Uniqueness of singular radial solutions for a class of quasilinear problems

Maria-Magdalena Boureanu

Abstract

We establish the uniqueness and the blow-up rate of the large positive solution of the singular boundary problem $-\Delta_p u = \lambda u^{p-1} - b(x)u^q$ in $B_R(x_0)$, $u|_{\partial B_R(x_0)} = +\infty$, where $B_R(x_0)$ is a ball domain of radius R centered at $x_0 \in \mathbb{R}^N$, $N \geq 3$, $\lambda > 0$ and the potential function b is a positive radially symmetric function. Our result extends the previous work by Ouyang and Xie from the case $p = 2$ to the case $p > 2$ and we prove that any large solution u must satisfy

$$\lim_{d(x)\to 0} \frac{u(x)}{K(b^*(\|x-x_0\|))^{-\beta}} = 1,$$

where $d(x) = \operatorname{dist}(x, \partial B_R(x_0))$, K is a constant defined by

$$K := \left[(p-1)[(\beta+1)C_0 - 1]\beta^{p-1}(C_0 b_0)^{(p-2)/2}\right]^{\frac{1}{q-p+1}},$$

with

$$\beta := \frac{p}{2(q-p+1)},\ q > p-1 > 1,\ b_0 := b(R) > 0,\ C_0 := \lim_{r\to R} \frac{(B(r))^2}{b^*(r)b(r)} \geq 1$$

and

$$B(r) := \int_r^R b(s)ds,\ b^*(r) = \int_r^R \int_s^R b(t)dtds.$$

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1 Introduction

We are studying the uniqueness and blow-up rates of the singular value problem:

$$\begin{cases} -\Delta_p u = \lambda u^{p-1} - b(x)u^q & \text{in } B_R(x_0), \\ u > 0 & \text{in } B_R(x_0), \\ u = \infty & \text{on } \partial B_R(x_0), \end{cases} \tag{1}$$

where $B_R(x_0)$ is the ball of radius R centered at $x_0 \in \mathbb{R}^N$, $N \geq 3$, $\lambda > 0$, $b \in C^{0,\mu}(\overline{B_R(x_0)})$, $0 < \mu < 1$, $b > 0$, $q > p-1 > 1$ and we denoted by Δ_p the p-Laplace operator given by

$$\Delta_p u = \operatorname{div}\left(|\nabla u|^{p-2}\nabla u\right).$$

The boundary condition in (1) is understood as $u(x) \to +\infty$ when $d(x) := \operatorname{dist}(x, \partial\Omega) \to 0^+$.

Definition 1. *A solution of* (1) *is called a large (or explosive) solution. Most precisely, by a large solution it is understood a strong solution u such that*

$$\lim_{d(x)\searrow 0} u(x) = +\infty.$$

Problem (1) may be viewed as a nonlinear perturbation of the quasilinear logistic equation

$$\Delta_p u = b(x)u^q.$$

The potential $b(x)$ measures the anisotropy and should be assumed to be nonnegative. The above equation was studied for example in [6] under the assumptions $q > p-1$ and $b(x) \geq b_1 > 0$. To the best of our knowledge, the limiting case corresponding to $b = 0$ at some points or even on the boundary has not been studied in the quasilinear case $p > 2$. It is a different situation in the semilinear case $p = 2$ and we refer to [5], [21] for more details. We notice that if $b(y_0) = 0$ with $y_0 \in \partial B_R(x_0)$, then this situation corresponds to a singular case, due to the fact that we are dealing with solutions that blow up on the boundary, which implies the "competition" between the vanishing potential b and the blow-up boundary nonlinearity u^q. We refer to [9] for further results on blow-up boundary solutions for the semilinear logistic equation.

Our purpose in this paper is to study the effect of the absorption perturbation λu^{p-1}. In the light of the above said, the hypothesis $b(x) > 0$ arises as a natural condition on the potential function b. We also suppose that b is satisfying $b(x) = b(\|x - x_0\|)$, therefore $b(r)$ is a real function, $b \in C^{0,\mu}([0,R];(0,\infty))$. In addition, we assume $B(r)/b(r) \in C^1([0,R])$ and $\lim_{r\to R} B(r)/b(r) = 0$, where $B(r) := \int_r^R b(s)ds$. The main result is stated in Theorem 7 and establishes the existence of a unique solution to problem (1) without any further assumption on the decay rate of $b(x)$ near the boundary $\partial\Omega$. The proof combines a related version of the maximum principle with a careful asymptotic analysis near the boundary, which enables us to deduce the uniqueness of the singular solution.

There is a vast literature on elliptic problems that have solutions which blow up at the boundary. Starting with the pioneering papers [3], [19], [10], [16], [12], problems related to large solutions have a long history, arise naturally from a number of different areas and are studied by many authors and in many contexts.

In 1916, in [3], Bieberbach studied the equation $\Delta u = e^u$ in the plane and in 1943, in [19], Rademacher studied the same equation in the space. Later on, singular value problems of this type were studied under the general form $\Delta u = f(u)$ in N-dimensional domains. Answering the questions of blow-up rates near $\partial\Omega$ and uniqueness of solutions has become the goal of more recent literature. In 1993-1994, it was shown that the problem $\Delta u = a(x)e^u$ with $u|_{\partial\Omega} = +\infty$ exhibits a unique solution in a smooth domain Ω together with an estimate of the form

$$u = \log d^{-2} + v(d)$$

in [11] (where $a(x) \geq a_0 > 0$ and $v = O(1)$ as $d \to 0^+$) and in [1] (where $a \equiv 1$ and $v = o(1)$ as $d \to 0^+$). We also point out the equation

$$\Delta u = a(x)u^{(N+2)/(N-2)},$$

first considered in [12] (with $a \equiv 1$) and later in [20], in a ball. Other papers consider the general form $\Delta u = a(x)u^p$ or the p-Laplace extension $\Delta_p u = a(x)u^q$, where $q > p - 1$ and $a > 0$. Recently, the uniqueness and blow-up rates of

$$-\Delta u = \lambda(x)u - a(x)u^p \tag{2}$$

are treated in [7], [8], [13], [17] and many other works can be found from their references. In 1999-2001, under the assumption $a(x) = C_0 d^\gamma + o(d^\gamma)$ as $d \to 0^+$ with $\gamma > 0$ and $C_0 > 0$, an explicit expression for the blow-up rates of this problem was proved in [7] and [8] as

$$u = (\alpha(\alpha+1)/C_0)^{1/(p-1)} d^{-\alpha}(1 + o(d)),$$

$\alpha = (\gamma+2)/(p-1)$. In addition, in [8], if $a(x) = C_0 d^\gamma(1 + C_1 d + o(d))$ as $d \to 0^+$, a further estimate of the blow-up rate is available, namely

$$u(x) = Ad^{-\alpha}(1 + B(s)d + o(d))$$

as $d \to 0^+$, where $B(s) = [(n-1)H(s) - (\alpha+1)C_1]/(\gamma + p + 3)$ with $H(s)$ standing for the mean curvature of $\partial\Omega$ in s and $s = s(x)$ is the projection of x over $\partial\Omega$.

In 2003, in [13], the main result improves the results obtained in [7] and [8] by allowing the weight function $a(x)$ to decay to zero on $\emptyset \neq \Gamma_\infty$ (which is an open or closed subset of $\partial\Omega$ and might have several components) with different rates, depending upon the particular point, or region, of Γ_∞. In the previous results, the decay rate of the weight function was assumed to be the same at any point of $\partial\Omega$. In this case, $a(x)$ might exhibit several different decays at $\partial\Omega$ and this hypothesis simplifies the mathematical analysis of the problem because it allows us to construct global sub and super-solutions in an open neighborhood of $\partial\Omega$.

