

Products of composition and differentiation operators on the weighted Bergman space

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Abstract

Motivated by a recent paper by S. Ohno we calculate Hilbert-Schmidt norms of products of composition and differentiation operators on the Bergman space A_α^2 , $\alpha > -1$ and the Hardy space H^2 on the unit disk. When the convergence of sequences (φ_n) of symbols to a given symbol φ implies the convergence of product operators $C_{\varphi_n}D^k$ is also studied. Finally, the boundedness and compactness of the operator $C_\varphi D^k : A_\alpha^2 \rightarrow A_\alpha^2$ are characterized in terms of the generalized Nevanlinna counting function.

1 Introduction and an auxiliary result

Let \mathbb{D} be the open unit disk in \mathbb{C} , $dm(z) = \frac{1}{\pi} r dr d\theta$ the normalized Lebesgue area measure on \mathbb{D} , $dm_\alpha(z) = (\alpha + 1)(1 - |z|^2)^\alpha dm(z)$, $\alpha > -1$, (note that $m_\alpha(\mathbb{D}) = 1$), and $H(\mathbb{D})$ the class of all analytic functions on \mathbb{D} .

The Bergman space $A_\alpha^p = A_\alpha^p(\mathbb{D})$, $p > 0$, $\alpha > -1$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{p,\alpha}^p = \int_{\mathbb{D}} |f(z)|^p dm_\alpha(z) < \infty.$$

With the norm $\|\cdot\|_{p,\alpha}$ the weighted Bergman space becomes a Banach space, when $p \geq 1$. If $p \in (0, 1)$, it is a Frechet space with the translation invariant metric

$$d(f, g) = \|f - g\|_{p,\alpha}^p.$$

The Hardy space $H^p = H^p(\mathbb{D})$, $p > 0$ consists of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{H^p}^p = \sup_{0 < r < 1} \frac{1}{2\pi} \int_0^{2\pi} |f(re^{i\theta})|^p d\theta < \infty.$$

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Since for every $f \in H^p$

$$\lim_{\alpha \rightarrow -1+0} \|f\|_{p,\alpha} = \|f\|_{H^p},$$

H^p can be viewed as A_{-1}^p .

Let D be the differentiation operator, i.e., $Df = f'$, $D^k f = f^{(k)}$, $k \in \mathbb{N}$ and C_φ the composition operator induced by a nonconstant analytic self-map φ of \mathbb{D} . For some classical results on composition operators, see, e.g., [2]. For some recent results, see, e.g., [1, 3, 8, 9, 10, 15, 17, 18, 20, 25, 28, 32, 33] and the references therein. Products of integral, differentiation and composition operators on spaces of analytic functions are studied in [4, 5, 11, 12, 13, 14, 16, 19, 29, 31, 34, 35].

The paper is organized as follows. Motivated by [19, Theorem 3.3] which studies products of composition and differentiation on H^2 , in Section 2 we calculate the exact value of Hilbert-Schmidt norms of products of composition and differentiation operators $C_\varphi D^k$, $k \in \mathbb{N}$ and DC_φ on the Bergman space A_α^2 , $\alpha > -1$ and the Hardy space H^2 .

In Section 3 we investigate when convergence of sequences (φ_n) of symbols to a given symbol φ implies the convergence of product operators $C_{\varphi_n} D^k$.

In the last section we characterize the boundedness and compactness of the operator $C_\varphi D^k : A_\alpha^2 \rightarrow A_\alpha^2$ in terms of the generalized Nevanlinna counting function.

We need the following lemma:

Lemma 1. Let $c_{n,\alpha} = \left(\frac{\Gamma(n+\alpha+2)}{\Gamma(n+1)\Gamma(\alpha+2)}\right)^{1/2}$, $\alpha \geq -1$. Then for $|x| < 1$, the following formula hold true

$$\sum_{n=k}^{\infty} (n(n-1)\cdots(n-k+1))^2 c_{n,\alpha}^2 x^{n-k} = \prod_{j=0}^{k-1} (\alpha+2+j) \left(\frac{x^k}{(1-x)^{\alpha+2+k}}\right)^{(k)}. \quad (1)$$

Proof. Let

$$f(x) = \sum_{n=k}^{\infty} \left(\prod_{j=0}^{k-1} (n-j)\right)^2 c_{n,\alpha}^2 x^{n-k}. \quad (2)$$

