The arithmetic of curves over two dimensional local fields

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Abstract

We study the class field theory of curves defined over two dimensional local fields. The approach used here is a combination of the work of Kato-Saito and Yoshida where the base field is one dimensional

1 Introduction

Let k_1 be a local field with finite residue field and let X be a proper smooth geometrically irreducible curve over k_1. In order to investigate the fundamental group $\pi_1^{ab}(X)$, Saito in [9] introduced the groups $SK_1(X)$ and $V(X)$ and then constructed the maps $\sigma : SK_1(X) \longrightarrow \pi_1^{ab}(X)$ and $\tau : V(X) \longrightarrow \pi_1^{ab}(X)^{g\acute{e}o}$, where $\pi_1^{ab}(X)^{g\acute{e}o}$ is defined by the exact sequence

$$0 \longrightarrow \pi_1^{ab}(X)^{g\acute{e}o} \longrightarrow \pi_1^{ab}(X) \longrightarrow Gal(k_1^{ab}/k_1) \longrightarrow 0$$

The most important results in this context are the following.

1) The quotient of $\pi_1^{ab}(X)$ by the closure of the image of σ as well as the cokernel of τ are isomorphic to $\widehat{\mathbb{Z}}^r$, where r is the rank of the curve.

2) For this integer r, there is an exact sequence

$$0 \longrightarrow (\mathbb{Q}/\mathbb{Z})^r \longrightarrow H^3(K, \mathbb{Q}/\mathbb{Z}(2)) \longrightarrow \underset{v\in P}{\oplus} \mathbb{Q}/\mathbb{Z} \longrightarrow \mathbb{Q}/\mathbb{Z} \longrightarrow 0,$$

Received by the editors May 2007.

Communicated by M. Van den bergh.

1991 *Mathematics Subject Classification* : 11G25, 14H25.

Key words and phrases : Bloch-Ogus complex, Generalized reciprocity map, Higher local fields, Curves over local fields.

where $K = K(X)$ is the function field of X and P indicates the set of closed points of X.

These results are obtained by Saito in [9]. Actually, Saito generalized previous work by Bloch which dealt only with the good reduction case [9, Introduction]. The method of Saito is based upon the class field theory of two-dimensional local ring having finite residue field. He shows these results for arbitrary curves except for the p -primary part in $\operatorname{char} k = p > 0$ [9, Section II-4]. The p -primary part has been proved by Yoshida in [12].

Douai in [3] pointed out that these results can be obtained in a different way. Indeed, one may consider, for any l prime to the residual characteristic, the group $Co\ker\sigma$ as the dual of the group W_0 of the monodromy weight filtration of $H^1(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell)$

$$H^1(\overline{X}, \mathbb{Q}_\ell/\mathbb{Z}_\ell) = W_2 \supseteq W_1 \supseteq W_0 \supseteq 0,$$

where $\overline{X} = X \otimes_{k_1} \overline{k_1}$ and $\overline{k_1}$ is an algebraic closure of k_1. This allows one to extend the preceding results to projective smooth surfaces [3].

The purpose of this paper is to use a combination of this approach and the theory of the monodromy-weight filtration of degenerating abelian varieties on local fields explained by Yoshida in his paper [12] to study curves over two-dimensional local fields (section 3).

Let X be a projective smooth curve defined over a two dimensional local field k. Let K be its function field and P be the set of closed points of X. For each $v \in P$, $k(v)$ denotes the residue field at $v \in P$. A finite etale covering $Z \to X$ of X is called a c.s covering if for any closed point x of X, $x \times_X Z$ is isomorphic to a finite sum of x. We denote by $\pi_1^{c.s}(X)$ the quotient group of $\pi_1^{ab}(X)$ which classifies abelian c.s coverings of X.

