A Quadratically Convergent Class of Modifications for Kovarik's Method

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Abstract

In this article, a single parametric class of modifications for Kovarik's method is proposed. It is proved that all methods in this class are quadratically convergent. Numerical comparison among methods of Kovarik, Petcu-Popa [5], and a special method in this class, chosen based on a specific value for the parameter, shows that Kovarik and Petcu-Popa's methods give almost similar convergence results. However, the special method converges faster and its iteration number is considerably lower than that of others. For Numerical experiments, there are used ten $n \times n$ test matrices with $n = 5, 10, 20, 50$, whose condition numbers vary in the interval $[2\,,\, 8.1e146]$.

1 Introduction

Suppose that $m \leq n$ and A is a $m \times n$ matrix of rank r. Kovarik's method [4] is given by the following iterations:

$$A_0 = A, \;\; K_k = (I - A_k A_k^T)(I + A_k A_k^T)^{-1}, \;\; A_{k+1} = (I + K_k)A_k, \;\; k \geq 0. \tag{1}$$

It is shown [6] that if

$$\|AA^T\|_2 < 1, \tag{2}$$

then $\|A_k A_k^T\|_2 < 1$, for all $k \geq 1$. Assumption (2) can be obtained by an appropriate scaling of matrix A, say, as the following:

$$A^{new} := \frac{1}{\sqrt{\|A\|_1 \, \|A\|_\infty + 1}} \, A. \tag{3}$$

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Therefore, without loss of generality, we assume that A satisfies (2).

It is proved [4,6] that the sequence $\{A_k\}$ defined by (1) converges to $A_\infty = \left[(AA^T)^{1/2}\right]^+ A$, in which A^+ is the Moore-Penrose pseudo inverse of A (see [2]), and the rows of A_∞ are "quasi-orthogonal". It is also proved that $\|K_0\|_2 < 1$ and $\|A_\infty - A_k\|_2 \le \|K_0\|_2^{2^k}$ which show Kovarik's method is quadratically convergent.

Several modifications have been proposed for Kovarik's method, depending upon using some approximations for $(I + A_kA_k^T)^{-1}$, which are generally linearly convergent [1,5,6]. Specially, in [5] Petcu and Popa approximated $(I + A_kA_k^T)^{-1}$ with $I - 0.5A_kA_k^T$ and introduced the modification of

$$A_0 = A,\ \ K_k = (I - A_kA_k^T)(I - 0.5A_kA_k^T),\ \ A_{k+1} = (I + K_k)A_k,\ \ k \ge 0 \quad (4)$$

for Kovarik's method. They proved that (4) is linearly convergent and using numerical tests showed that their method converges rapidly. In [1], we introduced the single parametric class of modifications

$$A_0 = A,\ \ K_k = (I - A_kA_k^T)(I - \alpha A_kA_k^T),\ \ A_{k+1} = (I + K_k)A_k,\ \ k \ge 0 \quad (5)$$

and proved that if $\alpha \in [0.21, 1)$, then all methods of class (5) are linearly convergent with asymptotic error constant $|2\alpha - 1|$. In particular, for $\alpha = 0.5$ we get Petcu-Popa's method (4). In this manner, we proved that method (4) is indeed quadratically convergent and is the only method with this property in class (5).

The current paper gives a variant of the approach given in [1]. Here, we introduce again a single parametric class of modifications for Kovarik's method and prove that all methods in this class are quadratically convergent.