It is easy to see that $f^{(n)}(x)$, $n \in \mathbb{Z}$ exists for $|x| < 1$ (when for $n < 0$ it denotes the n th antiderivative of f). Integrating (2) k times, we have that for $|x| < 1$

$$\int_0^x \int_0^{x_1} \cdots \int_0^{x_{k-1}} f(x_k) dx_k \cdots dx_1 = \sum_{n=k}^{\infty} \left(\prod_{j=0}^{k-1} (n-j)\right) c_{n,\alpha}^2 x^n = x^k g(x), \quad (3)$$

and

$$\int_0^x \int_0^{x_1} \cdots \int_0^{x_{k-1}} g(x_k) dx_k \cdots dx_1 = \sum_{n=k}^{\infty} c_{n,\alpha}^2 x^n = \frac{1}{(1-x)^{\alpha+2}} - \sum_{l=0}^{k-1} c_{l,\alpha}^2 x^l. \quad (4)$$

Differentiating (4) k times we obtain

$$g(x) = \frac{\prod_{j=0}^{k-1} (\alpha+2+j)}{(1-x)^{\alpha+2+k}}.$$

Replacing this in (3) and differentiating such obtained formula k times, (1) follows. ■

Remark 1. By using Leibnitz’s formula and some simple calculations, (1) can be written in the following form

$$\sum_{n=k}^{\infty} \left(\prod_{j=0}^{k-1} (n-j) \right)^2 c_{n,\alpha}^2 x^{n-k} = \frac{\sum_{l=0}^k C_l^k \prod_{s=l+1}^k s x^l (1-x)^{k-l} \Gamma(\alpha+k+l+2)}{\Gamma(\alpha+2)(1-x)^{\alpha+2+2k}}. \quad (5)$$

Note also that the polynomial in the numerator is positive on the interval $[0, 1)$.

2 Hilbert-Schmidt norm of $C_\varphi D^k$ (or DC_φ) : $A_\alpha^2 \rightarrow A_\alpha^2$

If \mathcal{H} is a separable Hilbert space, then the Hilbert-Schmidt norm $\|T\|_{HS}$ of an operator $T : \mathcal{H} \rightarrow \mathcal{H}$ is defined by:

$$\|T\|_{HS} = \left(\sum_{n=1}^{\infty} \|Te_n\|^2 \right)^{1/2}, \quad (6)$$

where $\{e_n\}$ is an orthonormal basis on \mathcal{H} . The right-hand side in (6) does not depend on the choice of basis. Hence, it is larger than the operator norm $\|T\|_{op}$ of T .

Let

$$\langle f, g \rangle_\alpha = \int_{\mathbb{D}} f(z) \overline{g(z)} dm_\alpha(z),$$

$\alpha \geq -1$, be the scalar product on A_α^2 and $e_n(z) = c_{n,\alpha} z^n, n \in \mathbb{N}_0$.

Since for $\alpha > -1$

$$\begin{aligned} \|e_n\|_{2,\alpha}^2 &= (\alpha+1)c_{n,\alpha}^2 \int_{\mathbb{D}} |z|^{2n} (1-|z|^2)^\alpha dm(z) \\ &= (\alpha+1)c_{n,\alpha}^2 \int_0^1 \rho^n (1-\rho)^\alpha d\rho = 1 \end{aligned}$$

and $\langle e_n, e_m \rangle_\alpha = 0$ when $m \neq n$, it follows that $(e_n)_{n \in \mathbb{N}_0}$ is an orthonormal basis for A_α^2 . The proof of this fact, for the case $\alpha = -1$, is simpler.

Theorem 1. The Hilbert-Schmidt norm of the operator $C_\varphi D^k$ on $A_\alpha^2, \alpha > -1$ is

$$\|C_\varphi D^k\|_{HS} = \left((\alpha+1) \int_{\mathbb{D}} \frac{P_\alpha(|\varphi(z)|^2)}{(1-|\varphi(z)|^2)^{\alpha+2+2k}} (1-|z|^2)^\alpha dm(z) \right)^{1/2}, \quad (7)$$

where

$$P_\alpha(x) = \frac{1}{\Gamma(\alpha+2)} \sum_{l=0}^k C_l^k \left(\prod_{s=l+1}^k s \right) x^l (1-x)^{k-l} \Gamma(\alpha+k+l+2). \quad (8)$$