To study the class field theory of the curve X, we construct the generalized reciprocity map

$$\sigma/\ell : SK_2(X)/\ell \longrightarrow \pi_1^{ab}(X)/\ell,$$

where $SK_2(X)/\ell = Co\ker\left\{K_3(K)/\ell \xrightarrow{\oplus\partial_v} \underset{v\in P}{\oplus} K_2(k(v))/\ell\right\}$ and $\tau/l : V(X)/\ell \longrightarrow \pi_1^{ab}(X)^{g\acute{e}o}/\ell$ for all ℓ prime to the residual characteristic. The group $V(X)$ is defined to be the kernel of the norm map $N : SK_2(X) \longrightarrow K_2(k)$ induced by the norm map $N_{k(v)/k^x} : K_2(k(v)) \longrightarrow K_2(k)$ for all v and $\pi_1^{ab}(X)^{g\acute{e}o}$ by the exact sequence

$$0 \longrightarrow \pi_1^{ab}(X)^{g\acute{e}o} \longrightarrow \pi_1^{ab}(X) \longrightarrow Gal(k^{ab}/k) \longrightarrow 0$$

The cokernel of σ/ℓ is the quotient group of $\pi_1^{ab}(X)/\ell$ that classifies completely split coverings of X, that is, $\pi_1^{c.s}(X)/\ell$.

We begin by proving the exactness of the following Kato-Saito sequence (Proposition 4.3)

$$\begin{aligned} 0 \longrightarrow \pi_1^{c.s}(X)/\ell \longrightarrow & \quad H^4(K, \mathbb{Z}/\ell(3)) \\ \longrightarrow & \underset{v\in P}{\oplus} H^3(k(v), \mathbb{Z}/\ell(2)) \longrightarrow \mathbb{Z}/\ell \longrightarrow 0 \end{aligned}$$

To determinate the group $\pi_1^{c.s}(X)/\ell$, we need to consider a semi stable model of the curve X (see Section 5) and the weight filtration on its special fiber. In fact, we will prove in (Proposition 5.1) that $\pi_1^{c.s}(X) \otimes \mathbb{Q}_\ell$ admits a quotient of type $\mathbb{Q}_l^r$, where r is the rank of the first crane of this filtration.

Now, to investigate the group $\pi_1^{ab}(X)^{g\acute{e}o}$, we use class field theory of two-dimensional local field and prove the vanishing of the group $H^2(k, \mathbb{Q}/\mathbb{Z})$ (theorem 3.1). This gives the isomorphism

$$\pi_1^{ab}(X)^{g\acute{e}o} \simeq \pi_1^{ab}\left(\overline{X}\right)_{G_k}$$

Finally, using the Grothendieck weight filtration on the group $\pi_1^{ab}\left(\overline{X}\right)_{G_k}$ and assuming the semi-stable reduction, we obtain the structure of the group $\pi_1^{ab}(X)^{g\acute{e}o}$ and some information about the map $\tau : V(X) \longrightarrow \pi_1^{ab}(X)^{g\acute{e}o}$.

This paper is organized as follows. Section 2 is devoted to some notations. Section 3 contains the proprieties of two-dimensional local field we need, such as, duality and the vanishing of the second cohomology group. In section 4, we construct the generalized reciprocity map and study the Bloch-Ogus complex associated to X. Finally, Section 5 is devoted to the group $\pi_1^{c.s}(X)$.

2 Notations

For an abelian group M, and a positive integer $n \geq 1$, M/n denotes the group M/nM. For a scheme Z, and a sheaf $\mathcal{F}$ over the étale site of Z, $H^i(Z, \mathcal{F})$ denotes the i-th étale cohomology group. The group $H^1(Z, \mathbb{Z}/\ell)$ is identified with the group of all continues homomorphisms $\pi_1^{ab}(Z) \longrightarrow \mathbb{Z}/\ell$. If ℓ is invertible on Z, $\mathbb{Z}/\ell(1)$ denotes the sheaf of l-th root of unity and for any integer i, we denote $\mathbb{Z}/\ell(i) = (\mathbb{Z}/\ell(1))^{\otimes i}$.

For a field L, $K_i(L)$ is the i-th Milnor group. It coincides with the $i-$th Quillen group for $i \leq 2$. For ℓ prime to *char* L, there is a Galois symbol

$$h^i_{\ell,L} \quad K_iL/\ell \longrightarrow H^i(L, \mathbb{Z}/\ell(i))$$

which is an isomorphism for $i = 0, 1, 2$ ($i = 2$ is Merkur'jev-Suslin).

3 On two-dimensional local field

A local field k is said to be $n-$dimensional *local* if there exists a sequence of fields k_i $(1 \leq i \leq n)$ such that
(i) each k_i is a complete discrete valuation field with k_{i-1} as the residue field of the valuation ring O_{k_i} of k_i, and
(ii) k_0 is a finite field.