Another improvement is made in 2005, by Ouyang-Xie [17]. Before their paper, all the blow-up rates were obtained by assuming $a(x) \sim C_0 d^\gamma$ near the boundary. In [17], the equation (2) was considered on a ball domain and the decay rate of the weight function was not assumed to be approximated by a distance function near the boundary $\partial\Omega$ anymore. They considered a to be a continuous function on a ball $B_R(x_0)$ such that $a(x) = a(\|x - x_0\|)$, $a(x) > 0$. They also assumed $A(r)/a(r) \in C^1([0, R])$ and $\lim_{r\to R} A(r)/a(r) = 0$, where $A(r) = \int_r^R a(s)ds$. Obtaining the accurate blow-up rate of solutions for such a function a required more subtle analysis of the problem.

It is very likely that even in this moment someone, somewhere, is trying to improve a mathematical result. As Kafka said in one of his writings: *"As long as you don't stop climbing, the stairs won't end, under your climbing feet they will go on growing upwards."* Indeed, our work has no end. With every answer, another question will be asked; with every article, another problem will be raised.

In fact, quite recently, Ouyang and Xie improved their result from [17] in the new paper [18] (2008). Climbing another step, they produced sharper results in a general domain by combining the localization method of [13] with the result of [17]. Considering Ω to be a bounded smooth domain in $\mathbb{R}^N$ (instead of a ball domain, as in [17]) and $a(x) > 0$ in Ω, for each $x \in \partial\Omega$ they defined the boundary normal sections

$$b_x(r) = a(x - r\boldsymbol{n}_x), \quad r \geq 0, \ r \sim 0, \tag{3}$$

where $\boldsymbol{n}_x$ stands for the outward unit normal vector at $x \in \partial\Omega$. For any $x_0 \in \partial\Omega$, they assumed that there exists $\tau > 0$ such that $a(x) \in C^1(\overline{B}_r(x_0) \cap \overline{\Omega})$, $b_{x_0}(r) \in C^1(0, \tau)$ with $b'_{x_0}(r) > 0$ for each $t \in (0, \tau)$, and

$$\lim_{x\in\partial\Omega,\ x\to x_0,\ r\to 0^+} b_x(r)/b_{x_0}(r) = 1. \tag{4}$$

They also assumed that $\lim_{r\to 0}[(B_{x_0}(r))^2]/[b^*_{x_0}(r)b_{x_0}(r)] \geq 1$ and $\lim_{r\to 0} B_{x_0}(r)/b_{x_0}(r) = 0$, where $B_{x_0}(r) = \int_0^r b_{x_0}(s)ds$, $b^*_{x_0}(r) = \int_0^r B_{x_0}(s)ds$ and they managed to establish the exact boundary blow-up rate for the solution of (2).

Meantime, a parallel study was conducted by López-Gómez in the same spirit, though under different hypotheses. In [14] (2006), López-Gómez first considered the problem (2) on a ball domain $B_R(x_0)$ with $a \in C([0, R]) \cap C^1((0, R])$ satisfying the hypotheses $a(0) = 0$, $a'(r) \geq 0$ and there exists $\delta > 0$ such that $a \in C^2((0, \delta])$, $a'(r) > 0$ and $(\log a)''(r) < 0$ for each $r \in (0, \delta]$. He showed the existence of a unique positive solution and established the exact boundary blow-up rate for the solution of (2) on the ball $B_R(x_0)$. A similar result remains true for annuli. Combining these results with the localization method of [13], he also produced sharper results in a general domain. He worked with the boundary normal sections introduced by the formula (3) under hypotheses just like the ones described in this paragraph, but in addition he imposed the condition (4). Note that in his study appears the associated one-dimensional problem

$$\begin{cases} u'' = gu^p, & r > 0, \\ u(0) = \infty, & u(\infty) = 0 \end{cases}$$

whose analysis seems to be addressed in [14] for the first time. We must remark that the methods used in [18] resemble those used in [14] (in both papers the localization method introduced in [13] is used together with the results previously obtained in a ball or in an annulus domain).

On the other hand, even in the present paper another step is climbed, and we turn again our attention to [17]. But we will not try to obtain sharper results in the same direction as in [18] or [14]. Instead we will make an extension of the results from [17]. Notice that the equation (2) may be considered as being the equation presented in the problem (1) in the particular case $p = 2$. As we have seen, singular boundary value problems such (1) have been intensively studied in the case where $p = 2$. Our main theorem extends the result obtained in [17] to the case $p > 2$ by maintaining similar hypotheses on the weight function. The interest in studying this p-Laplace extension is due to the growing attention for the study of the p-Laplace operator Δ_p in the last decades. This attention is motivated by the fact that it arises in various applications, for example, in Fluid Mechanics, in the study of flowing through porous environments or glacial sliding, in the mathematical model of the torsional creep, in the mathematical approach of the Hele-Shaw flow of "power-law fluids", in Quantum Mechanics etc.

We do not know yet if our main result may be sharpened by working on a more general domain instead of a ball domain, as Ouyang and Xie did in [18]. Their method might fail when applied for $p > 2$, since they used repeatedly the result of [17] for different balls or annuli domains. Difficulties occur when we try to sharpen our result this way, because our nonuniform blow-up rates depend on the radius of the ball domain, which was not the case for $p = 2$. For now, this problem remains open, or, in other words, it becomes another step to be climbed.

In what concerns the structure, our paper is divided as follows. Section 2 is dedicated mainly to the study of the equation

$$-\Delta_p u = \lambda u^{p-1} - b(x)u^q$$

without the assumption of a blow-up on the boundary. We establish a uniqueness result (Theorem 2), a positiveness result (Corollary 1) and finally we combine them in Theorem 5 with an existence result. The first theorem in Section 2 deals with the asymptotic properties of the potential function of the nonlinear term $b(x)$ as x approaches the boundary, while the last theorem, Theorem 6, states the existence of a solution for problem (1) between a sub and a super-solution. These theorems will be especially helpful in Section 3, where we show the existence and the uniqueness of a large solution to our problem (1).

2 Preliminary results

In this section we collect some important results which will be used in the proof of our main result. The first theorem is borrowed from [17] while Theorems 2-6 are extending results we already know from the case $p = 2$ to the case $p > 2$ (actually it is an extension from the Laplace operator to the p-Laplace operator).

Theorem 1. *(Ouyang and Xie [17, Lemma 2.1]). Let $b(r) : [0, R] \to [0, \infty)$ be a continuous function such that $b(r) > 0$ for $r \in [0, R)$. Define*

$$B(r) := \int_r^R b(s)ds, \qquad b^*(r) := \int_r^R B(s)ds.$$

If $g(r) = B(r)/b(r)$ is differentiable in $[0, R]$ and $\lim_{r\to R} g(r) = 0$, $\lim_{r\to R} g'(r) \le 0$, then we have

(b1) $\dfrac{B(r)}{b(r)} \to 0$ *as* $r \to R$,

(b2) $\dfrac{b^*(r)}{B(r)} \to 0$ *as* $r \to R$,

(b3) $\lim_{r\to R} \dfrac{(B(r))^2}{b^*(r)b(r)} = C_0 \ge 1.$

Next, consider the problem

$$\begin{cases} -\Delta_p u = \lambda u^{p-1} - b(x)u^q & \text{in } \Omega, \\ 0 \not\equiv u \ge 0 & \text{in } \Omega, \\ u = \Phi & \text{on } \partial\Omega, \end{cases} \tag{5}$$

where $\Omega \subset \mathbb{R}^N$, $N \ge 3$, is a bounded, connected, open set with smooth boundary, $0 \not\equiv \Phi \in C^{0,\mu}(\partial\Omega)$, $\lambda \ge 0$, $b \in C^{0,\mu}(\overline{\Omega})$, $0 < \mu < 1$, $0 \not\equiv b \ge 0$, $q > p - 1 > 1$ and we denoted by Δ_p the p-Laplace operator.

Theorem 2. *If $\underline{u}$, $\overline{u} \in C^2(\overline{\Omega})$ are both positive in $\overline{\Omega}$ such that*

$$-\Delta_p \underline{u} \le \lambda \underline{u}^{p-1} - b(x)\underline{u}^q \quad in\ \Omega,$$

$$-\Delta_p \overline{u} \ge \lambda \overline{u}^{p-1} - b(x)\overline{u}^q \quad in\ \Omega,$$

and $\underline{u}(x) \le \Phi \le \overline{u}(x)$ on $\partial\Omega$, then $\underline{u}(x) \le \overline{u}(x)$ on $\overline{\Omega}$.

