A fixed point property characterizing inner amenable locally compact semigroups

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Abstract

For a locally compact semigroup $\mathfrak{S}$, we study a fixed point property in terms of left Banach $\mathfrak{S}$-modules; we also use this property to give a characterization for inner amenability of $\mathfrak{S}$.

1 Introduction

Throughout this paper, $\mathfrak{S}$ denotes a *locally compact semigroup*; i.e., a semigroup with a locally compact Hausdorff topology whose binary operation is jointly continuous. Let $\mathfrak{X}$ be a *left Banach* $\mathfrak{S}$*-module*, i.e. a Banach space $\mathfrak{X}$ equipped with a map from $\mathfrak{S} \times \mathfrak{X}$ into $\mathfrak{X}$, denoted by $(x, \xi) \longmapsto x.\xi$ $(x \in \mathfrak{S}, \xi \in \mathfrak{X})$ such that

$$x.(y.\xi) = (xy).\xi$$

for all $x, y \in \mathfrak{S}$ and $\xi \in \mathfrak{X}$, the map $x \longmapsto x.\xi$ is continuous of $\mathfrak{S}$ into $\mathfrak{X}$ for all $\xi \in \mathfrak{X}$, the map $\xi \longmapsto x.\xi$ is a bounded linear operator on $\mathfrak{X}$ for all $x \in \mathfrak{S}$, and there is a constant $K > 0$ with

$$\|x.\xi\| \leq K \|\xi\|$$

for all $x \in \mathfrak{S}$ and $\xi \in \mathfrak{X}$. In this case, we define

$$(\xi^*.\mu)(\xi) = \int_{\mathfrak{S}} \xi^*(x.\xi)\, d\mu(x),$$

*The research of the second author was in part supported by a grant from IPM (No. 81430032).
Received by the editors May 2008.
Communicated by F. Bastin.
2000 *Mathematics Subject Classification* : 43A07, 43A10, 43A20, 46H05.
Key words and phrases : Fixed point property, inner amenability, inner invariant mean, left Banach modules, locally compact semigroup, weak* operator topology.

and

$$(\mu.\xi^{**})(\xi^*) = \xi^{**}(\xi^*.\mu);$$

also, we define the operator Λ_μ on $\mathfrak{X}^{**}$ by

$$\Lambda_\mu \xi^{**} = \mu.\xi^{**}$$

for all $\xi \in \mathfrak{X}$, $\xi^* \in \mathfrak{X}^*$, $\xi^{**} \in \mathfrak{X}^{**}$ and $\mu \in M(\mathfrak{S})$, the Banach algebra of all complex Radon measures on $\mathfrak{S}$ with the convolution product $*$ and the total variation norm. Any left Banach $\mathfrak{S}$-module $\mathfrak{X}$ equipped with the map $(\mu, \xi) \longmapsto \mu.\xi$ $(\xi \in \mathfrak{X}, \mu \in M(\mathfrak{S}))$ can be considered as a left Banach $M(\mathfrak{S})$-module. Let $\mathcal{B}(\mathfrak{X}^{**})$ denote the Banach space of bounded linear operators on $\mathfrak{X}^{**}$. By the *weak* operator topology* on $\mathcal{B}(\mathfrak{X}^{**})$, we shall mean the locally convex topology of $\mathcal{B}(\mathfrak{X}^{**})$ determined by the family

$$\{q(\xi^{**}, \xi^*) : \xi^{**} \in \mathfrak{X}^{**}, \xi^* \in \mathfrak{X}^*\}$$

of seminorms on $\mathcal{B}(\mathfrak{X}^{**})$, where

$$q(\xi^{**}, \xi^*)(T) = | \, T\xi^{**}(\xi^*) \, | \quad \text{for all } \; T \in \mathcal{B}(\mathfrak{X}^{**}).$$

The space of all measures $\mu \in M(\mathfrak{S})$ for which the maps $x \longmapsto \delta_x * |\mu|$ and $x \longmapsto |\mu| * \delta_x$ from $\mathfrak{S}$ into $M(\mathfrak{S})$ are weakly continuous is denoted by $M_a(\mathfrak{S})$ (or $\widetilde{L}(\mathfrak{S})$ as in [2]), where δ_x denotes the Dirac measure at x. It is well-known that $M_a(\mathfrak{S})$ is a closed two-sided L-ideal of $M(\mathfrak{S})$; see [2] or [6].

We denote by $\mathcal{P}(M_a(\mathfrak{S}), \mathfrak{X}^{**})$ the closure of the set

$$\{\Lambda_\mu : \mu \in P_1(M_a(\mathfrak{S}))\,\}$$

in the weak* operator topology of $\mathcal{B}(\mathfrak{X}^{**})$, where $P_1(M_a(\mathfrak{S}))$ denotes the set of all probability measures in $M_a(\mathfrak{S})$. Let us remark that $P_1(M_a(\mathfrak{S}))$ with the convolution multiplication is a semigroup. In particular, the set $\{\Lambda_\mu : \mu \in P_1(M_a(\mathfrak{S}))\,\}$ is a subsemigroup of the semigroup $\mathcal{B}(\mathfrak{X}^{**})$ with the ordinary multiplication of linear operators, and as easily verified, so is its closure $\mathcal{P}(M_a(\mathfrak{S}), \mathfrak{X}^{**})$ in the weak* operator topology of $\mathcal{B}(\mathfrak{X}^{**})$.