For such a field, and for ℓ prime to Char(k), the well-known isomorphism

$$H^{n+1}(k, \mathbb{Z}/\ell(n)) \simeq \mathbb{Z}/\ell \tag{3.1}$$

holds. If in addition $i \in \{0, ..., n+1\}$, we have the next perfect duality:

$$H^i(k, \mathbb{Z}/\ell(j)) \times H^{n+1-i}(k, \mathbb{Z}/\ell(n-j) \longrightarrow H^{n+1}(k, \mathbb{Z}/\ell(n)) \simeq \mathbb{Z}/\ell \tag{3.2}$$

In the case $n = 2$, the class field theory for such fields is summarized as follows: There is a map $h : K_2(k) \longrightarrow Gal(k^{ab}/k)$ which generalizes the classical reciprocity map for usually local fields. This map induces an isomorphism $K_2(k)/N_{L/k}K_2(L) \simeq Gal(L/k)$ for each finite abelian extension L of k. Furthermore, the canonical pairing

$$H^1(k, \mathbb{Q}_l/\mathbb{Z}_l) \times K_2(k) \longrightarrow H^3(k, \mathbb{Q}_l/\mathbb{Z}_l(2)) \simeq \mathbb{Q}_l/\mathbb{Z}_l \tag{3.3}$$

induces the injective homomorphism

$$H^1(k, \mathbb{Q}_l/\mathbb{Z}_l) \longrightarrow Hom(K_2(k), \mathbb{Q}_l/\mathbb{Z}_l) \tag{3.4}$$

It is well-known that the group $H^2(M, \mathbb{Q}/\mathbb{Z})$ vanishes when M is a finite field or usually local field. Next, we prove the same result for two-dimensional local field.

Theorem 3.1. *If k is a two-dimensional local field of characteristic zero, then the group $H^2(k, \mathbb{Q}/\mathbb{Z})$ vanishes.*

Proof. We proceed as in the proof of theorem 4 of [11]. It is enough to prove that $H^2(k, \mathbb{Q}_l/\mathbb{Z}_l)$ vanishes for all l and when k contains the group μ_l of l-th roots of unity. First, we prove that multiplication by l is injective, that is, we have to show that the coboundary map

$$H^1(k, \mathbb{Q}_l/\mathbb{Z}_l) \xrightarrow{\delta} H^2(k, \mathbb{Z}/l\mathbb{Z})$$

is injective.

By assumption on k, we have

$$H^2(k, \mathbb{Z}/l\mathbb{Z}) \simeq H^2(k, \mu_l) \simeq \mathbb{Z}/\ell$$

The last isomorphism is well-known for one-dimensional local field and was generalized to non archimedean and locally compact fields by Shatz in [7]. Now we show that $\delta \neq 0$;

By class field theory of two dimensional local field, the cohomology group $H^1(k, \mathbb{Q}_l/\mathbb{Z}_l)$ can be identified with the group of continuous homomorphisms $K_2(k) \xrightarrow{\Phi} \mathbb{Q}_l/\mathbb{Z}_l$.

Now, $\delta(\Phi) = 0$ if and only if Φ is a $l-$th power. Moreover, Φ is a $l-$th power if and only if Φ is trivial on μ_l. Thus, it is sufficient to construct an homomorphism $K_2(k) \longrightarrow \mathbb{Q}_l/\mathbb{Z}_l$ which is non trivial on μ_l.

Let i be the maximal natural number such that k contains a primitive l^i-th root of unity. Then, the image ξ of a primitive l^i-th root of unity under the composite map

$$k^x/k^{xl} \simeq H^1(k, \mu_l) \simeq H^1(k, \mathbb{Z}/l\mathbb{Z}) \longrightarrow H^1(k, \mathbb{Q}_l/\mathbb{Z}_l)$$

is not zero. Thus, the injectivity of the map

$$H^1(k, \mathbb{Q}_l/\mathbb{Z}_l) \longrightarrow Hom(K_2(k), \mathbb{Q}_l/\mathbb{Z}_l)$$

gives rise to a character which is non trivial on μ_l. ■

Remark 3.2. This proof is somehow analogous to the proof of Proposition 7 in [5].