Proof. By using the definition of the Hilbert-Schmidt norm, the monotone convergence theorem, Lemma 1 and formula (5), we obtain

$$\begin{aligned} \|C_\varphi D^k\|_{HS}^2 &= \sum_{n=0}^\infty \|C_\varphi D^k(e_n)\|_{2,\alpha}^2 = \sum_{n=0}^\infty c_{n,\alpha}^2 \left(\prod_{j=0}^{k-1} (n-j)\right)^2 \|\varphi^{n-k}\|_{2,\alpha}^2 \\ &= (\alpha + 1) \sum_{n=k}^\infty c_{n,\alpha}^2 \left(\prod_{j=0}^{k-1} (n-j)\right)^2 \int_{\mathbb{D}} |\varphi(z)|^{2n-2k} (1 - |z|^2)^\alpha dm(z) \\ &= (\alpha + 1) \int_{\mathbb{D}} \frac{P_\alpha(|\varphi(z)|^2)}{(1 - |\varphi(z)|^2)^{\alpha+2+2k}} (1 - |z|^2)^\alpha dm(z), \end{aligned}$$

from which the result follows. ■

Corollary 1. *The Hilbert-Schmidt norm of the operator $C_\varphi D$ on A_α^2 , $\alpha > -1$ is*

$$\|C_\varphi D\|_{HS} = \left((\alpha + 1)(\alpha + 2) \int_{\mathbb{D}} \frac{1 + (\alpha + 2)|\varphi(z)|^2}{(1 - |\varphi(z)|^2)^{\alpha+4}} (1 - |z|^2)^\alpha dm(z) \right)^{1/2}. \tag{9}$$

Moreover, the operator $C_\varphi D : A_\alpha^2 \rightarrow A_\alpha^2$ is Hilbert-Schmidt if and only if

$$\int_{\mathbb{D}} \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^{\alpha+4}} dm(z) < \infty. \tag{10}$$

Proof. Theorem 1 and Lemma 1 with $k = 1$, imply formula (9). Since $1 \leq 1 + (\alpha + 2)|\varphi(z)|^2 \leq \alpha + 3$ we get that the integrals in (9) and (10) are comparable, from which the second statement follows. ■

The next corollary can be proved in the same way. Hence we omit its proof.

Corollary 2. *The Hilbert-Schmidt norm of the operator $C_\varphi D^2$ on A_α^2 , $\alpha > -1$ is*

$$\|C_\varphi D^2\|_{HS} = \left(C_\alpha \int_{\mathbb{D}} \frac{2 + (4\alpha + 12)|\varphi(z)| + (\alpha^2 + 5\alpha + 6)|\varphi(z)|^2}{(1 - |\varphi(z)|^2)^{\alpha+6}} dm_\alpha(z) \right)^{1/2},$$

where $C_\alpha = (\alpha + 1)(\alpha + 2)(\alpha + 3)$.

Moreover, the operator $C_\varphi D^2 : A_\alpha^2 \rightarrow A_\alpha^2$ is Hilbert-Schmidt if and only if

$$\int_{\mathbb{D}} \frac{(1 - |z|^2)^\alpha}{(1 - |\varphi(z)|^2)^{\alpha+6}} dm(z) < \infty. \tag{11}$$

Similarly, by Lemma 1 with the case $\alpha = -1$, we can prove the following theorem (for the case $k = 1$ the proof is close to the proof of Theorem 3.3 in [19]):

Theorem 2. *The Hilbert-Schmidt norm of the operator $C_\varphi D^k$ on H^2 is*

$$\|C_\varphi D^k\|_{HS} = \left(\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} \frac{P_{-1}(|\varphi(re^{i\theta})|^2)}{(1 - |\varphi(re^{i\theta})|^2)^{2k+1}} d\theta \right)^{1/2},$$

where $P_{-1}(x)$ is polynomial (8) with $\alpha = -1$.

Corollary 3. *The Hilbert-Schmidt norm of the operator $C_\varphi D$ on H^2 is*

$$\|C_\varphi D\|_{HS} = \left(\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + |\varphi(re^{i\theta})|^2}{(1 - |\varphi(re^{i\theta})|^2)^3} d\theta \right)^{1/2}.$$

Moreover, the operator $C_\varphi D : H^2 \rightarrow H^2$ is Hilbert-Schmidt if and only if

$$\sup_{0 \leq r < 1} \int_0^{2\pi} \frac{d\theta}{(1 - |\varphi(re^{i\theta})|^2)^3} < \infty.$$

3 Hilbert-Schmidt norm of $DC_\varphi : A_\alpha^2 \rightarrow A_\alpha^2$

Here we consider the operator DC_φ . We will not consider the general case $D^k C_\varphi$, $k \in \mathbb{N}$ since the corresponding formulae are not written in a simple way.