2 A Quadratically Convergent Class of Modifications for Kovarik's Method

To obtain a modification for Kovarik's method, it is necessary to consider the convergence of (1) that leads to the convergence of the singular values sequence $\sigma_j^{(k)}$, $j = 1, \ldots, r$, of A_k given by $\sigma_j^{(k+1)} = g(\sigma_j^{(k)}), k \ge 0$, with $g(t) = 2t/(1 + t^2)$ [1,6]. We try to approximate $g(t)$ in the interval $[-1, 1]$ using a polynomial. In order to do so, consider the function $f(t) = 2/(1 + t)$, $t \in [0, 1]$, and note that $f(1) = 1$. We approximate the function $f(t)$ with a quadratic polynomial passing through the point (1,1). Suppose that this quadratic polynomial is $p(t) = a_0 + a_1t + a_2t^2$. As $p(1) = 1$, we have $a_2 = 1 - a_0 - a_1$. Therefore, $p(t) = a_0 + a_1t - (1-a_0-a_1)t^2$. With this $p(t)$, we obtain the following polynomial approximation for the function $g(t)$:

$$g(t) = \frac{2t}{1+t^2} \approx tp(t^2) = t[1 + \alpha(1 - t^2)(1 - \beta t^2)] = h(t),$$

in which α and β are parameters. Consequently, a class of approximations for $g(t)$ in $[-1,1]$ has been obtained. Nonnegative fixed points of $h(t)$ are $t = 0, 1, 1/\sqrt{\beta}$. As $t = 1$ is supposed to be the unique positive fixed point of $h(t)$ in $[0, 1]$, we have $\beta \in (0, 1)$.

Suppose that $t_0 > 0$. If $e_k = t_k - 1$ denotes the error of the kth iteration of the sequence $t_{k+1} = h(t_k)$, and if $\lim_{k\to\infty} t_k = 1$, then

$$e_{k+1} = (1 - 2\alpha + 2\alpha\beta)e_k + (-3\alpha + 7\alpha\beta)e_k^2 + (-\alpha + 9\alpha\beta)e_k^3 + \cdots .$$

For the iterative process $t_{k+1} = h(t_k)$ to be quadratically convergent with the asymptotic error constant $|c|$, the parameters α and β must be chosen such that

$$\begin{cases} 1 - 2\alpha + 2\alpha\beta = 0 \\ \quad -3\alpha + 7\alpha\beta = c, \end{cases}$$

and therefore $\alpha = 7/(8-2c)$, $\beta = (3+c)/7$. Since $\beta \in (0,1)$, we have $c \in (-3, 4)$ and hence

$$\begin{aligned} h(t) &= t[1 + \alpha(1-t^2)(1-\beta t^2)] = t[1 + \alpha\beta(1-t^2)(\tfrac{1}{\beta} - t^2)] \\ &= t\left[1 + \tfrac{c+3}{8-2c}(1-t^2)(\tfrac{7}{3+c} - t^2)\right]. \end{aligned} \tag{6}$$

Finally, the single parametric class of modifications

$$A_0 = A, \quad K_k = \tfrac{c+3}{8-2c}(I - A_k A_k^T)(\tfrac{7}{3+c}\, I - A_k A_k^T), \quad A_{k+1} = (I + K_k)A_k, \quad k \geq 0 \tag{7}$$

for Kovarik's method is obtained. In particular, Petcu-Popa's method (4) corresponds to $c = 0.5$. In the case of convergence, each method of class (7) is quadratically convergent. In the next section, we study convergence properties of class of methods (7) and impose some conditions on the parameter c to be always convergent.

3 Study of the Convergence

To study the convergence of the single parametric class of methods (7), we should first examine the convergence of the sequence

$$x_{k+1} = h(x_k), \quad k \geq 0, \tag{8}$$

in which $h(x)$ is the same function of (6). As $h(x)$ is an odd function, we consider its behaviour only in the interval $[0, 1]$. The sequence (8) starts from an initial approximation $x_0 \in (0, 1]$. If there exists $x^* = \lim_{k\to\infty} x_k$, then x^* must be a fixed point of $h(x)$ so that $x^* \in \{0, 1, 1/\sqrt{\beta}\}$.