4 Curves over two dimensional local field

Let k be a two dimensional local field of characteristic zero and X be a smooth projective curve defined over k.
Recall that $K = K(X)$ is the function field of X and P is set of closed points of X, and for $v \in P$, $k(v)$ is the residue field at $v \in P$.

The residue field of k is one-dimensional local field and is denoted by k_1
Let $n \geq 1$ and $\mathcal{H}^n(\mathbb{Z}/\ell(3))$ be the Zariskien sheaf associated to the presheaf $U \longrightarrow H^n(U, \mathbb{Z}/\ell(3))$. Its cohomology is calculated by the Bloch-Ogus resolution. So, we have the two exact sequences:

$$H^3(K, \mathbb{Z}/\ell(3)) \longrightarrow \underset{v\in P}{\oplus} H^2(k(v), \mathbb{Z}/\ell(2)) \longrightarrow H^1\left(X_{Zar}, \mathcal{H}^3(\mathbb{Z}/\ell(3))\right) \longrightarrow 0 \tag{4.1}$$

$$0 \longrightarrow H^0(X_{Zar}, \mathcal{H}^4(\mathbb{Z}/\ell(3))) \longrightarrow H^4(K, \mathbb{Z}/\ell(3)) \longrightarrow \underset{v\in P}{\oplus} H^3(k(v), \mathbb{Z}/\ell(2)) \tag{4.2}$$

4.1 The reciprocity map

We define the group $SK_2(X)/\ell$ by

$$SK_2(X)/\ell = Co\ker\left\{K_3(K)/\ell \overset{\oplus\partial_v}{\longrightarrow} \underset{v\in P}{\oplus} K_2(k(v))/\ell\right\},$$

where $\partial_v : K_3(K) \longrightarrow K_2(k(v))$ is the boundary map in K-Theory. It will play a key role in class field theory for X as observed by Saito in the introduction of [9]. In this section, we construct a map

$$\sigma/\ell : SK_2(X)/\ell \longrightarrow \pi_1^{ab}(X)/\ell$$

which describes the class field theory of X.

The definition of $SK_2(X)/\ell$ leads to the exact sequence

$$K_3(K)/\ell \longrightarrow \underset{v\in P}{\oplus} K_2(k(v))/\ell \longrightarrow SK_2(X)/\ell \longrightarrow 0$$

On the other hand, it is known that the following diagram is commutative

$$\begin{array}{ccc} K_3(K)/\ell & \longrightarrow \underset{v\in P}{\oplus} & K_2(k(v))/\ell \\ \downarrow h^3 & & \downarrow h^2 \\ H^3(K, \mathbb{Z}/\ell(3)) & \longrightarrow \underset{v\in P}{\oplus} & H^2(k(v), \mathbb{Z}/\ell(2)), \end{array}$$

where h^2, h^3 are the Galois symbols. Taking in account the exact sequence (4.1), we get the existence of a morphism

$$h : SK_2(X)/\ell \longrightarrow H^1\left(X_{Zar}, \mathcal{H}^3(\mathbb{Z}/\ell(2))\right)$$

This morphism fits in the following commutative diagram

$$\begin{array}{ccccccccc} 0 \longrightarrow & K_3(K)/\ell & \longrightarrow \underset{v\in P}{\oplus} & K_2(k(v))/\ell & \rightarrow & SK_2(X)/\ell & \longrightarrow 0 \\ & \downarrow h^3 & & \downarrow h^2 & & \downarrow h & \\ 0 \longrightarrow & H^3(K,\mathbb{Z}/\ell(2)) & \longrightarrow \underset{v\in P}{\oplus} & H^2(k(v),\mathbb{Z}/\ell(2)) & \rightarrow & H^1\left(X_{Zar},\mathcal{H}^3(\mathbb{Z}/\ell(2))\right) & \longrightarrow 0 \end{array}$$