Proof. We use a similar method as in proof of Lemma 1.1 in Marcus-Véron [15] (see also [4], [5] and [7]), that goes back to Benguria-Brezis-Lieb [2]. By our hypotheses

$$\Delta_p \underline{u} + \lambda \underline{u}^{p-1} - b(x)\underline{u}^q \ge 0 \quad \text{in } \Omega \tag{6}$$

and

$$\Delta_p \overline{u} + \lambda \overline{u}^{p-1} - b(x)\overline{u}^q \le 0 \quad \text{in } \Omega. \tag{7}$$

Let w_1, w_2 be nonnegative C^2 functions on $\overline{\Omega}$ vanishing near $\partial\Omega$. We multiply (6) by w_1 and (7) by w_2 and we integrate. Thus

$$\int_\Omega \left(w_1 \Delta_p \underline{u} + \lambda \underline{u}^{p-1} w_1 - b(x)\underline{u}^q w_1 \right) dx \ge \int_\Omega \left(w_2 \Delta_p \overline{u} + \lambda \overline{u}^{p-1} w_2 - b(x)\overline{u}^q w_2 \right) dx.$$

Applying integration by parts we obtain

$$-\int_\Omega \left[|\nabla \underline{u}|^{p-2} \nabla \underline{u} \nabla w_1 - |\nabla \overline{u}|^{p-2} \nabla \overline{u} \nabla w_2 \right] dx +$$
$$\int_{\partial\Omega} \left[|\nabla \underline{u}|^{p-2} \frac{\partial \underline{u}}{\partial \nu} w_1 - |\nabla \overline{u}|^{p-2} \frac{\partial \overline{u}}{\partial \nu} w_2 \right] d\sigma \ge$$
$$\int_\Omega [b(x)(\underline{u}^q w_1 - \overline{u}^q w_2)]\, dx - \lambda \int_\Omega \left[\underline{u}^{p-1} w_1 - \overline{u}^{p-1} w_2 \right] dx.$$

Since w_1, w_2 are vanishing near $\partial\Omega$,

$$-\int_\Omega \left[|\nabla \underline{u}|^{p-2}\nabla \underline{u}\nabla w_1 - |\nabla \overline{u}|^{p-2}\nabla \overline{u}\nabla w_2\right] dx \geq \int_\Omega [b(x)(\underline{u}^q w_1 - \overline{u}^q w_2)]\, dx - \lambda \int_\Omega \left[\underline{u}^{p-1}w_1 - \overline{u}^{p-1}w_2\right] dx. \quad (8)$$

Let $\varepsilon_2 > \varepsilon_1 > 0$ and denote

$$v_1 := \left[\frac{(\underline{u}+\varepsilon_1)^{2p-2} - (\overline{u}+\varepsilon_2)^{2p-2}}{(\underline{u}+\varepsilon_1)^{p-1}}\right]^+,$$

$$v_2 := \left[\frac{(\underline{u}+\varepsilon_1)^{2p-2} - (\overline{u}+\varepsilon_2)^{2p-2}}{(\overline{u}+\varepsilon_2)^{p-1}}\right]^+.$$

Replacing w_i by v_i, $(i = 1, 2)$ in (8) and denoting

$$\Omega_+(\varepsilon_1, \varepsilon_2) := \{x \in \Omega \mid \ \underline{u}+\varepsilon_1 > \overline{u}+\varepsilon_2\}$$

we note that the integrands in (8) (with $w_i = v_i$) vanish outside this set. We arrive at

$$-\int_{\Omega_+(\varepsilon_1,\varepsilon_2)} \left[|\nabla \underline{u}|^{p-2}\nabla \underline{u}\nabla v_1 - |\nabla \overline{u}|^{p-2}\nabla \overline{u}\nabla v_2\right] dx \geq$$
$$\geq \int_{\Omega_+(\varepsilon_1,\varepsilon_2)} [b(x)(\underline{u}^q v_1 - \overline{u}^q v_2)]\, dx - \lambda \int_{\Omega_+(\varepsilon_1,\varepsilon_2)} \left[\underline{u}^{p-1}v_1 - \overline{u}^{p-1}v_2\right] dx. \quad (9)$$

On the left-hand side of (9) we obtain

$$-\int_{\Omega_+(\varepsilon_1,\varepsilon_2)} \left[|\nabla \underline{u}|^{p-2}\nabla \underline{u}\nabla v_1 - |\nabla \overline{u}|^{p-2}\nabla \overline{u}\nabla v_2\right] dx =$$
$$= -(p-1)\int_{\Omega_+(\varepsilon_1,\varepsilon_2)} (\underline{u}+\varepsilon_1)^{p-2}\left[1 + \left(\frac{\overline{u}+\varepsilon_2}{\underline{u}+\varepsilon_1}\right)^{2p-2}\right] |\nabla \underline{u}|^p dx -$$
$$-(p-1)\int_{\Omega_+(\varepsilon_1,\varepsilon_2)} (\overline{u}+\varepsilon_2)^{p-2}\left[1 + \left(\frac{\underline{u}+\varepsilon_1}{\overline{u}+\varepsilon_2}\right)^{2p-2}\right] |\nabla \overline{u}|^p dx +$$
$$+2(p-1)\int_{\Omega_+(\varepsilon_1,\varepsilon_2)} \nabla \underline{u}\nabla \overline{u}\left[(\overline{u}+\varepsilon_2)^{p-2}\left(\frac{\overline{u}+\varepsilon_2}{\underline{u}+\varepsilon_1}\right)^{p-1} |\nabla \underline{u}|^{p-2} + \right.$$
$$\left. +(\underline{u}+\varepsilon_1)^{p-2}\left(\frac{\underline{u}+\varepsilon_1}{\overline{u}+\varepsilon_2}\right)^{p-1} |\nabla \overline{u}|^{p-2}\right] dx.$$

Hence

$$-\int_{\Omega_+(\varepsilon_1,\varepsilon_2)} \left[|\nabla \underline{u}|^{p-2}\nabla \underline{u}\nabla v_1 - |\nabla \overline{u}|^{p-2}\nabla \overline{u}\nabla v_2\right] dx =$$
$$= -(p-1)\int_{\Omega_+(\varepsilon_1,\varepsilon_2)} (\underline{u}+\varepsilon_1)^{p-2}\left(\frac{\overline{u}+\varepsilon_2}{\underline{u}+\varepsilon_1}\right)^{2p-2} |\nabla \underline{u}|^{p-2}\left|\nabla \underline{u} - \nabla \overline{u}\frac{\underline{u}+\varepsilon_1}{\overline{u}+\varepsilon_2}\right|^2 dx -$$

$$-(p-1)\int_{\Omega_+(\varepsilon_1,\varepsilon_2)}(\overline{u}+\varepsilon_2)^{p-2}\left(\frac{\underline{u}+\varepsilon_1}{\overline{u}+\varepsilon_2}\right)^{2p-2}|\nabla\underline{u}|^{p-2}\left|\nabla\overline{u}-\nabla\underline{u}\frac{\overline{u}+\varepsilon_2}{\underline{u}+\varepsilon_1}\right|^2 dx-$$
$$-(p-1)\int_{\Omega_+(\varepsilon_1,\varepsilon_2)}\left[(\underline{u}+\varepsilon_1)^{p-2}\left(|\nabla\underline{u}|^{p-2}-|\nabla\overline{u}|^{p-2}\left(\frac{\underline{u}+\varepsilon_1}{\overline{u}+\varepsilon_2}\right)^{p-2}\right)\cdot\right.$$
$$\left.\cdot\left(|\nabla\underline{u}|^2-|\nabla\overline{u}|^2\left(\frac{\overline{u}+\varepsilon_2}{\underline{u}+\varepsilon_1}\right)^{2p-4}\right)\right]dx.$$