Now, we impose some conditions on β (and equivalently on c) under which the sequence (8) is convergent. In doing so, we find an interval $[0, b]$ containing $[0, 1]$ so that $h : [0, b] \longrightarrow [0, b]$ and the sequence (8) converges to $x^* = 1$ for each initial approximation $x_0 \in (0, b]$. As we need $h(b) \leq b$, we must have $1 - (1+\beta)b^2 + \beta b^4 \leq 0$ and because $\beta \in (0, 1)$, we get $b \in (1, 1/\sqrt{\beta})$. Notice that for these values $h(x) > x$ in $(0, 1)$ and $h(x) < x$ in $(1, b)$. With some simple calculations it can be shown that (see figure 1):

- Suppose that $0 < c < 4$. In the interval $[0, 1]$, the function $h(x)$ has a maximum at $x_{\max} = \sqrt{\frac{15-2c}{15+5c}}$, with $h(x_{\max}) > 1$, and a minimum at $x_{\min} = 0$, with $h(x_{\min}) = 0$. It is strictly ascending on $[0, x_{\max}]$ and strictly descending on $[x_{\max}, 1]$. In the interval $[1, 1/\sqrt{\beta}]$, $h(x)$ has a maximum at $y_{\max} = 1/\sqrt{\beta}$, with $h(y_{\max}) = 1/\sqrt{\beta}$, and a minimum at $y_{\min} = 1$, with $h(y_{\min}) = 1$. Moreover, $h(x)$ is strictly ascending on this interval.
- Suppose that $-3 < c < 0$. In the interval $[0, 1]$, the function $h(x)$ has a maximum at $u_{\max} = 1$, with $h(u_{\max}) = 1$, and a minimum at $u_{\min} = 0$, with $h(u_{\min}) = 0$. It is strictly ascending on this interval. In the interval

$[1, 1/\sqrt{\beta}]$, $h(x)$ has a maximum at $v_{\max} = 1/\sqrt{\beta}$, with $h(v_{\max}) = 1/\sqrt{\beta}$, and a minimum at $v_{\min} = \sqrt{\frac{15-2c}{15+5c}}$, with $h(v_{\min}) < 1$. It is strictly descending on $[1, v_{\min}]$ and strictly ascending on $[v_{\min}, 1/\sqrt{\beta}]$.

- Considering $h(x)$ is an odd function, we can find similar statements on intervals $[-1/\sqrt{\beta}, -1]$ and $[-1, 0]$.
- Since we want the function $h(x)$ to be from $[0, 1/\sqrt{\beta}]$ onto $[0, 1/\sqrt{\beta}]$, the parameter c must be chosen such that:
  - ▷ If $0 < c < 4$, then we need $h(x_{\max}) < h(y_{\max})$, hence $c \in (0, 2]$.
  - ▷ If $-3 < c < 0$, then we need $h(u_{\min}) < h(v_{\min})$, hence $c \in [-2, 0)$.

From the above considerations, we conclude that parameter c must be chosen in the interval $[-2, 2]$ so that $h(x)$ is the function from $[0, 1/\sqrt{\beta}]$ onto $[0, 1/\sqrt{\beta}]$.

In the following theorems, we show that sequence (8) converges to $x^* = 1$ for all $x_0 \in (0, 1/\sqrt{\beta})$ (and therefore for all $x_0 \in (0, 1]$).

Theorem 1. *Suppose $0 < c \le 2$. For each $x_0 \in (0, 1/\sqrt{\beta})$ there exists an index k_0 such that sequence (8) falls in the interval $[1, 1/\sqrt{\beta})$ for all $k \ge k_0$ and converges to $x^* = 1$ monotonically.*

