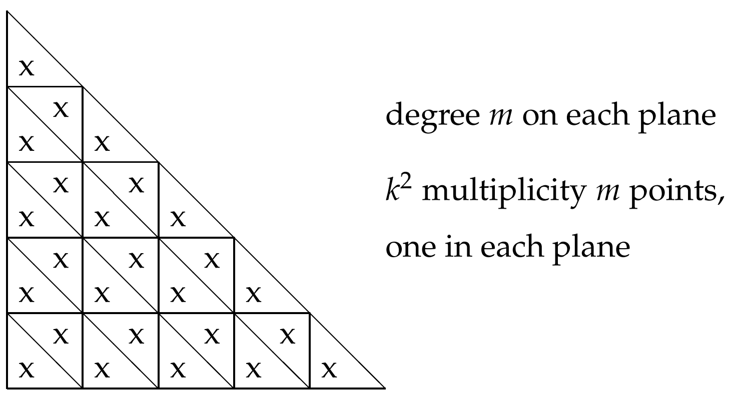
Note that the unique curve in the limit of $\mathcal{L}_{3m}(m^9)$ consists of the unique cubic curve in $\mathcal{L}_3(1^9)$ counted with multiplicity $m$.

**Remark 6.8.** Lemma 6.6 is useful when there are cyclical configurations of surfaces that overlap. In this case, in each of the cycles of surfaces, there is a unique section up to scalar satisfying the matching conditions. However these two sections will not agree on the overlap. Hence we conclude that any section satisfying the matching conditions must be zero.

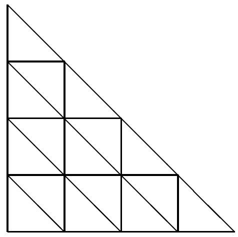
We have seen in Lemmas 6.3, 6.5 and 6.7 that the linear system $\mathcal{L}_{km}(m^{k^2})$ has the expected dimension for $k = 1, 2, 3$. We can now prove the more general statement, which is slightly better than Nagata's conjecture in this case, but it is weaker than Harbourne-Hirschowitz.

**Theorem 6.9.** *The system $\mathcal{L}_{km}(m^{k^2})$ has the expected dimension, in particular it is empty for $k \geq 4$.*

*Proof.* As above, we consider the system associated to the line bundle $\mathcal{O}(m)$ on the $k$-fold Veronese $V_k$. Degenerating, we form a total planar degeneration to $k^2$ planes, and on each plane we have the linear system of curves of degree $m$. We degenerate the $k^2$ points by putting one in general position on each plane of the degeneration; for example, a $k = 5$ example is illustrated below.



The cases $k = 1, 2$ and 3 follow by the lemmas 6.3, 6.5 and 6.7 respectively. For $k \geq 4$, the system is expected to be empty. For $k = 4$, we have the following configuration:



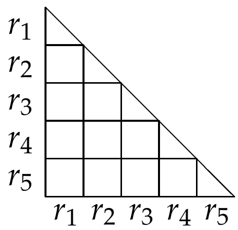
By Lemma 6.6, there is a unique divisor satisfying the matching conditions on the six planes adjacent to each of the three interior vertices. However for any two of these interior vertices, there are adjacent planes in common. The divisors will not agree on these common adjacent planes. Hence the system is empty as expected.

Finally for $k > 4$, if we form the same type of configuration, by induction, the *top* $(k-1)^2$ planes already cannot support a divisor. The system will thus be empty. ∎

## 7 Line bundles on quadrics degenerations of the Veronese

One can acquire some more flexibility in the limiting line bundle on the configuration by using the quadrics degeneration of the Veronese that we introduced in §3. Recall that this is the triangular configuration of $\binom{d}{2}$ quadrics, meeting along lines, with $d$ planes on the 'hypotenuse' of the configuration. Let us coordinatize the configuration, and index the surfaces in the configuration as $T_{ij}$, with $i \geq 1$, $j \geq 1$, and $i + j \leq d + 1$; the quadrics are the surfaces with $i + j \leq d$, and the planes are the surfaces $T_{i,d+1-i}$. We have that $T_{ij}$ meets $T_{k\ell}$ along a line if and only if either $i = k$ and $|j - \ell| = 1$ or $j = \ell$ and $|i - k| = 1$.