We can rewrite the above equality as

$$-\int_{\Omega_+(\varepsilon_1,\varepsilon_2)}\left[|\nabla\underline{u}|^{p-2}\nabla\underline{u}\nabla v_1-|\nabla\overline{u}|^{p-2}\nabla\overline{u}\nabla v_2\right]dx=I_1+I_2+I_3.$$

Then, relation (9) becomes

$$I_1+I_2+I_3\geq\int_{\Omega_+(\varepsilon_1,\varepsilon_2)}[b(x)(\underline{u}^q v_1-\overline{u}^q v_2)]\,dx-\lambda\int_{\Omega_+(\varepsilon_1,\varepsilon_2)}\left[\underline{u}^{p-1}v_1-\overline{u}^{p-1}v_2\right]dx. \tag{10}$$

As $\varepsilon_2\to 0$, I_3 converges to

$$-(p-1)\int_{\Omega_+(0,0)}\underline{u}^{p-2}\left(|\nabla\underline{u}|^{p-2}-|\nabla\overline{u}|^{p-2}\right)\left(|\nabla\underline{u}|^2-|\nabla\overline{u}|^2\right)dx,$$

which is non-positive. Also, note that I_1 and I_2 are both non-positive. Since

$$v_1\to\frac{\underline{u}^{2p-2}-\overline{u}^{2p-2}}{\underline{u}^{p-1}}\quad\text{when }\varepsilon_2\to 0$$

and

$$v_2\to\frac{\underline{u}^{2p-2}-\overline{u}^{2p-2}}{\overline{u}^{p-1}}\quad\text{when }\varepsilon_2\to 0,$$

the first term on the right-hand side of (10) converges to

$$\int_{\Omega_+(0,0)}b(x)(\underline{u}^{q-p+1}-\overline{u}^{q-p+1})(\underline{u}^{2p-2}-\overline{u}^{2p-2})dx$$

and the second term on the right-hand side of (10) converges to 0, as ε_2 goes to 0. Unless $\Omega_+(0,0)$ is empty, the limiting value of the right-hand side of (10) is positive. Therefore we would have a contradiction, unless $\Omega_+(0,0)$ has the measure 0, i.e. $\underline{u}\leq\overline{u}$ in Ω. ■

Definition 2. *If $\underline{u}$ (respectively $\overline{u}$) satisfies the conditions in Theorem 2 and $\underline{u}\leq\Phi$ (respectively $\overline{u}\geq\Phi$) on $\partial\Omega$, then $\underline{u}$ (respectively $\overline{u}$) is called a sub-solution (respectively super-solution) of* (5).

Theorem 3. *Let $\Omega\subset\mathbb{R}^N$, $N\geq 3$, be a bounded, connected, open set with smooth boundary. Assume $0\not\equiv b\in C^{0,\mu}(\overline{\Omega})$ and $0\not\equiv\Phi\in C^{0,\mu}(\partial\Omega)$ $(0<\mu<1)$, are both non-negative functions and $q>p-1>1$. Then the boundary value problem*

$$\begin{cases}\Delta_p u=b(x)u^q & \text{in }\Omega,\\ 0\not\equiv u\geq 0 & \text{in }\Omega,\\ u=\Phi & \text{on }\partial\Omega,\end{cases}\tag{11}$$

has a unique classical solution, which is positive in Ω.

Proof. To prove the existence of solution we will apply the sub and super-solutions method. We note that $\underline{u} = 0$ is a sub-solution of (11), while $\overline{u} = n$ is a super-solution of (11) if n is large enough. Then problem (11) has at least a solution $u_\Phi \geq 0$. Taking into account the regularity of b, a standard boot-strap argument based on Schauder and Hölder regularity shows that $u_\Phi \in C^2(\Omega) \cap C(\overline{\Omega})$. The uniqueness of u_Φ is given by Theorem 2, for $\lambda \equiv 0$. Next, we discuss the positivity of u_Φ.

Since $\Phi \not\equiv 0$, there exists $x_1 \in \Omega$ such that $u_\Phi(x_1) > 0$. Since Ω is connected, it is sufficient to show that $u_\Phi > 0$ in $B_{\overline{r}}(x_1)$, where $\overline{r} = d(x_1)$, in order to prove that $u_\Phi > 0$ in Ω. Without loss of generality we can assume $x_1 = 0$. By the continuity of u_Φ, there exists $\underline{r} \in (0, \overline{r})$ such that $u_\Phi(x) > 0$ for all $|x| \leq \underline{r}$. Hence $\min_{|x|=\underline{r}} u_\Phi(x) =: \rho > 0$. We define

$$M := \max_{\Omega} b, \quad \eta := \int_{\rho}^{\rho+1} t^{q/(1-p)} dt, \quad \nu(\varepsilon) := \int_{\varepsilon}^{\rho+1} t^{q/(1-p)} dt, \quad (\text{for } \ 0 < \varepsilon < \rho)$$

and

$$A(\underline{r}, \overline{r}) := \{x \in \mathbb{R}^N | \quad \underline{r} < |x| < \overline{r}\}.$$

Thus, to conclude that $u_\Phi > 0$ in Ω all we need to show is that $u_\Phi > 0$ in $A(\underline{r}, \overline{r})$. For this aim we give the following lemma which will be a very useful tool in our proof.

Lemma 1. *For $\varepsilon > 0$ well chosen and $p > 2$, the solution to the problem*

$$\begin{cases} -\Delta_p v = M & \text{in } A(\underline{r}, \overline{r}), \\ v(x) = \eta & \text{as } |x| = \underline{r}, \\ v(x) = \nu(\varepsilon) & \text{as } |x| = \overline{r}, \end{cases} \tag{12}$$

increases in $A(\underline{r}, \overline{r})$.

Proof. Let us consider the corresponding problem in one dimension:

$$(p-1)(v')^{p-2} v'' + \frac{N-1}{r} (v')^{p-1} = -M, \tag{13}$$

where by $(v'(r))^k$ we understand $|v'(r)|^k \cdot \operatorname{sign}(v(r))$. We denote

$$w := (v')^{p-1}.$$

Then equation (13) becomes

$$\frac{dw}{dr} + \frac{N-1}{r} w = -M.$$

By the method of variation of constants, the formula of the general solution is

$$w(r) = \frac{c(r)}{r^{N-1}},$$

where $c = c(r) \in C^1$. By replacing this formula in our equation we obtain

$$\frac{c'(r)}{r^{N-1}} = -M.$$

So $c(r) = -\frac{M}{N}r^N + c_0$, where c_0 is a constant of integration. In conclusion, our general solution is

$$|v'|^{p-1} \cdot \operatorname{sign}(v'(r)) = w(r) = \frac{1}{r^{N-1}}\left(c_0 - \frac{M}{N}r^N\right).$$

We choose c_0 such that

$$c_0 > \frac{M}{N}\overline{r}^N$$

and we consider the function

$$v(r) = \eta +$$
$$+(\overline{r}-\underline{r})\int_{\rho}^{\frac{r-\underline{r}}{\overline{r}-\underline{r}}+\rho}\left\{\frac{c_0}{[(\overline{r}-\underline{r})(s-\rho)+\underline{r}]^{N-1}} - \frac{M}{N}[((\overline{r}-\underline{r})(s-\rho)+\underline{r})]\right\}^{\frac{1}{p-1}} ds,$$

where $r \in [\underline{r}, \overline{r}]$.