By Merkur'jev-Suslin, the map h^2 is an isomorphism, which implies that h is surjective. Furthermore, the spectral sequence

$$H^p\left(X_{Zar},\mathcal{H}^q(\mathbb{Z}/\ell(3))\right) \Rightarrow H^{p+q}(X,\mathbb{Z}/\ell(3))$$

induces the exact sequence

$$\begin{aligned} 0 \longrightarrow H^1\left(X_{Zar},\mathcal{H}^3(\mathbb{Z}/\ell(3))\right) &\xrightarrow{e} H^4(X,\mathbb{Z}/\ell(3)) \\ \longrightarrow H^0\left(X_{Zar},\mathcal{H}^4(\mathbb{Z}/\ell(3))\right) &\longrightarrow H^2\left(X_{Zar},\mathcal{H}^3(\mathbb{Z}/\ell(3))\right) = 0 \end{aligned} \tag{4.3}$$

Composing h and e, we get the map

$$SK_2(X)/\ell \longrightarrow H^4(X,\mathbb{Z}/\ell(3)).$$

Finally, the group $H^4(X,\mathbb{Z}/\ell(3))$ is identified to the group $\pi_1^{ab}(X)/\ell$ by the duality [4,II, th 2.1]

$$H^4(X,\mathbb{Z}/\ell(3)) \otimes H^1(X,\mathbb{Z}/\ell) \longrightarrow H^5(X,\mathbb{Z}/\ell(3)) \simeq H^3(k,\mathbb{Z}/\ell(2)) \simeq \mathbb{Z}/\ell$$

Hence, we obtain the map

$$\sigma/\ell : SK_2(X)/\ell \longrightarrow \pi_1^{ab}(X)/\ell$$

Remark 4.1. By the exact sequence (4.2), the group $H^0\left(X_{Zar},\mathcal{H}^4(\mathbb{Z}/\ell(3))\right)$ coincides with the kernel of the map

$$H^4(K,\mathbb{Z}/\ell(3)) \longrightarrow \underset{v\in P}{\oplus} H^3(k(v),\mathbb{Z}/\ell(2)).$$

Besides, by (4.3) and localization in étale cohomology

$$\begin{aligned} \underset{v\in P}{\oplus} H^2(k(v),\mathbb{Z}/\ell(2)) \longrightarrow H^4(X,\mathbb{Z}/\ell(3)) \longrightarrow \\ H^4(K,\mathbb{Z}/\ell(3)) \longrightarrow \underset{v\in P}{\oplus} H^3(k(v),\mathbb{Z}/\ell(2)) \end{aligned}$$

we see that $H^1\left(X_{Zar},\mathcal{H}^4(\mathbb{Z}/\ell(3))\right)$ is the image of the Gysin map

$$\underset{v\in P}{\oplus} H^2(k(v),\mathbb{Z}/\ell(2)) \xrightarrow{g} H^4(X,\mathbb{Z}/\ell(3)).$$

Consequently, the morphism g factorize through $H^1\left(X_{Zar},\mathcal{H}^4(\mathbb{Z}/\ell(3))\right)$

$$\begin{array}{ccc} \underset{v\in P}{\oplus} H^2(k(v),\mathbb{Z}/\ell(2)) & \xrightarrow{g} & H^4(X,\mathbb{Z}/\ell(3)) \\ \searrow & & \nearrow \\ & H^1\left(X_{Zar},\mathcal{H}^4(\mathbb{Z}/\ell(3))\right) & \end{array}$$

Then, we derive the following commutative diagram

$$\begin{array}{ccccc}
K_3(K)/\ell & \rightarrow & \underset{v\in P}{\oplus} K_2(k(v))/\ell & \rightarrow & SK_2(X)/\ell \longrightarrow 0 \\
\downarrow h^3 & & \downarrow h^2 & & \downarrow h \\
H^3(K,\mathbb{Z}/\ell(3)) & \rightarrow & \underset{v\in P}{\oplus} H^2(k(v),\mathbb{Z}/\ell(2)) & \rightarrow & H^1\left(X_{Zar},\mathcal{H}^4(\mathbb{Z}/\ell(3))\right) \longrightarrow 0 \\
 & & \downarrow g & \swarrow e & \\
 & & \pi_1^{ab}(X)/l = H^4(X,\mathbb{Z}/\ell(3)) & &
\end{array}$$