Theorem 3. *The Hilbert-Schmidt norm of the operator DC_φ on A_α^2 , $\alpha > -1$ is*

$$\|DC_\varphi\|_{HS} = \left((\alpha + 1)(\alpha + 2) \int_{\mathbb{D}} \frac{1 + (\alpha + 2)|\varphi(z)|^2}{(1 - |\varphi(z)|^2)^{\alpha+4}} |\varphi'(z)|^2 (1 - |z|^2)^\alpha dm(z) \right)^{1/2}.$$

Proof. We have

$$\begin{aligned} \|DC_\varphi\|_{HS}^2 &= \sum_{n=0}^{\infty} \|DC_\varphi(e_n)\|_{2,\alpha}^2 = \sum_{n=0}^{\infty} c_{n,\alpha}^2 n^2 \|\varphi^{n-1} \varphi'\|_{2,\alpha}^2 \\ &= (\alpha + 1) \sum_{n=0}^{\infty} c_{n,\alpha}^2 n^2 \int_{\mathbb{D}} |\varphi(z)|^{2n-2} |\varphi'(z)|^2 (1 - |z|^2)^\alpha dm(z). \end{aligned}$$

From this, by using the monotone convergence theorem and Lemma 1 with $k = 1$ the result follows. ■

For the case $\alpha = -1$, the following result holds true.

Theorem 4. *The Hilbert-Schmidt norm of the operator DC_φ on H^2 is*

$$\|DC_\varphi\|_{HS} = \left(\sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} \frac{1 + |\varphi(re^{i\theta})|^2}{(1 - |\varphi(re^{i\theta})|^2)^3} |\varphi'(re^{i\theta})|^2 d\theta \right)^{1/2}.$$

Proof. We have

$$\|DC_\varphi\|_{HS}^2 = \sum_{n=0}^{\infty} \|DC_\varphi(e_n)\|_{H^2}^2 = \sum_{n=0}^{\infty} \sup_{0 \leq r < 1} \frac{1}{2\pi} \int_0^{2\pi} n^2 |\varphi(re^{i\theta})|^{2n-2} (\varphi'(re^{i\theta}))^2 d\theta.$$

From this, since the supremum and sum can interchange their positions (in view of the positivity and continuity of the integral means appearing there), using the monotone convergence theorem and Lemma 1 (case $\alpha = -1$) the formula follows. ■

4 Continuity with respect to symbols of composition operators

By Theorem 1, similar to the proof of Theorem 1 in [27], the following theorem can be proved. We omit the proof.

Theorem 5. Let $\alpha \geq -1$, $Q_{\alpha,k}(x) = P_{\alpha}(x)/(1-x)^{\alpha+2+2k}$, φ be an analytic self-map of \mathbb{D} and suppose that $(\varphi_n)_{n \in \mathbb{N}}$ is a sequence of analytic self-maps of \mathbb{D} such that $\varphi_n \rightarrow \varphi$ a.e. on \mathbb{D} ,

$$\int_{\mathbb{D}} Q(|\varphi(z)|^2) dm_{\alpha}(z) < \infty, \quad (12)$$

and

$$\lim_{n \rightarrow \infty} \int_{\mathbb{D}} Q(|\varphi_n(z)|^2) dm_{\alpha}(z) = \int_{\mathbb{D}} Q(|\varphi(z)|^2) dm_{\alpha}(z). \quad (13)$$

Then the sequence $(C_{\varphi_n} D^k)_{n \in \mathbb{N}}$ of Hilbert-Schmidt composition operators converges in Hilbert-Schmidt norm to the composition operator $C_{\varphi} D^k$.