Notice that

$$v(\underline{r}) = \eta$$

and

$$v'(r) = \left(\frac{c_0}{r^{N-1}} - \frac{M}{N}r\right)^{\frac{1}{p-1}} > 0, \quad r \in [\underline{r}, \overline{r}].$$

Therefore $[v'(r)]^{p-1} = w(r)$ and v is a solution of the initial value problem

$$\begin{cases} (p-1)(v')^{p-2}v'' + \frac{N-1}{r}(v')^{p-1} = -M \\ v(\underline{r}) = \eta. \end{cases}$$

Since v is increasing, $v(\overline{r}) > v(\underline{r})$, thus $v(\overline{r}) \in (\eta, +\infty)$. Moreover,

$$\int_{\varepsilon}^{\rho+1} \frac{dt}{t^{\frac{q}{p-1}}} \to +\infty \quad \text{as } \varepsilon \searrow 0,$$

for $q > p-1 > 1$ and we can write

$$v(\overline{r}) \in (\int_{\rho}^{\rho+1} t^{q/(1-p)}dt, \int_{0+}^{\rho+1} t^{q/(1-p)}dt).$$

For $\varepsilon \in (0, \rho)$ defined implicitly as

$$v(\overline{r}) = \int_{\varepsilon}^{\rho+1} t^{q/(1-p)}dt,$$

v fulfills the two boundary conditions $v(\underline{r}) = \eta$ and $v(\overline{r}) = \nu(\varepsilon)$. In conclusion, for ε chosen as above, the function v provides us an increasing solution to the problem (12). ∎

For $\varepsilon > 0$ chosen so the conclusion of the above lemma holds, let u^* be the function defined implicitly as follows

$$\int_{u^*+\varepsilon}^{\rho+1} t^{q/(1-p)}dt = v(x) \quad \text{for all } x \in A(\underline{r}, \overline{r}), \tag{14}$$

where v is the solution of Lemma 1. We have

$$\Delta_p v = q(u^* + \varepsilon)^{-q-1}|\nabla u^*|^p - (u^* + \varepsilon)^{-q}\Delta_p u^*.$$

Then

$$M = -\Delta_p v \le (u^* + \varepsilon)^{-q}\Delta_p u^* \le (u^*)^{-q}\Delta_p u^*$$

and we deduce

$$\Delta_p u^* \ge b(x)(u^*)^q \quad \text{in } A(\underline{r}, \overline{r}). \tag{15}$$

It is easy to check that

$$u^*(x) = \rho - \varepsilon < u_\Phi(x), \quad \text{for } |x| = \underline{r} \tag{16}$$

and

$$u^*(x) = 0 \le u_\Phi(x), \quad \text{for } |x| = \overline{r}. \tag{17}$$

Combining relations (15), (16), (17) and using Theorem 2 for the problem

$$\begin{cases} -\Delta_p u = -b(x)u^q & \text{in } A(\underline{r}, \overline{r}), \\ u = u_\Phi & \text{on } \partial A(\underline{r}, \overline{r}), \end{cases}$$

we obtain that $u^* \le u_\Phi$ in $A(\underline{r}, \overline{r})$. By (14) and Lemma 1 we deduce that u^* is decreasing in $A(\underline{r}, \overline{r})$. Since $u^*(x) = 0$ for $|x| = \overline{r}$, we conclude that $u^*(x) > 0$ in $A(\underline{r}, \overline{r})$ and our proof is complete. ∎

The above theorem has an important corollary:

Corollary 1. *Let $\Omega \subset \mathbb{R}^N$, $N \ge 3$, be a bounded, connected, open set with smooth boundary. Assume $0 \not\equiv b \in C^{0,\mu}(\overline{\Omega})$, $0 < \mu < 1$, $b(x) \ge 0$, $a \ge 0$ and u_1 is a non-negative classical solution of the equation*

$$\Delta_p u + au^{p-1} = b(x)u^q \quad \text{in } \Omega$$

such that $u_1 \not\equiv 0$ on $\partial\Omega$. Then u_1 is positive in Ω.

Proof.
We consider the problem (11) with b taken as above and $0 \not\equiv \Phi \in C^{0,\mu}(\partial\Omega)$ chosen such that $0 \le \Phi \le u_1$ on $\partial\Omega$. By applying Theorem 3 we obtain that the problem (11) has a unique solution, say u_0 and, moreover, $u_0 > 0$ in Ω. By the choice of Φ, u_1 is a super-solution of (11), thus $u_1 \ge u_0 > 0$ in Ω. ∎

Remark 1. *For the proof of Theorem 3 we borrowed some techniques used in the proof of Theorem A.1. in Cîrstea-Rădulescu [4], where they considered the case $p = 2$ and a much more general function instead of our function u^q. Although in our work we are not directly interested in the conditions that are more general, we cannot merely ignore them. Therefore we will offer below a theorem which extends the results of Theorem A.1. in [4], under the notice that its proof follows exactly the same steps as in our Theorem 3.*

Theorem 4. *Let $\Omega \subset \mathbb{R}^N$, $N \ge 3$, be a bounded, connected, open set with smooth boundary. Assume $0 \not\equiv b \in C^{0,\mu}(\overline{\Omega})$ and $0 \not\equiv \Phi \in C^{0,\mu}(\partial\Omega)$ $(0 < \mu < 1)$ are both non-negative functions and $f \in C^1([0,\infty))$ is a positive, non-decreasing function on $(0,\infty)$ such that $f(0) = 0$. Then the boundary value problem*

$$\begin{cases} \Delta_p u = b(x)f(u) & \text{in } \Omega, \\ 0 \not\equiv u \geq 0 & \text{in } \Omega, \\ u = \Phi & \text{on } \partial\Omega, \end{cases}$$

has a unique classical solution, which is positive in Ω*.*

We give now the corresponding corollary:

Corollary 2. *Let* $\Omega \subset \mathbb{R}^N$*,* $N \geq 3$*, be a bounded, connected, open set with smooth boundary. Assume* $0 \not\equiv b \in C^{0,\mu}(\overline{\Omega})$*,* $0 < \mu < 1$*,* $b(x) \geq 0$*,* $a \in \mathbb{R}$ *and* $f \in C^1([0,\infty))$ *is a positive, non-decreasing function on* $(0,\infty)$ *such that* $f(0) = 0$*. If* u_1 *is a non-negative classical solution of the equation*

$$\Delta_p u + au^{p-1} = b(x)f(u) \quad \text{in } \Omega$$

such that $u_1 \not\equiv 0$ *on* $\partial\Omega$*, then* u_1 *is positive in* Ω*.*

Proof. Let $0 \not\equiv \Phi \in C^{0,\mu}(\partial\Omega)$ be such that $0 \leq \Phi \leq u_1$ on $\partial\Omega$. We consider the problem

$$\begin{cases} \Delta_p u = |a|u^{p-1} + \|b\|_\infty f(u) & \text{in } \Omega, \\ 0 \not\equiv u \geq 0 & \text{in } \Omega, \\ u = \Phi & \text{on } \partial\Omega. \end{cases} \tag{18}$$

We note that

$$\Delta_p u = g(x)h(u) \quad \text{in } \Omega,$$

where

$$g(x) = \|b\|_\infty \quad \text{and } h(u) = \frac{|a|u^{p-1}}{\|b\|_\infty} + f(u) \in C^1([0,\infty)).$$

Since g is a known function which is independent of the unknown function u and h satisfies $h'(u) \geq 0,\ h > 0$ on $(0,\infty),\ h(0) = 0$, we are under the hypotheses of Theorem 4. Applying Theorem 4 we find that the problem (18) has a unique solution, say u_0 and, moreover, $u_0 > 0$ in Ω. By the choice of Φ, u_1 is a super-solution of (18), thus $u_1 \geq u_0 > 0$ in Ω. ■

Returning to the problem we were discussing before, with $f = u^q$, we are able to prove the next theorem:

Theorem 5. *The problem* (5) *admits a unique positive solution.*

Proof. For the existence of solution we apply again the sub and super-solutions method. We note that $\underline{u} = 0$ is a sub-solution of (5), while $\overline{u} = n$ is a super-solution of (5) if n is large enough. Then problem (5) has a solution which is unique by Theorem 2 and positive by Corollary 1. ■

Remark 2. *If* θ *is a non-negative solution of the problem* (1)*, then the problem* (5) *possesses a unique non-negative solution for each* $\Phi \in C(\partial\Omega)$ *denoted by* u_Φ *and* $u_\Phi \leq \theta$ *in* Ω*.*

Finally, we give our last preliminary result:

Theorem 6. *Let $\Omega \subset \mathbb{R}^N$ be a bounded, connected, open set with smooth boundary. If $\underline{u}$, $\overline{u} \in C^2(\Omega)$ are both positive in Ω such that*

$$-\Delta_p \underline{u} \leq \lambda \underline{u}^{p-1} - b(x)\underline{u}^q \quad \text{in } \Omega,$$

$$-\Delta_p \overline{u} \geq \lambda \overline{u}^{p-1} - b(x)\overline{u}^q \quad \text{in } \Omega,$$

$$\lim_{d(x)\to 0} \underline{u}(x) = +\infty, \qquad \lim_{d(x)\to 0} \overline{u}(x) = +\infty,$$

and $\underline{u}(x) \leq \overline{u}(x)$ in Ω, then there exists at least one solution $u \in C^2(\Omega)$ to (1) *satisfying $\underline{u}(x) \leq u \leq \overline{u}(x)$ in Ω.*

The proof follows exactly the same arguments as in the proof of Theorem 3.2 in López-Gómez [13] and it is omitted for brevity: we apply Theorem 5 to problem (5) in domains $\Omega_n := \{x \in \Omega | \ \ d(x) > 1/n\}$ with $u = (\underline{u} + \overline{u})/2$ on $\partial\Omega_n$ and we make $n \to \infty$ through a diagonal process. The limit of the diagonal sequence provides us with a solution satisfying all the required conditions.

3 Main result

The main result of this paper is given by the following theorem:

Theorem 7. *Consider the radially symmetric quasilinear elliptic equation:*

$$\begin{cases} -\Delta_p u = \lambda u^{p-1} - b(\|x - x_0\|) \cdot u^q & \text{in } B_R(x_0), \\ u > 0 & \text{in } B_R(x_0), \\ u = \infty & \text{on } \partial B_R(x_0), \end{cases} \tag{19}$$

where $B_R(x_0)$ is the ball of radius R centered at $x_0 \in \mathbb{R}^N$, $N \geq 3$, $\lambda > 0$, $q > p-1 > 1$, $b \in C^{0,\mu}([0,R])$ satisfying $b > 0$ in $[0,R]$, $0 < \mu < 1$, $B(r)/b(r) \in C^1([0,R])$ and $\lim_{r\to R} B(r)/b(r) = 0$, where $B(r) := \int_r^R b(s)ds$. Then there exists a unique solution u satisfying

$$\lim_{d(x)\to 0} \frac{u(x)}{K(b^*(\|x - x_0\|))^{-\beta}} = 1,$$

where $d(x) = \operatorname{dist}(x, \partial B_R(x_0))$, K is a constant defined by

$$K := \left[(p-1)[(\beta+1)C_0 - 1]\beta^{p-1}(C_0 b_0)^{(p-2)/2}\right]^{\frac{1}{q-p+1}},$$

with

$$\beta := \frac{p}{2(q-p+1)}, \; b_0 := b(R) > 0, \; C_0 := \lim_{r\to R} \frac{(B(r))^2}{b^*(r)b(r)} \geq 1$$

and

$$b^*(r) = \int_r^R \int_s^R b(t)dtds.$$

Proof. We consider the corresponding singular problem in one dimension

$$\begin{cases} -(p-1)(\psi')^{p-2}\psi'' - \frac{N-1}{r}(\psi')^{p-1} = \lambda\psi^{p-1} - b(r)\psi^q & \text{in } (0,R), \\ \lim_{r\to R}\ \psi(R) = \infty, \\ \psi'(0) = 0. \end{cases} \tag{20}$$

We will show that for each $\varepsilon > 0$, problem (20) admits a positive large solution ψ_ε such that

$$1-\varepsilon \le \liminf_{r\to R} \frac{\psi_\varepsilon(r)}{K(b^*(r))^{-\beta}} \le \limsup_{r\to R} \frac{\psi_\varepsilon(r)}{K(b^*(r))^{-\beta}} \le 1+\varepsilon.$$

Then for each $\varepsilon > 0$ the function

$$u_\varepsilon(x) := \psi_\varepsilon(r), \quad r := \|x - x_0\|$$

provides us a radially symmetric positive large solution of (19) and the solution satisfies

$$1-\varepsilon \le \liminf_{d(x)\to 0} \frac{u_\varepsilon(x)}{K(b^*(\|x-x_0\|))^{-\beta}} \le \limsup_{d(x)\to 0} \frac{u_\varepsilon(x)}{K(b^*(\|x-x_0\|))^{-\beta}} \le 1+\varepsilon. \tag{21}$$

In order to do this we construct a super-solution and a sub-solution of (20). For each $\varepsilon > 0$ we claim that

$$\overline{\psi}_\varepsilon(r) := A + C\left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta},$$

provides us a super-solution, where $A > 0$, $C > 0$ will be determined later. We find

$$\begin{aligned} \overline{\psi}\,'_\varepsilon(r) &= 2C\frac{r}{R^2}(b^*(r))^{-\beta} - \beta C\left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta-1}(b^*(r))' \\ &= 2C\frac{r}{R^2}(b^*(r))^{-\beta} + \beta C\left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta-1}B(r) > 0 \end{aligned}$$

and

$$\begin{aligned} \overline{\psi}\,''_\varepsilon(r) &= 2C\frac{1}{R^2}(b^*(r))^{-\beta} - 4\beta C\frac{r}{R^2}(b^*(r))^{-\beta-1}(b^*(r))' + \\ &\quad + \beta(\beta+1)C\left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta-2}[(b^*(r))']^2 - \beta C\left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta-1}(b^*(r))'' \\ &= 2C\frac{1}{R^2}(b^*(r))^{-\beta} + 4\beta C\frac{r}{R^2}(b^*(r))^{-\beta-1}B(r) + \\ &\quad + \beta(\beta+1)C\left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta-2}(B(r))^2 - \beta C\left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta-1}b(r). \end{aligned}$$

We have $\overline{\psi}_\varepsilon(r) \to \infty$ as $r \to R$ because $b^*(r) \to 0$ as $r \to R$. We also have $\overline{\psi}\,'_\varepsilon(r) \to 0$ as $r \to 0$. Therefore $\overline{\psi}_\varepsilon(r)$ is a super-solution of (20) if, and only if,

$$-(p-1)(\overline{\psi}\,'_\varepsilon(r))^{p-2}\overline{\psi}\,''_\varepsilon(r) - \frac{N-1}{r}(\overline{\psi}\,'_\varepsilon(r))^{p-1} \ge \lambda(\overline{\psi}_\varepsilon(r))^{p-1} - b(r)(\overline{\psi}_\varepsilon(r))^q, \tag{22}$$

i.e.

$$
\begin{aligned}
&-(p-1)\left(2C\frac{r}{R^2}(b^*(r))^{-\beta}+\beta C\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta-1}B(r)\right)^{p-2}\cdot\\
&\left(2C\frac{1}{R^2}(b^*(r))^{-\beta}+4\beta C\frac{r}{R^2}(b^*(r))^{-\beta-1}B(r)+\right.\\
&\left.+\beta(\beta+1)C\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta-2}(B(r))^2-\beta C\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta-1}b(r)\right)-\\
&-\frac{N-1}{r}\left(2C\frac{r}{R^2}(b^*(r))^{-\beta}+\beta C\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta-1}B(r)\right)^{p-1}\geq\\
&\geq\lambda\left(A+C\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta}\right)^{p-1}-b(r)\left(A+C\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta}\right)^{q}.
\end{aligned}
$$