Proof. As $x_{\max} < 1$ and $h(x_{\max}) > 1$, and $h(x)$ is strictly ascending in $[0, x_{\max}]$ there exists a $z_0 \in (0, x_{\max})$ such that $h(z_0) = 1$. Moreover, $h(x) < 1$ for $x \in [0, z_0)$, $h(x) \ge 1$ for $x \in (z_0, 1/\sqrt{\beta}]$. We show that there exists an index k_0 such that $x_{k_0} \ge 1$. If we have $x_0 \in (z_0, 1/\sqrt{\beta})$, then $x_1 = h(x_0) > 1$ is obtained and we can take $k_0 = 1$. Now suppose $x_0 \in (0, z_0)$. If we have $x_k < z_0$ for all k, then sequence $\{x_k\}$ is ascending and upper bounded. Therefore, it has a limit such that $x^* = \lim_{k\to\infty} x_k < z_0$. On the other hand, we have $h(x^*) = x^*$. This is a contradiction because $0 < z_0 < 1$ and nonnegative fixed points of $h(x)$ are only $0, 1, 1/\sqrt{\beta}$. Therefore, there exists $\tilde{k}_0$ so that $x_{\tilde{k}_0} \ge z_0$. In this case we can take $k_0 = \tilde{k}_0 + 1$. According to the above considerations, we have:

$$\begin{aligned} k < k_0 &\implies x_k \in (0, z_0),\ x_{k+1} > x_k, \\ k = k_0 &\implies x_k = 1, \\ k > k_0 &\implies x_k \in (1, 1/\sqrt{\beta}),\ x_{k+1} < x_k. \end{aligned}$$

These relationships show that sequence $\{x_k\}$, after some iterations, falls in $[1, 1/\sqrt{\beta})$ and converges to $x^* = 1$ monotonically. ■

Similarly, it can be proved that

Theorem 2. *Suppose $-2 \le c < 0$. For each $x_0 \in (0, 1/\sqrt{\beta})$ there exists an index k_0 such that sequence (8) falls in the interval $(0, 1]$ for all $k \ge k_0$ and converges to $x^* = 1$ monotonically.*

We can summarize our findings to the following theorem:

Theorem 3. *If $c \in [-2, 2]$ and $b \in (0, 1/\sqrt{\beta})$, in which $\beta = (c+3)/7$, then $x^* = 1$ is the unique nonzero fixed point of $h(x)$ in the interval $[0, b]$ and sequence (8) converges to x^* for all $x_0 \in (0, b]$ (and hence for all $x_0 \in (0,1]$). Moreover, the order of convergence is two.*

The above theorem shows that all methods from class (7) are only convergent for $c \in [-2, 2]$ and their convergence order is two.

4 Numerical Results

In this section, we compare methods (1), (4), and (7) numerically. The methods are programmed using MATLAB software and tested by a PC with PIV processor at 2.8 MHz and 1 Gb RAM (the stop criterion is $\|A_{k+1} - A_k\|_1 < 10^{-6}\|A_{k+1}\|_1$). Also, there are used ten $n \times n$ test matrices [3] with $n = 5, 10, 20, 50$, whose condition numbers vary in the interval $[2\,, 8.1e146]$. The matrices **F**, **H**, **I**, and **J** are well-conditioned, whereas others are ill-conditioned.

Matrix A: Hankel matrix of type I with entries $a_{ij} = (i+j)!$.
Matrix B: Hankel matrix of type II with entries $a_{ij} = 1/(i+j)!$.
Matrix C: Lotkin matrix with entries $a_{1j} = 1$ and $a_{ij} = 1/(i+j-1)$.
Matrix D: Hilbert matrix with entries $a_{ij} = 1/(i+j-1)$.
Matrix E: Pascal matrix with entries $a_{i1} = a_{1j} = 1$ and $a_{ij} = a_{i-1,j} + a_{i,j-1}$.
Matrix F: Dingdong matrix with entries $a_{ij} = 0.5/(n-i-j+1.5)$.
Matrix G: Vandermonde matrix with entries $a_{ij} = i^j$.
Matrix H: Cauchy matrix with entries $a_{ij} = 1/(i-j+0.5)$.
Matrix I: Absolute matrix with entries $a_{ij} = |i-j|$.
Matrix J: Lehmer matrix with entries $a_{ij} = a_{ji} = i/j$.