We can form a line bundle on this partial quadrics degeneration by putting a line bundle on each surface such that on each double curve the restriction of the two bundles agree. This can be done by choosing $d$ integers $r_1, r_2, \ldots, r_d$, and for $i + j \leq d$ putting the bundle of bidegree $(r_i, r_{d+1-j})$ on the quadric $T_{ij}$; on the plane $T_{i,d+1-i}$ one puts the bundle of degree $r_i$. This can be conveniently visualized with the following picture referring to the case $d = 5$:




One can prove that this line bundle is the limit of the line bundle $\mathcal{O}_{\mathbb{P}^2}(r)$, with $r = r_1 + r_2 + \ldots + r_d$. This can be seen by using the theory of toric degenerations (see §2) but also by performing a series of blow–up, blow-down and twist operations as indicated in §3. We leave the details to the reader.

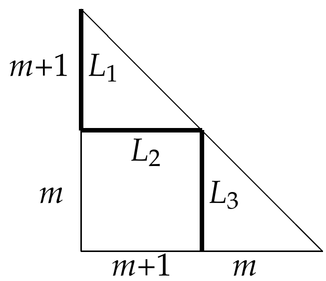
We want to illustrate the technique, invoking Lemmas 6.3 and 6.4, by proving the following theorem:

**Theorem 7.1.** *The system $\mathcal{L}_{km+1}(m^{k^2})$ has the expected dimension. In particular, it is empty for $k \geq 6$.*

*Proof.* First note that the expected dimension is $v = mk(5-k)/2 + 2$. We will degenerate $V_k$ to a union of $k$ planes and $\binom{k}{2}$ quadrics placing one point on each plane and two points on each quadric and the bundle degeneration is as indicated above: for each value of $k$, we will see which values of $r_1, \dots, r_k$ is convenient to take. In particular, for $k \leq 4$ we will take $r_1 = m+1, r_2 = \ldots = r_k = m$.

The $k = 1$ case is proved by noting that the linear system $\mathcal{L}_{m+1}(m)$ has the expected dimension $2m+2$. This is easy to see in many ways, but let us use Lemma 6.3, to demonstrate the method. The second case of this lemma tells us that this linear system is isomorphic to the linear system restricted to two general lines of $\mathbb{P}^2$. This is a linear system on two lines, each of degree $m+1$, which meet at one point. The space of sections on each line has dimension $m+2$, and the matching of the sections at the point of intersection gives one condition, so that the space of sections on the lines has dimension $2m+3$. Hence the projective dimension of the linear system is $2m+2$ as expected.

Case $k = 2$. Here the expected dimension is $v = 3m+2$. We claim that in this case the linear system is determined by the restriction to the three lines $L_1$, $L_2$, and $L_3$ outlined in the picture below.



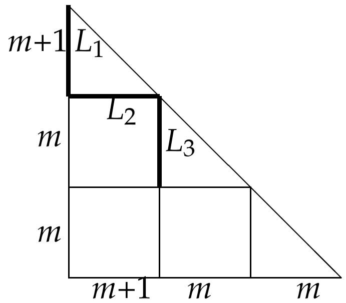

$k = 2$

Indeed, consider first the upper plane $T_{1,2}$. Applying case (2) of Lemma 6.3, one gets a unique divisor in the linear system $\mathcal{L}_{m+1}(m)$ that restricts to the given divisor on $L_1 + L_2$. For the lower quadric $T_{1,1}$, invoking Lemma 6.4 (fourth case) we conclude the existence of a unique divisor in $\mathcal{L}_{(m,m+1)}(m^2)$ that restricts to the given divisor on $L_2 + L_3$. Similarly for the lower plane $T_{2,1}$, using Lemma 6.3 again, we get a unique divisor restricting to the given divisor on $L_3$.

This proves the claim that the divisor on the union of the three surfaces is determined by the divisor on the union of the three lines. Hence the dimension of $\mathcal{L}_{2m+1}(m^4)$ is given by the number of conditions imposed by fixing linear systems on three lines that agree at two points of intersection. The three systems have degrees $m+1$, $m+1$, and $m$ on $L_1$, $L_2$, and $L_3$ respectively, and so we have vector spaces on the three lines of dimension $m+2$, $m+2$, and $m+1$; the two points of intersection give two conditions, and hence the projective dimension is

$\dim(\mathcal{L}_{2m+1}(m^4)) = (m+2)+(m+2)+(m+1)-2-1 = 3m+2 = v$ as expected.