We multiply both sides of this inequality by $(b^*(r))^{(\beta+1)(p-1)+1}\cdot(B(r))^{-p}$. Then

$$
\begin{aligned}
&-(p-1)\left[2C\frac{r}{R^2}\cdot\frac{b^*(r)}{B(r)}+\beta C\left(\frac{r}{R}\right)^2\right]^{p-2}\cdot\\
&\cdot\left[2C\frac{1}{R^2}\cdot\frac{(b^*(r))^2}{(B(r))^2}+4\beta C\frac{r}{R^2}\cdot\frac{b^*(r)}{B(r)}+\beta(\beta+1)C\left(\frac{r}{R}\right)^2-\beta C\left(\frac{r}{R}\right)^2\frac{b^*(r)\cdot b(r)}{(B(r))^2}\right]-\\
&-\frac{N-1}{r}\cdot\frac{b^*(r)}{B(r)}\left[2C\frac{r}{R^2}\cdot\frac{b^*(r)}{B(r)}+\beta C\left(\frac{r}{R}\right)^2\right]^{p-1}\geq\\
&\geq\lambda\left(\frac{b^*(r)}{B(r)}\right)^p\left[A(b^*(r))^\beta+C\left(\frac{r}{R}\right)^2\right]^{p-1}-\\
&-b(r)(b^*(r))^{-\beta q}\frac{(b^*(r))^{(\beta+1)(p-1)+1}}{(B(r))^p}\left[A(b^*(r))^\beta+C\left(\frac{r}{R}\right)^2\right]^{q}.
\end{aligned}
$$

Taking into account that $\beta=\frac{p}{2(q-p+1)}$ and rearranging we have

$$
\begin{aligned}
&-(p-1)\left[2C\frac{r}{R^2}\cdot\frac{b^*(r)}{B(r)}+\beta C\left(\frac{r}{R}\right)^2\right]^{p-2}\cdot\\
&\cdot\left[2C\frac{1}{R^2}\cdot\frac{(b^*(r))^2}{(B(r))^2}+4\beta C\frac{r}{R^2}\cdot\frac{b^*(r)}{B(r)}+\beta(\beta+1)C\left(\frac{r}{R}\right)^2-\beta C\left(\frac{r}{R}\right)^2\frac{b^*(r)\cdot b(r)}{(B(r))^2}\right]-\\
&-\frac{N-1}{r}\cdot\frac{b^*(r)}{B(r)}\left[2C\frac{r}{R^2}\cdot\frac{b^*(r)}{B(r)}+\beta C\left(\frac{r}{R}\right)^2\right]^{p-1}\geq\\
&\geq\lambda\left(\frac{b^*(r)}{B(r)}\right)^p\left[A(b^*(r))^\beta+C\left(\frac{r}{R}\right)^2\right]^{p-1}-\\
&-(b(r))^{1-p/2}\left(\frac{b^*(r)\cdot b(r)}{(B(r))^2}\right)^{p/2}\left[A(b^*(r))^\beta+C\left(\frac{r}{R}\right)^2\right]^{q}.
\end{aligned}
$$

When $r\to R$, using Theorem 1 we obtain

$$
-(p-1)(\beta C)^{p-2}\left(\beta(\beta+1)C-\beta C\frac{1}{C_0}\right)\geq -b_0^{1-p/2}C_0^{-p/2}C^q
$$

therefore

$$C \geq \left[(p-1)[(\beta+1)C_0-1]\beta^{p-1}(C_0 b_0)^{(p-2)/2}\right]^{\frac{1}{q-p+1}}.$$

We choose C,

$$C=(1+\varepsilon)\left[(p-1)[(\beta+1)C_0-1]\beta^{p-1}(C_0 b_0)^{(p-2)/2}\right]^{\frac{1}{q-p+1}}=(1+\varepsilon)K.$$

With this choice of C made, the inequality (22) is satisfied in a left neighborhood of R and choosing A sufficiently large the inequality is satisfied in the whole interval $[0,R]$. Hence $\overline{\psi}_\varepsilon$ is a super-solution of problem (20).

Now let us construct a sub-solution with the same blow-up rate as the super-solution constructed before. For each $\varepsilon>0$ sufficiently small we claim that

$$\underline{\psi}_\varepsilon(r):=\max\left\{0,D+E\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta}\right\},$$

provides us a sub-solution, where $D<0$ and $E>0$ will be determined later. We denote

$$f_D(r):=D+E\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta}.$$

Then

$$f_D'(r)=2E\frac{r}{R^2}(b^*(r))^{-\beta}+\beta E\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta-1}B(r)>0 \text{ in } (0,R).$$

Hence $f_D(r)$ is increasing and

$$\lim_{r\to R} f_D(r)=+\infty;\quad \lim_{r\to 0} f_D(r)=D<0.$$

By the continuity of $f_D(r)$ and the intermediate-value theorem, there exists a unique $Z=Z(D)\in(0,R)$ such that

$$D+E\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta}<0,\quad \text{when } r\in[0,Z(D))$$

and

$$D+E\left(\frac{r}{R}\right)^2(b^*(r))^{-\beta}>0,\quad \text{when } r\in[Z(D),R].$$

Moreover, $Z(D)$ is decreasing and

$$\lim_{D\to-\infty} Z(D)=R,\quad \lim_{D\to 0} Z(D)=0.$$

By the definition of $\underline{\psi}_\varepsilon(r)$ and $Z(D)$,

$$\underline{\psi}_\varepsilon(r)\equiv 0\quad \text{in } [0,Z(D)).$$

It follows that

$$-(p-1)(\underline{\psi}_\varepsilon'(r))^{p-2}\underline{\psi}_\varepsilon''(r)-\frac{N-1}{r}(\underline{\psi}_\varepsilon'(r))^{p-1}\leq\lambda(\underline{\psi}_\varepsilon(r))^{p-1}-b(r)(\underline{\psi}_\varepsilon(r))^q \tag{23}$$

holds in $[0, Z(D))$. Then $\underline{\psi}_\varepsilon(r)$ is a sub-solution of (20) if the inequality above holds in $[Z(D), R]$. We have

$$-\frac{N-1}{r}(\underline{\psi}'_\varepsilon(r))^{p-1} \leq 0 \leq \lambda(\underline{\psi}_\varepsilon(r))^{p-1}, \quad \text{for each } r \in [Z(D), R].$$

Therefore (23) is satisfied in $[Z(D), R]$ if

$$-(p-1)(\underline{\psi}'_\varepsilon(r))^{p-2}\underline{\psi}''_\varepsilon(r) \leq -b(r)(\underline{\psi}_\varepsilon(r))^q, \quad \text{for each } r \in [Z(D), R],$$

i.e.

$$\begin{aligned}
&-(p-1)\left(2E\frac{r}{R^2}(b^*(r))^{-\beta} + \beta E\left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta-1}B(r)\right)^{p-2} \cdot \\
&\left(2E\frac{1}{R^2}(b^*(r))^{-\beta} + 4\beta E\frac{r}{R^2}(b^*(r))^{-\beta-1}B(r) + \right.\\
&\left.+\beta(\beta+1)E\left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta-2}(B(r))^2 - \beta E\left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta-1}b(r)\right) \\
&\leq -b(r)\left(D + E\left(\frac{r}{R}\right)^2 (b^*(r))^{-\beta}\right)^q.
\end{aligned}$$

We multiply both sides of this inequality by $(b^*(r))^{(\beta+1)(p-1)+1} \cdot (B(r))^{-p}$ and use the fact that $\beta = \frac{p}{2(q-p+1)}$.

$$\begin{aligned}
&-(p-1)\left[2E\frac{r}{R^2}\cdot\frac{b^*(r)}{B(r)} + \beta E\left(\frac{r}{R}\right)^2\right]^{p-2} \cdot \\
&\cdot\left[2E\frac{1}{R^2}\cdot\frac{(b^*(r))^2}{(B(r))^2} + 4\beta E\frac{r}{R^2}\cdot\frac{b^*(r)}{B(r)} + \beta(\beta+1)E\left(\frac{r}{R}\right)^2 - \beta E\left(\frac{r}{R}\right)^2\frac{b^*(r)\cdot b(r)}{(B(r))^2}\right] \leq \\
&\leq -(b(r))^{1-p/2}\left(\frac{b^*(r)\cdot b(r)}{(B(r))^2}\right)^{p/2}\left[D(b^*(r))^\beta + E\left(\frac{r}{R}\right)^2\right]^q.
\end{aligned}$$

Let $r \to R$. Using Theorem 1 we obtain

$$-(p-1)(\beta E)^{p-2}\left(\beta(\beta+1)E - \beta E\frac{1}{C_0}\right) \leq -b_0^{1-p/2}C_0^{-p/2}E^q$$

therefore

$$E \leq \left[(p-1)[(\beta+1)C_0 - 1]\beta^{p-1}(C_0 b_0)^{(p-2)/2}\right]^{\frac{1}{q-p+1}}.$$

We choose E,

$$E = (1-\varepsilon)\left[(p-1)[(\beta+1)C_0 - 1]\beta^{p-1}(C_0 b_0)^{(p-2)/2}\right]^{\frac{1}{q-p+1}} = (1-\varepsilon)K.$$

With this choice of E made, the inequality (22) is satisfied in a left neighborhood of R, say $[R-\delta, R]$, where $\delta = \delta(\varepsilon) > 0$. Then we choose D such that $Z(D) = R - \delta(\varepsilon)$ and it follows that $\underline{\psi}_\varepsilon$ is a sub-solution of problem (20).