To determine the best value of the parameter c belonging to $[-2, 2]$, we applied methods of class (7), for $c = -2 + 0.5\delta$, $\delta = 0, 1, \ldots, 8$, to above matrices and noticed that the selection of $c = 2$ gives the least number of iterations in all cases. In the sequel therefore, $c = 2$ is used in (7). We note that there are only n more additions in each iteration of (7) than that of (4).

Numerical results are shown in table 1. The quantities therein denote the number of iterations (the entries of matrix **A** with $n = 50$ are so large that they can not be computed and therefore all methods breakdown). All our test matrices are scaled according to (3), so that (2) holds. By taking table 1 into consideration, we note that these methods do not considerably differ for well-conditioned matrices **F**, **H**, **I**, and **J**, whereas methods (4) and (7) do not use any inverse. But for ill-conditioned matrices, the iteration number of method (7) (with $c = 2$) is considerably lower than that of others.

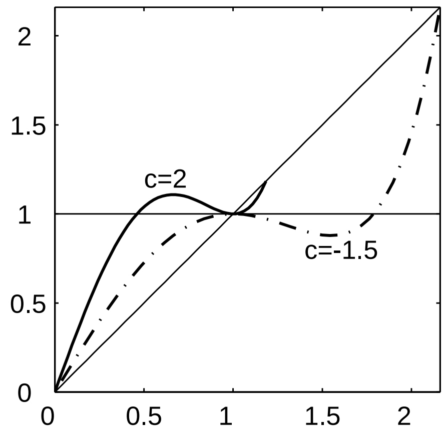
Conclusion

In this paper we presented a single parametric class of modifications (7) for Kovarik's method. We proved that for a parameter c in (7) belonging to $[-2, 2]$, all of methods in the above class would be quadratically convergent.

Table 1. Number of Iterations

	n	(1)	(4)	(7)		n	(1)	(4)	(7)
A	5	32	31	23	**F**	5	6	6	7
	10	76	75	54		10	7	6	6
	20	185	179	125		20	7	6	7
	50	-	-	-		50	8	7	8
B	5	34	33	26	**G**	5	19	19	16
	10	83	82	59		10	46	45	34
	20	175	179	121		20	108	108	77
	50	488	484	336		50	304	308	211
C	5	24	23	19	**H**	5	6	6	7
	10	50	49	37		10	7	6	6
	20	65	66	47		20	7	6	7
	50	69	67	50		50	8	7	8
D	5	24	24	19	**I**	5	8	8	8
	10	49	49	35		10	11	10	10
	20	63	62	48		20	13	12	11
	50	70	66	48		50	15	15	13
E	5	18	18	15	**J**	5	9	8	9
	10	37	36	28		10	11	11	10
	20	74	74	52		20	13	13	12
	50	131	130	94		50	16	15	14

Figure 1. Function h(x) for c=-1.5,2



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References

[1] H. Esmaeili, *A class of modifications for Kovarik's method*, Bull. Belg. Math. Soc. Simon-Stevin 15, 2008, 377-384.

[2] G. H. Golub, C. F. van Loan, *Matrix computations*, The John's Hopkins University Press, Baltimore, 1983.

[3] N. J. Higham, *The test matrix toolbox for MATLAB (version 3.0)*, Numerical Analysis Report No. 276, Manchester Center for Computational Mathematics, Manchester, England, 1995.

[4] Z. Kovarik, *Some iterative methods for improving orthogonality*, SIAM J. Numer. Anal. 7, 1970, 386-389.

[5] D. Petcu, C. Popa, *A new version of Kovarik's approximate orthogonalization algorithm without matrix inversion*, International J. Computer Mathematics 82, 2005, 1235-1246.

[6] C. Popa, *A method for improving orthogonality of rows and columns of matrices*, International J. Computer Mathematics 77, 2001, 469-480.

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