Remark 2. Note that in view of equation (7) Theorem 1 can be written in the following form: Let φ be an analytic self-map of \mathbb{D} and suppose that $(\varphi_n)_{n \in \mathbb{N}}$ is a sequence of analytic self-maps of \mathbb{D} such that $C_{\varphi} D^k$ and $(C_{\varphi_n} D^k)_{n \in \mathbb{N}}$ are Hilbert-Schmidt operators, $\varphi_n \rightarrow \varphi$ a.e. on \mathbb{D} , and $\lim_{n \rightarrow \infty} \|C_{\varphi_n} D^k\|_{HS} = \|C_{\varphi} D^k\|_{HS}$. Then $\lim_{n \rightarrow \infty} \|C_{\varphi_n} D^k - C_{\varphi} D^k\|_{HS} = 0$.

Theorem 6. If $C_{\varphi} D^k$ is a Hilbert-Schmidt operator, then

$$\lim_{n \rightarrow \infty} \|C_{\varphi_n} D^k - C_{\varphi} D^k\|_{HS} = 0 \quad (14)$$

if and only if $\|C_{\varphi_n} D^k\|_{HS} \rightarrow \|C_{\varphi} D^k\|_{HS}$ and $\|\varphi_n - \varphi\|_{2,\alpha} \rightarrow 0$ as $n \rightarrow \infty$.

Proof. If (14) holds, then clearly $\|C_{\varphi_n} D^k\|_{HS} \rightarrow \|C_{\varphi} D^k\|_{HS}$ as $n \rightarrow \infty$. On the other hand,

$$\begin{aligned} \|\varphi_n - \varphi\|_{2,\alpha} &= C \|(C_{\varphi_n} D^k - C_{\varphi} D^k)(z^{k+1})\|_{2,\alpha} \\ &\leq \|C_{\varphi_n} D^k - C_{\varphi} D^k\| \|z^{k+1}\|_{2,\alpha} \\ &\leq C \|C_{\varphi_n} D^k - C_{\varphi} D^k\|_{HS}, \end{aligned}$$

from which it follows that $\|\varphi_n - \varphi\|_{2,\alpha} \rightarrow 0$.

Now assume $\|C_{\varphi_n} D^k\|_{HS} \rightarrow \|C_{\varphi} D^k\|_{HS}$ and $\|\varphi_n - \varphi\|_{2,\alpha} \rightarrow 0$ as $n \rightarrow \infty$, but

$$\|C_{\varphi_n} D^k - C_{\varphi} D^k\|_{HS} \not\rightarrow 0.$$

Then there are a subsequence $(C_{\varphi_{n_j}} D^k)_{j \in \mathbb{N}}$ and an $\varepsilon_0 > 0$ such that

$$\|C_{\varphi_{n_j}} D^k - C_{\varphi} D^k\|_{HS} \geq \varepsilon_0 > 0, \quad j \in \mathbb{N}.$$

Since $\|\varphi_{n_j} - \varphi\|_{2,\alpha} \rightarrow 0$, as $j \rightarrow \infty$, there is a subsequence $(\varphi_{n_{j_l}})_{l \in \mathbb{N}}$ converging to φ a.e. By Theorem 5 we have

$$\lim_{l \rightarrow \infty} \|C_{\varphi_{n_{j_l}}} D^k - C_{\varphi} D^k\|_{HS} = 0,$$

which is a contradiction. ■

Theorem 7. *Let φ be an analytic self-map of \mathbb{D} and suppose that $(\varphi_n)_{n \in \mathbb{N}}$ is a sequence of analytic self-maps of \mathbb{D} such that $\|\varphi_n - \varphi\|_{2,\alpha} \rightarrow 0$ as $n \rightarrow \infty$. If there is a measurable function $\chi : \mathbb{D} \rightarrow [0, 1]$ such that for every $n \in \mathbb{N}$, $|\varphi_n| \leq |\chi|$ a.e. on \mathbb{D} and*

$$\int_{\mathbb{D}} Q_{\alpha,k}(|\chi(z)|^2) dm_{\alpha}(z) < \infty,$$

then the sequence $(C_{\varphi_n} D^k)_{n \in \mathbb{N}}$ converges in Hilbert-Schmidt norm to $C_{\varphi} D^k$.

The proof of Theorem 7 is similar to the proof of Theorem 3 in [27] and is omitted.

The next theorem is proved similar to Theorem 4 in [27].