The surjectivity of the map h implies that the cokernel of

$$\sigma/\ell : SK_2(X)/\ell \longrightarrow \pi_1^{ab}(X)/\ell$$

coincides with the cokernel of e which is $H^0\left(X_{Zar},\mathcal{H}^4(\mathbb{Z}/\ell(3))\right)$. Hence $Co\ker\sigma/\ell$ is the dual of the kernel of the map

$$H^1(X,\mathbb{Z}/\ell) \longrightarrow \prod_{v\in P} H^1(k(v),\mathbb{Z}/\ell) \tag{4.4}$$

4.2 The Kato-Saito exact sequence

Definition 4.2. Let Z be a Noetherian scheme. A finite etale covering $f : W \rightarrow Z$ is called a c.s covering if for any closed point z of Z , $z \times_Z W$ is isomorphic to a finite scheme-theoretic sum of copies of z. We denote by $\pi_1^{c.s}(Z)$ the quotient group of $\pi_1^{ab}(Z)$ which classifies abelian c.s coverings of Z.

Hence, the group $\pi_1^{c.s}(X)/\ell$ is the dual of the kernel of the map

$$H^1(X,\mathbb{Z}/\ell) \longrightarrow \prod_{v\in P} H^1(k(v),\mathbb{Z}/\ell)$$

as in [9, section 2, definition and sentence just below]. Now, we are in position to calculate the homologies of the Bloch-Ogus complex associated to X.

Generalizing [10, Theorem 7], we obtain the following.

Proposition 4.3. *Let X be a projective smooth curve defined over k. Then for all ℓ, we have the following exact sequence*

$$\begin{array}{ccc}
0 \longrightarrow \pi_1^{c.s}(X)/\ell & \longrightarrow & H^4(K,\mathbb{Z}/\ell(3)) \\
 & \longrightarrow & \underset{v\in P}{\oplus} H^3(k(v),\mathbb{Z}/\ell(2)) \longrightarrow \mathbb{Z}/\ell \longrightarrow 0.
\end{array}$$

Proof. Consider the localization sequence on X

$$\underset{v\in P}{\oplus} H^2(k(v),\mathbb{Z}/\ell(2)) \overset{g}{\longrightarrow} H^4(X,\mathbb{Z}/\ell(3)) \longrightarrow H^4(K,\mathbb{Z}/\ell(3))$$
$$\longrightarrow \underset{v\in P}{\oplus} H^3(k(v),\mathbb{Z}/\ell(2)) \longrightarrow H^5(X,\mathbb{Z}/\ell(3)) \longrightarrow 0$$

We know that the cokernel of the Gysin map g coincides with $\pi_1^{c.s}(X)/\ell$ and we use the isomorphism $H^5(X,\mathbb{Z}/\ell(3)) \simeq \mathbb{Z}/\ell$. ∎

5 The group $\pi_1^{c.s}(X)$

In his paper [9], Saito does not prove the $p-$ primary part in the char $k = p > 0$ case. This case was done by Yoshida in [12]. His method is based on the theory of monodromy-weight filtration of degenerating abelian varieties on local fields. In the current paper, we use this approach to investigate the group $\pi_1^{c.s}(X)$. As mentioned by Yoshida in [12, section 2] Grothendieck's theory of monodromy-weight filtration on Tate module of abelian varieties are valid where the residue field is arbitrary perfect field.

We assume the semi-stable reduction and choose a regular model $\mathcal{X}$ of X over $SpecO_k$, by which we mean a two dimensional regular scheme with a proper birational morphism $f : \mathcal{X} \longrightarrow SpecO_k$ such that $\mathcal{X} \otimes_{O_k} k \simeq X$ and if $\mathcal{X}_s$ designates the special fiber $\mathcal{X} \otimes_{O_k} k_1$, then $Y = (\mathcal{X}_s)_{réd}$ is a curve defined over the residue field k_1 such that any irreducible component of Y is regular and it has ordinary double points as singularity.

Let $\overline{Y} = Y \otimes_{k_1} \overline{k_1}$, where $\overline{k_1}$ is an algebraic closure of k_1 and $\overline{Y}^{[p]} = \bigcup_{i_/<i_1<\cdots<i_p} \overline{Y_{i_/}} \cap \overline{Y_{i_1}} \cap \cdots \cap \overline{Y_{i_p}}$, $(\overline{Y_i})_{i\in I}$ =collection of irreducible components of $\overline{Y}$.