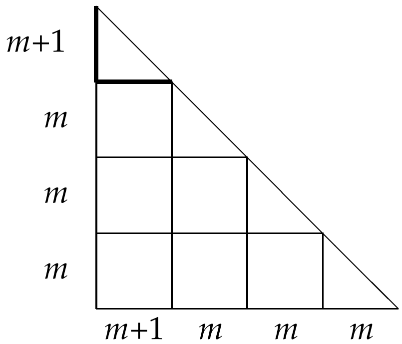
Case $k = 3$. Here the expected dimension is again $v = 3m+2$. We claim that the linear system is determined by the restriction to the three lines $L_1$, $L_2$, and $L_3$ as in the picture:




$k = 3$

By the previous case is enough to prove that the divisor $D$ in the upper three surfaces determines a unique one on each of the three surfaces from the lowest row. Indeed we get a unique divisor on the quadric $T_{2,1}$ using Lemma 6.4 (third case); then the restriction to the right line determines a unique divisor on the plane $T_{3,1}$ using Lemma 6.3 (second case), and the restriction to the left line together with the restriction of $D$ determines the divisor on $T_{1,1}$ using Lemma 6.4 (fourth case).

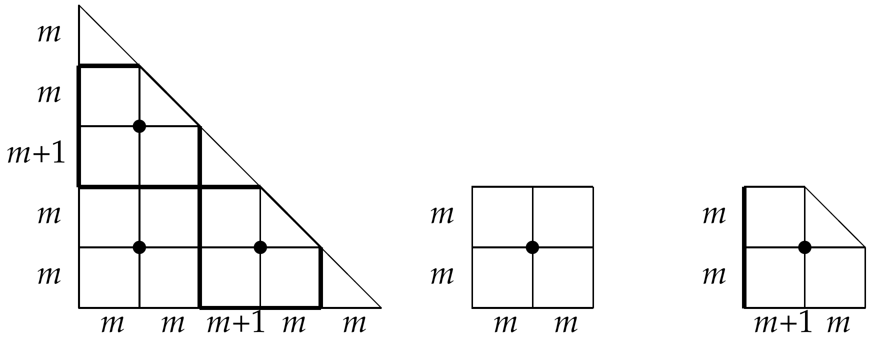
Case $k = 4$. The virtual dimension is $v = 2m+2$. We want to determine a unique divisor that restricts to the fixed two lines in the configuration below.




$k = 4$

On the plane $T_{1,4}$ the divisor is fixed by Lemma 6.3. On the six surfaces $T_{ij}$ with $i \geq 2$, we have a unique section, that agrees at the intersection point of $T_{1,4}$ and $T_{2,3}$, by Lemma 6.6 (see also the argument in the proof of Lemma 6.7). This then determines the sections on the lines of intersection between $T_{1,j}$ and $T_{2,j}$ for $j = 1,2,3$. At this point the divisor on $T_{1,3}$ is determined by Lemma 6.4; then the divisor on $T_{1,2}$ is determined as well; and finally the divisor on $T_{1,1}$ is determined.

Case $k = 5$. Here the virtual dimension is $v = 2$ and therefore we expect to find three points in the configuration such that the values of the section at these three points determine the section uniquely. We fix the degrees on the planes as indicated in the leftmost figure below, i.e. we take $r_1 = r_2 = r_4 = r_5 = m, r_3 = m+1$:




$k = 5$

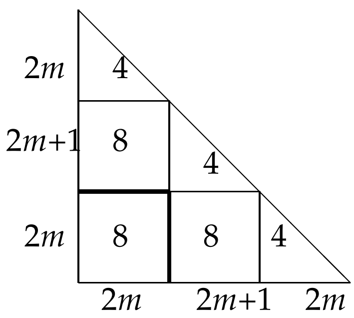
First consider the union of the four quadrics that form the lower left square $T_{1,1}$, $T_{1,2}$, $T_{2,1}$ and $T_{2,2}$; these are indicated in the middle picture of the figure. On each quadric we have the linear system of curves of bidegree $(m, m)$ with two points of multiplicity $m$, each of which is determined by the restriction to a line. We are therefore in the hypothesis of Lemma 6.6, and so we conclude that the space of sections is one dimensional being determined by the value at the interior point. Note that this section determines the section on the boundaries.