Theorem 8. *For every pair of distinct symbols φ, ψ , let $\chi = \max\{|\varphi|, |\psi|\}$. Then the following upper estimates hold*

$$\|C_{\varphi} D^k - C_{\psi} D^k\|_{HS} \leq M \left(\int_{\mathbb{D}} Q_{\alpha,k}(|\chi(z)|^2) dm_{\alpha}(z) \right)^{1/2} \|\varphi - \psi\|_{\infty},$$

where the constant M is independent of φ and ψ . Therefore, for each $R > 0$, the map $\varphi \rightarrow C_{\varphi} D^k$ is Lipschitz continuous on the set

$$\mathcal{S}_R = \left\{ \varphi \mid \int_{\mathbb{D}} Q_{\alpha,k}(|\chi(z)|^2) dm_{\alpha}(z) \leq R \right\}.$$

5 A characterization of the boundedness and compactness of

$C_{\varphi} D^k$

Here we characterize the boundedness and compactness of $C_{\varphi} D^k : A_{\alpha}^2 \rightarrow A_{\alpha}^2$ in terms of the generalized Nevanlinna counting function. Before we formulate the main result in this section, we need some notation and an auxiliary result.

In view of the following asymptotic relationship

$$1 - |z| \asymp \ln(1/|z|), \quad |z| \rightarrow 1, \tag{15}$$

it is easy to see that the Bergman space A_{α}^p is equivalent with the space of all $f \in H(\mathbb{D})$ such that

$$\|f\|_{p,\alpha,1}^p = \int_{\mathbb{D}} |f(z)|^p \left(\ln \frac{1}{|z|} \right)^{\alpha} dm(z) < \infty.$$

The generalized Nevanlinna counting function $N_{\varphi,\gamma}$, $\gamma > 0$ for φ , is defined as follows

$$N_{\varphi,\gamma}(w) = \sum_{z \in \varphi^{-1}(w)} [\ln(1/|z|)]^{\gamma}, \quad w \in \mathbb{D} \setminus \{\varphi(0)\}.$$

It is introduced by Shapiro in [21] to study composition operators from a weighted Bergman space to itself.

By a well known characterization of the Bergman space (see, for example, [22, 23, 30]) and the asymptotics in (15) it follows that

$$\|f\|_{A_\alpha^2}^2 \asymp |f(0)|^2 + \int_{\mathbb{D}} |f'(z)|^2 \left(\ln \frac{1}{|z|}\right)^{\alpha+2} dm(z).$$

From this and by applying the Stanton's formula

$$\|f \circ \varphi\|_{H^p}^p = |f(\varphi(0))|^p + \frac{p^2}{2} \int_{\mathbb{D}} |f(w)|^{p-2} |f'(w)|^2 N_{\varphi,1}(w) dm(w),$$

with $p = 2$, on the dilations $f_r(z) = f(rz)$, $r \in (0, 1)$, it follows that

$$\|f \circ \varphi\|_{A_\alpha^2}^2 \asymp |f(\varphi(0))|^2 + \int_{\mathbb{D}} |f'(w)|^2 N_{\varphi,\alpha+2}(w) dm(w). \quad (16)$$

It is also known ([21]) that if φ is an analytic self-map of \mathbb{D} with $\varphi(0) \neq 0$, $\alpha \geq 1$ and if $0 < r < |\varphi(0)|$, then

$$N_{\varphi,\alpha}(0) \leq \frac{1}{r^2} \int_{|z| \leq r} N_{\varphi,\alpha}(z) dm(z). \quad (17)$$

The following result is proved in a standard way (see, for example, the proofs of the corresponding results in [2, 6, 24, 25, 26]).

Lemma 2. *The operator $C_\varphi D^k : A_\alpha^2 \rightarrow A_\alpha^2$ is compact if and only if $C_\varphi D^k : A_\alpha^2 \rightarrow A_\alpha^2$ is bounded and for any bounded sequence $(f_n)_{n \in \mathbb{N}}$ in A_α^2 converging to zero uniformly on compacts of \mathbb{D} as $n \rightarrow \infty$, we have $\lim_{n \rightarrow \infty} \|C_\varphi D^k f_n\|_{A_\alpha^2} = 0$.*

Theorem 9. *Let φ be an analytic self-map of \mathbb{D} , $k \in \mathbb{N}$ and $\alpha > -1$. Then*

(a) $C_\varphi D^k : A_\alpha^2 \rightarrow A_\alpha^2$ is bounded if and only if

$$N_{\varphi,\alpha+2}(w) = O\left([\ln(1/|w|)]^{\alpha+2+2k}\right). \quad (18)$$