Let $|\overline{\Gamma}|$ be a realization of the dual graph $\overline{\Gamma}$. The group $H^1\left(|\overline{\Gamma}|, \mathbb{Q}_l\right)$ coincides with the group $W_0\left(H^1\left(\overline{Y}, \mathbb{Q}_l\right)\right)$ of all elements of weight 0 for the filtration

$$H^1(\overline{Y}, \mathbb{Q}_\ell) = W_1 \supseteq W_0 \supseteq 0$$

of $H^1(\overline{Y}, \mathbb{Q}_\ell)$ deduced from the spectral sequence

$$E_1^{p,q} = H^q(\overline{Y}^{[p]}, \mathbb{Q}_\ell) \Longrightarrow H^{p+q}(\overline{Y}, \mathbb{Q}_\ell)$$

For details see [2], [3] and [6].

Now, if in addition we assume that the irreducible components and double points of $\overline{Y}$ are defined over k_1, then the dual graph $\overline{\Gamma}$ of $\overline{Y}$ goes down to k_1 and we obtain the injection

$$W_0(H^1\left(\overline{Y}, \mathbb{Q}_l\right)) \subseteq H^1\left(Y, \mathbb{Q}_l\right) \hookrightarrow H^1\left(X, \mathbb{Q}_l\right)$$

Proposition 5.1. *The group $\pi_1^{c.s}(X) \otimes \mathbb{Q}_l$ admits a quotient of type $\mathbb{Q}_l^r$, where r is the $\mathbb{Q}_l$ – rank of the group $H^1\left(|\overline{\Gamma}|, \mathbb{Q}_l\right)$*

Proof. We know (4.4) that $\pi_1^{c.s}(X) \otimes \mathbb{Q}_l$ is the dual of the kernel of the map

$$\alpha : H^1\left(X, \mathbb{Q}_l\right) \longrightarrow \prod_{v\in P} H^1\left(k\left(v\right), \mathbb{Q}_l\right)$$

We will prove that $W_0(H^1\left(\overline{Y}, \mathbb{Q}_l\right)) \subseteq Ker\alpha$. The group $W_0 = W_0(H^1\left(\overline{Y}, \mathbb{Q}_l\right))$ is calculated as the homology of the complex

$$H^0(\overline{Y}^{[0]}, \mathbb{Q}_\ell) \longrightarrow H^0(\overline{Y}^{[1]}, \mathbb{Q}_\ell) \longrightarrow 0$$

Hence

$$W_0 = H^0(\overline{Y}^{[1]}, \mathbb{Q}_\ell) / \operatorname{Im}\{H^0(\overline{Y}^{[0]}, \mathbb{Q}_\ell) \longrightarrow H^0(\overline{Y}^{[1]}, \mathbb{Q}_\ell)\}.$$

Thus, it suffices to prove the vanishing of the composing map

$H^0(\overline{Y}^{[1]}, \mathbb{Q}_\ell) \longrightarrow W_0 \subseteq H^1(Y, \mathbb{Q}_l) \hookrightarrow H^1(X, \mathbb{Q}_l) \longrightarrow H^1(k(v), \mathbb{Q}_l)$
for all $v \in P$.

Let z_v be the $0-$ cycle in $\overline{Y}$ obtained by specializing v, which induces a map $z_v^{[1]} \longrightarrow \overline{Y}^{[1]}$. Consequently, the map $H^0(\overline{Y}^{[1]}, \mathbb{Q}_\ell) \longrightarrow H^1(k(v), \mathbb{Q}_l)$ factors as follows

$$\begin{array}{ccc} H^0(\overline{Y}^{[1]}, \mathbb{Q}_\ell) & \longrightarrow & H^1(k(v), \mathbb{Q}_l) \\ & \searrow \qquad \nearrow & \\ & H^0(z_v^{[1]}, \mathbb{Q}_\ell) & \end{array}$$

But the trace $z_v^{[1]}$ of $\overline{Y}^{[1]}$ on z_v is empty. This implies that $H^0(z_v^{[1]}, \mathbb{Q}_\ell)$ vanishes. ■