Next we analyse the union of the 4 surfaces $T_{3,1}$, $T_{3,2}$ $T_{4,1}$ and $T_{4,2}$. By the above, we see that the section is determined on the two leftmost vertical lines. This is indicated by the rightmost picture in the figure. We are again in the hypothesis of Lemma 6.6 with these four surfaces; hence the section is uniquely determined by its value at the interior point.

Similarly, the section on the four surfaces $T_{1,3}$, $T_{1,4}$, $T_{2,3}$ and $T_{2,4}$ (with the lower two horizontal lines determined) is determined by the value at the interior point.

Finally the sections on the planes $T_{1,5}$ $T_{5,1}$ and $T_{3,3}$ are now determined using Lemma 6.3.

Case $k = 6$. This is the first case where we must show that the system is empty. Here we depart from the above pattern a bit and consider the following degeneration of the plane into six surfaces, three re-embedded quadrics and three Veronese surfaces, with degrees indicated, that sum to $6m + 1$:




$k = 6$

The number of points on each quadric is 8, while the number on each Veronese is 4; note that the total number is 36 as required.

Focus on the lower left quadric $T_{1,1}$, where we have the linear system of curves of bidegree $(2m, 2m)$ on a quadric, with 8 points of multiplicity $m$. This has a space of sections of dimension one, namely a unique divisor, the $m$-fold curve in the linear system of bidegree $(2, 2)$ through the 8 points. The restriction of this space of sections to both the right double curve and the top double curve has dimension one.

Now consider the quadric $T_{2,1}$ just to the right of this, and consider the restriction to the double curve on the left with $T_{1,1}$. This restriction of sections is onto (the sheaf is the sheaf of degree $2m$ on that vertical curve), and the kernel has dimension one (as a vector space), with the similar analysis as above. Therefore the space of sections here that could agree with an element of the dimension one space of sections coming from $T_{1,1}$ has dimension two, one coming from the restriction and one coming from the kernel.

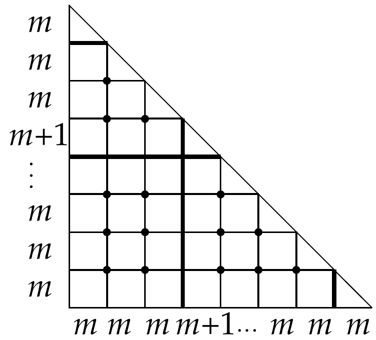
This same analysis holds for the quadric $T_{1,2}$: there is a dimension two space of sections there that restrict to some element of the dimension one space of sections of the double curve where this quadric meets the lower left quadric.

The space of sections now on these three quadrics has dimension three: 2 each on the two quadrics, but there is a condition that the sections agree at the point of intersection, which is the interior point of the configuration.

Now look at one of the corner planes, e.g., $T_{1,3}$. The system there is of degree $2m$, with four $m$-fold points. We know that this is the linear system composed with the pencil of conics through the four points. Therefore the restriction of this system to the double line is the system (of vector space dimension $m + 1$) of the intersections with the pencil of conics. If $m \geq 2$, no element of the 2-dimensional space of sections on the adjoining quadric will match with any such element on the double line. (The ambient space has vector space dimension $2m + 1$, and we have the restriction of a 2-dimensional space from the quadric and the $(m + 1)$-dimensional space from the plane, which will not intersect away from 0 if $m \geq 2$.)

We conclude that the section must be zero on that corner plane, also by symmetry on the other corner plane; then it must be zero as well on the quadrics, and finally on the center plane $T_{2,2}$.

Case $k \geq 7$. The virtual dimension is $v < 0$ and we must show the system is empty. We use a degeneration with $r_1 = r_2 = r_3 = m$, $r_4 = m + 1$, and $r_i = m$ for $i \geq 5$:




$k \geq 7$

The sections on the $3k-12$ lower left quadrics must be zero, using the overlapping interior vertices argument. Then on the eight surfaces just above these, we must also have zero sections; this applies as well to all of the surfaces to the right of these, except the final corner plane. This leaves only the two corner planes $T_{1,k}$ and $T_{k,1}$, and the final plane $T_{4,k-3}$; sections on these are now seen to be zero as well. ■

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