(b) $C_\varphi D^k : A_\alpha^2 \rightarrow A_\alpha^2$ is compact if and only if

$$N_{\varphi,\alpha+2}(w) = o\left([\ln(1/|w|)]^{\alpha+2+2k}\right), \quad \text{as } |w| \rightarrow 1. \quad (19)$$

Proof. (a) For $u \in \mathbb{D}$ set

$$f_u(z) = \frac{(1 - |u|^2)^{\frac{\alpha+2}{2}}}{(1 - \bar{u}z)^{\alpha+2}}, \quad z \in \mathbb{D}. \quad (20)$$

It is known that $\|f_u\|_{A_\alpha^2} = 1$, for each $u \in \mathbb{D}$. By using (16) we have

$$\begin{aligned} \|C_\varphi D^k\|^2 &\geq \|C_\varphi D^k f_u\|_{A_\alpha^2}^2 = \|f_u^{(k)} \circ \varphi\|_{A_\alpha^2}^2 \\ &\asymp |f_u^{(k)}(\varphi(0))|^2 + \int_{\mathbb{D}} |f_u^{(k+1)}(w)|^2 N_{\varphi, \alpha+2}(w) dm(w) \\ &\geq C \int_{\mathbb{D}} \frac{(1 - |u|^2)^{\alpha+2}}{|1 - \bar{u}w|^{2\alpha+6+2k}} |u|^{2(k+1)} N_{\varphi, \alpha+2}(w) dm(w). \end{aligned}$$

By the change

$$w = \varphi_u(z) = \frac{u - z}{1 - \bar{u}z}$$

and (17) we obtain

$$\begin{aligned} \|C_\varphi D^k\|^2 &\geq C \int_{\mathbb{D}} \frac{|1 - \bar{u}z|^{2\alpha+2+2k}}{(1 - |u|^2)^{\alpha+2+2k}} |u|^{2(k+1)} N_{\varphi, \alpha+2}(\varphi_u(z)) dm(z) \\ &\geq C \int_{|z| \leq 1/2} \frac{|1 - \bar{u}z|^{2\alpha+2+2k}}{(1 - |u|^2)^{\alpha+2+2k}} |u|^{2(k+1)} N_{\varphi, \alpha+2}(\varphi_u(z)) dm(z) \\ &\geq C \frac{|u|^{2(k+1)}}{(1 - |u|^2)^{\alpha+2+2k}} \int_{|z| \leq 1/2} N_{\varphi, \alpha+2}(\varphi_u(z)) dm(z) \\ &\geq C \frac{|u|^{2(k+1)}}{(1 - |u|^2)^{\alpha+2+2k}} N_{\varphi, \alpha+2}(u), \end{aligned} \tag{21}$$

when when $|\varphi_u(\varphi(0))| > 1/2$, which holds for $|u|$ sufficiently close to 1.

From (21) and (15) we obtain (18).

Now assume that (18) holds. Then for each $r \in (0, 1)$, there is $M > 0$ such that

$$\sup_{r < |w| < 1} N_{\varphi, \alpha+2}(w) / [\ln(1/|w|)]^{\alpha+2+2k} \leq M. \tag{22}$$

We have

$$\begin{aligned} \|C_\varphi D^k f\|^2 &\asymp |f^{(k)}(\varphi(0))|^2 + \int_{\mathbb{D}} |f^{(k+1)}(w)|^2 N_{\varphi, \alpha+2}(w) dm(w) \\ &= |f^{(k)}(\varphi(0))|^2 + \left(\int_{\mathbb{D} \setminus r\mathbb{D}} + \int_{r\mathbb{D}} \right) |f^{(k+1)}(w)|^2 N_{\varphi, \alpha+2}(w) dm(w). \end{aligned} \tag{23}$$

By a well known estimate (see, e.g., [7]), we have that

$$|f^{(k)}(\varphi(0))|^2 \leq \frac{C}{(1 - |\varphi(0)|)^{\alpha+2+2k}} \|f\|_{A_\alpha^2}^2. \tag{24}$$

Since $N_{\varphi, \alpha+2}$ is bounded on $r\mathbb{D}$, similarly we obtain

$$\int_{r\mathbb{D}} |f^{(k+1)}(w)|^2 N_{\varphi, \alpha+2}(w) dm(w) \leq C \sup_{|w| \leq r} |f^{(k+1)}(w)|^2 \tag{25}$$

$$\leq \frac{C}{(1 - |\varphi(0)|)^{\alpha+4+2k}} \|f\|_{A_\alpha^2}^2. \tag{26}$$