A sub-solution and a super-solution have been constructed for this problem. Since

$$1-\varepsilon \leq \lim_{r\to R} \frac{\underline{\psi}_\varepsilon(r)}{K(b^*(r))^{-\beta}} \leq \lim_{r\to R} \frac{\overline{\psi}_\varepsilon(r)}{K(b^*(r))^{-\beta}} \leq 1+\varepsilon,$$

then, by Theorem 6, there exists a solution ψ_ε of (20) such that

$$1-\varepsilon \leq \liminf_{r\to R} \frac{\psi_\varepsilon(r)}{K(b^*(r))^{-\beta}} \leq \limsup_{r\to R} \frac{\psi_\varepsilon(r)}{K(b^*(r))^{-\beta}} \leq 1+\varepsilon.$$

Proof of uniqueness. The proof of uniqueness basically follows the proofs in [7], [8] and [17]. We claim that for any arbitrary solution u of (19) we have

$$\lim_{d(x)\to 0} \frac{u(x)}{K(b^*(\|x-x_0\|))^{-\beta}} = 1.$$

Then for any pair of solutions $u,\ v$ of (19)

$$\lim_{d(x)\to 0} \frac{u(x)}{v(x)} = 1. \tag{24}$$

In order to prove our claim, we fix an arbitrary solution u of problem (19) and we will use what we have just proved, that for any $\varepsilon > 0$ there exists a radially symmetric positive solution u_ε of problem (19) satisfying relation (21). We choose $0 < \delta < R/3$ small, we fix $0 < \tau < \delta/4$ and we introduce the region

$$Q_\tau := \left\{ x|\ \tau < d(x) < \frac{\delta}{2} \right\}.$$

Let $M_1 \geq \max_{\|x-x_0\|\leq(R-\delta/4)} u(x)$ be large. For every $\tau \in (0,\ \delta/4)$, we denote by $\overline{V}_\varepsilon$,

$$\overline{V}_\varepsilon(x) := u_\varepsilon\left(x+\tau\frac{x-x_0}{\|x-x_0\|}\right) + M_1 = \psi_\varepsilon(\|x-x_0\|+\tau) + M_1.$$

Note that $\overline{V}_\varepsilon(x) \to \infty$ as $x \to \partial B_{R-\tau}(x_0)$. We also have $\overline{V}_\varepsilon(x) \geq M_1 \geq u(x)$ as $x \to \partial B_{R-\delta/2}(x_0)$. Since $\overline{V}_\varepsilon(x) \geq u(x)$ for $x \in \partial Q_\tau$, $\tau \in (0,\delta/4)$, then $\overline{V}_\varepsilon$ is a super-solution to

$$\begin{cases} -\Delta_p v = \lambda v^{p-1} - b v^q & \text{in } Q_\tau, \\ v = u & \text{on } \partial Q_\tau. \end{cases} \tag{25}$$

Moreover, the auxiliary problem (25) has $v = u$ as its unique solution. We conclude that $u(x) \leq \overline{V}_\varepsilon(x) = \psi_\varepsilon(\|x-x_0\|+\tau) + M_1$ for every $x \in Q_\tau,\ 0<\tau<\delta/4$. Letting $\tau \to 0$, we obtain $u(x) \leq \psi_\varepsilon(\|x-x_0\|) + M_1 = u_\varepsilon(x) + M_1$, for every $x \in Q_\tau,\ 0<\tau<\delta/4$. Thus

$$\limsup_{d(x)\to 0} \frac{u(x)}{K(b^*(\|x-x_0\|))^{-\beta}} \leq \limsup_{d(x)\to 0} \frac{u_\varepsilon(x)}{K(b^*(\|x-x_0\|))^{-\beta}} \leq 1+\varepsilon.$$

Further we will try to find a sub-solution with the same blow-up rate as the super-solution above. For $0 < \delta < R/3$ small and $0 < \tau < \delta/4$ fixed we introduce the annuli region

$$A_{R-\delta,\,R+\tau} := \{x|\ R-\delta < \|x-x_0\| < R+\tau\}.$$

Let $M_2 \geq \max_{R-(\delta+\delta/4)\leq\|x-x_0\|\leq(R-\delta)} u(x)$ be large. For every $\tau \in (0,\ \delta/4)$, we denote by $\underline{V}_\varepsilon$,

$$\underline{V}_\varepsilon(x) := \max\left\{u_\varepsilon\left(x-\tau\frac{x-x_0}{\|x-x_0\|}\right)-M_2,\ 0\right\} = \max\left\{\psi_\varepsilon(\|x-x_0\|-\tau)-M_2,\ 0\right\}.$$

Since $\underline{V}_\varepsilon(x) \leq u(x)$ for $x \in \partial A_{R-\delta,\,R+\tau}$, $\tau \in (0,\delta/4)$, then $\underline{V}_\varepsilon$ is a sub-solution to

$$\begin{cases} -\Delta_p v = \lambda v^{p-1} - b v^q & \text{in } A_{R-\delta,\,R+\tau}, \\ v = u & \text{on } \partial A_{R-\delta,\,R+\tau}. \end{cases}$$

In the same fashion as above, we obtain

$$1-\varepsilon \leq \liminf_{d(x)\to 0}\frac{u_\varepsilon(x)}{K(b^*(\|x-x_0\|))^{-\beta}} \leq \liminf_{d(x)\to 0}\frac{u(x)}{K(b^*(\|x-x_0\|))^{-\beta}}.$$

Thus

$$1-\varepsilon \leq \liminf_{d(x)\to 0}\frac{u(x)}{K(b^*(\|x-x_0\|))^{-\beta}} \leq \limsup_{d(x)\to 0}\frac{u(x)}{K(b^*(\|x-x_0\|))^{-\beta}} \leq 1+\varepsilon.$$

Letting $\varepsilon \to 0$,

$$\lim_{d(x)\to 0}\frac{u(x)}{K(b^*(\|x-x_0\|))^{-\beta}} = 1.$$

Let u and v be two large positive solutions of problem (19). Then u and v satisfy relation (24). For every $\varepsilon > 0$ we can find $\delta > 0$ small enough such that

$$(1-\varepsilon)v(x) \leq u(x) \leq (1+\varepsilon)v(x)$$

when $0 < d(x) \leq \delta$. Note that $\underline{w}(x) := (1-\varepsilon)v(x)$ and $\overline{w}(x) := (1+\varepsilon)v(x)$ are a sub-solution, respectively a super-solution to

$$\begin{cases} -\Delta_p w = \lambda w^{p-1} - b w^q & \text{in } B_{R-\delta}(x_0), \\ w = u & \text{on } \partial B_{R-\delta}(x_0). \end{cases}$$

This problem has $w = u$ as its unique solution. Since

$$(1-\varepsilon)v(x) \leq u(x) \leq (1+\varepsilon)v(x)$$

holds in $B_{R-\delta}(x_0)$, then by Theorem 2 the relation above also holds in $B_R(x_0)$. Letting $\varepsilon \to 0$ we obtain $u = v$ and our proof is complete. ■

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Department of Mathematics,
University of Craiova, 200585 Craiova, Romania
E-mail address: `mmboureanu@yahoo.com`