Let $V(X)$ be the kernel of the norm map $N : SK_2(X) \longrightarrow K_2(k)$ induced by the norm map $N_{k(v)/k^x} : K_2(k(v)) \longrightarrow K_2(k)$ for all v . Then, we obtain a map $\tau/l : V(X)/\ell \longrightarrow \pi_1^{ab}(X)^{géo}/\ell$ and a commutative diagram

$$\begin{array}{ccccc} V(X)/\ell & \longrightarrow & SK_2(X)/\ell & \rightarrow & K_2(k)/\ell \\ \downarrow \tau/l & & \downarrow \sigma/\ell & & \downarrow h/l \\ \pi_1^{ab}(X)^{géo}/\ell & \longrightarrow & \pi_1^{ab}(X)/\ell & \rightarrow & Gal(k^{ab}/k)/l \end{array}$$

where the map $h/l : K_2(k)/l \longrightarrow Gal(k^{ab}/k)/l$ is the one obtained by class field theory of k (section 3). From this diagram we see that the group $Co\ker \tau/l$ is isomorphic to the group $Co\ker \sigma/\ell$. Next, we investigate the map τ/l.

We start by the following result which is a consequence of the structure of the two-dimensional local field k.

Lemma 5.2. *There is an isomorphism*

$$\pi_1^{ab}(X)^{géo} \simeq \pi_1^{ab}(\overline{X})_{G_k},$$

where $\pi_1^{ab}(\overline{X})_{G_k}$ is the group of coinvariants under $G_k = Gal(k^{ab}/k)$.

Proof. As in the proof of Lemma 4.3 of [12], this is an immediate consequence of (Theorem 3.1). ■

Finally, we are in position to infer the structure of the group $\pi_1^{ab}(X)^{géo}$

Theorem 5.3. *The group $\pi_1^{ab}(X)^{géo} \otimes \mathbb{Q}_l$ is isomorphic to $\widehat{\mathbb{Q}_l}^r$ and the map $\tau : V(X) \longrightarrow \pi_1^{ab}(X)^{géo}$ is a surjection onto $(\pi_1^{ab}(X)^{géo})_{tor}$.*

Proof. By the preceding lemma, we have the isomorphism $\pi_1^{ab}(X)^{géo} \simeq \pi_1^{ab}(\overline{X})_{G_k}$. On the other hand, the group $\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_\ell$ admits the filtration [12,Lemma 4.1 and section 2]

$$W_0(\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l) = \pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l \supseteq W_{-1}(\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l) \supseteq W_{-2}(\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l)$$

But, by the assumption, the curve X admits a semi-stable reduction, then the group $Gr_0(\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l) = W_0(\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l)/W_{-1}(\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l)$ has the following structure

$$0 \longrightarrow Gr_0(\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l)_{tor} \longrightarrow Gr_0(\pi_1^{ab}(\overline{X})_{G_k} \otimes \mathbb{Q}_l) \longrightarrow \widehat{\mathbb{Q}_l}^{r'} \longrightarrow 0,$$

where r' is the $k - rank$ of X. This is confirmed by Yoshida [12, section 2], independently of the finiteness of the residue field of k considered in his paper. The integer r' is equal to the integer $r = H^1(|\overline{\Gamma}|, \mathbb{Q}_l) = H^1(|\Gamma|, \mathbb{Q}_l)$ by assuming that the irreducible components and double points of $\overline{Y}$ are defined over k_1.

Furthermore, the exact sequence

$$0 \longrightarrow W_{-1}(\pi_1^{ab}(\overline{X})_{G_k}) \longrightarrow \pi_1^{ab}(\overline{X})_{G_k} \longrightarrow Gr_0(\pi_1^{ab}(\overline{X})_{G_k}) \longrightarrow 0$$

and (Proposition 5.1) allow us to conclude that the group $W_{-1}(\pi_1^{ab}(\overline{X})_{G_k})$ is finite and the map $\tau : V(X) \longrightarrow \pi_1^{ab}(X)^{géo}$ is a surjection onto $(\pi_1^{ab}(X)^{géo})_{tor}$ as established by Yoshida in [12] for curve over usually local fields. ∎

Remark 5.4. If we use the same method of Saito to study curves over two-dimensional local fields, we need class field theory of two-dimensional local ring having a one-dimensional local field as residue field. This is already done in [1]. Hence, one can follow Saito 's method to obtain the same results.

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