Let $f(z) = \sum_{n=0}^{\infty} a_n z^n$, then we have

$$\begin{aligned}
& \int_{\mathbb{D} \setminus r\mathbb{D}} |f^{(k+1)}(w)|^2 N_{\varphi, \alpha+2}(w) dm(w) \\
& \leq \sup_{r < |w| < 1} \frac{N_{\varphi, \alpha+2}(w)}{[\ln(1/|w|)]^{\alpha+2+2k}} \int_{\mathbb{D} \setminus r\mathbb{D}} |f^{(k+1)}(w)|^2 \left(\ln \frac{1}{|w|}\right)^{\alpha+2+2k} dm(w) \\
& \leq M \sum_{n=k+1}^{\infty} \prod_{j=0}^k (n-j)^2 |a_n|^2 \int_0^1 r^{2n-2k-1} \left(\ln \frac{1}{r}\right)^{\alpha+2+2k} dr \\
& \leq M \sum_{n=k+1}^{\infty} n^{2+2k} |a_n|^2 \int_0^1 t^{n-k-1} \left(\ln \frac{1}{t}\right)^{\alpha+2+2k} dt \\
& \leq M \sum_{n=k+1}^{\infty} n^{2+2k} |a_n|^2 \int_0^{\infty} e^{-(n-k)s} s^{\alpha+2+2k} ds \\
& = M \sum_{n=k+1}^{\infty} n^{2+2k} |a_n|^2 \frac{\Gamma(\alpha+3+2k)}{(n-k)^{\alpha+3+2k}} \leq CM \sum_{n=0}^{\infty} \frac{|a_n|^2}{n^{\alpha+1}}, \tag{27}
\end{aligned}$$

where we have used the changes $t = r^2$ and $e^{-s} = t$.

On the other hand, by a direct calculation and Stirling's formula, we have

$$\|f\|_{A_{\alpha}^2}^2 = \sum_{n=0}^{\infty} \frac{|a_n|^2}{c_{n,\alpha}^2} \asymp \sum_{n=0}^{\infty} \frac{|a_n|^2}{n^{\alpha+1}}. \tag{28}$$

From (24), (26), (27) and (28) the boundedness of $C_{\varphi} D^k : A_{\alpha}^2 \rightarrow A_{\alpha}^2$ follows.

(b) Assume that $|u_n| \rightarrow 1$ as $n \rightarrow \infty$. Then it is clear that $f_{u_n}(z)$ is a bounded sequence in A_{α}^2 converging to zero uniformly on compacts. Hence

$$\lim_{n \rightarrow \infty} \|C_{\varphi} D^k f_{u_n}\| = 0.$$

On the other hand as in (21) it follows that

$$\|C_{\varphi} D^k f_{u_n}\|^2 \geq C \frac{|u_n|^{2(k+1)}}{(1 - |u_n|^2)^{\alpha+2+2k}} N_{\varphi, \alpha+2}(u_n), \tag{29}$$

from which (19) holds.

Now assume that (19) holds and that $(f_n)_{n \in \mathbb{N}}$ is a bounded sequence in A_{α}^2 converging to zero on compacts of \mathbb{D} . Then by Cauchy's inequality it follows that $f'_n \rightarrow 0$ uniformly on compacts of \mathbb{D} . We also have that for every $\varepsilon > 0$ there is an r such that (22) holds where instead of M is ε , whenever $|w| > r$. On the other hand, we have that $|f_n^{(k+1)}(\varphi(0))| \rightarrow 0$ as $n \rightarrow \infty$, and from (25)

$$\lim_{n \rightarrow \infty} \int_{r\mathbb{D}} |f_n^{(k+1)}(w)|^2 N_{\varphi, \alpha+2}(w) dm(w) \leq C \lim_{n \rightarrow \infty} \max_{|w| \leq r} |f_n^{(k+1)}(w)| = 0.$$

Applying (23) to the function f_n , letting in such obtained inequality $n \rightarrow \infty$, and using above mentioned facts, it follows that

$$\limsup_{n \rightarrow \infty} \|C_{\varphi} D^k f_n\|^2 \leq C\varepsilon.$$

Hence this limit is zero, and consequently the operator $C_{\varphi} D^k : A_{\alpha}^2 \rightarrow A_{\alpha}^2$ is compact. \blacksquare

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