Definition 1.1. Let $\mathfrak{S}$ be a locally compact semigroup, $\mathfrak{X}$ be a left Banach $\mathfrak{S}$-module, and $\mathcal{M} \subseteq M(\mathfrak{S})$. We say that $\mathfrak{X}$ has the *$\mathcal{M}$-fixed point property* if there exists $\Lambda \in \mathcal{P}(M_a(\mathfrak{S}), \mathfrak{X}^{**})$ such that

$$\Lambda_\mu \Lambda = \Lambda \Lambda_\mu \qquad (\mu \in \mathcal{M}).$$

Our aim in this work is to study this property and its relation to inner amenability of locally compact semigroups.

2 The results

We commence with the following result.

Lemma 2.1. *Let $\mathfrak{S}$ be a locally compact semigroup and $\mathcal{M} \subseteq M(\mathfrak{S})$. Suppose that there exists a net $(\mu_\alpha)_{\alpha\in A}$ in $P_1(M_a(\mathfrak{S}))$ such that $\|\mu * \mu_\alpha - \mu_\alpha * \mu\| \longrightarrow 0$ for all $\mu \in \mathcal{M}$. Then every left Banach $\mathfrak{S}$-module $\mathfrak{X}$ has the $\mathcal{M}$-fixed point property.*

Proof. First, note that the operator algebra $\mathcal{B}(\mathfrak{X}^{**})$ can be identified with the dual space $(\mathfrak{X}^{**}\widehat{\otimes}\mathfrak{X}^*)^*$ of the projective tensor product $\mathfrak{X}^{**}\widehat{\otimes}\mathfrak{X}^*$ in a natural way; see for example Corollary VIII.2.2 of [5]. In particular, the weak* operator topology of $\mathcal{B}(\mathfrak{X}^{**})$ coincides with the weak* topology of $(\mathfrak{X}^{**} \otimes \mathfrak{X}^*)^*$ on bounded subsets of $\mathcal{B}(\mathfrak{X}^{**})$, and therefore $\mathcal{P}(M_a(\mathfrak{S}), \mathfrak{X}^{**})$ is compact in the weak* operator topology of $\mathcal{B}(\mathfrak{X}^{**})$.

Next, we may find $\Lambda \in \mathcal{P}(M_a(\mathfrak{S}), \mathfrak{X}^{**})$ with $\|\Lambda\| \leq K$ and a subnet (μ_δ) of (μ_α) such that

$$\Lambda_{\mu_\delta} \longrightarrow \Lambda$$

in the weak* operator topology; where K is a constant satisfying

$$\| x.\xi \| \leq K \| \xi \|$$

for all $x \in \mathfrak{S}$ and $\xi \in \mathfrak{X}$. For each $\mu \in \mathcal{M}$, we therefore have

$$\Lambda_{\mu_\delta}\Lambda_\mu \longrightarrow \Lambda\Lambda_\mu \quad \text{and} \quad \Lambda_\mu\Lambda_{\mu_\delta} \longrightarrow \Lambda_\mu\Lambda$$

in the weak* operator topology. Also

$$|(\Lambda_{\mu_\delta}\Lambda_\mu\xi^{**} - \Lambda_\mu\Lambda_{\mu_\delta}\xi^{**})(\xi^*)| \leq K\, \|\mu_\delta * \mu - \mu * \mu_\delta\|\ \|\xi^{**}\|\ \|\xi^*\| \longrightarrow 0,$$

for all $\xi^* \in \mathfrak{X}^*$ and $\xi^{**} \in \mathfrak{X}^{**}$, and hence

$$\Lambda_{\mu_\delta}\Lambda_\mu - \Lambda_\mu\Lambda_{\mu_\delta} \longrightarrow 0$$

in the weak* operator topology. Consequently, $\Lambda\Lambda_\mu = \Lambda_\mu\Lambda$. □

Let us recall that $M_a(\mathfrak{S})^{**}$ with the first Arens product $\odot$ defined by

$$(F \odot G)(f) = F(G\,f)$$

for $f \in M_a(\mathfrak{S})^*$ and $F, G \in M_a(\mathfrak{S})^{**}$, is a Banach algebra, where

$$(G\,f)(\mu) = G(f\,\mu)$$

and

$$(f\mu)(\nu) = f(\mu * \nu)$$

for all $\mu, \nu \in M_a(\mathfrak{S})$.

For each $\mu \in M_a(\mathfrak{S})$, let μ also denote the functional in $M_a(\mathfrak{S})^{**}$ defined by the formula $f \mapsto f(\mu)$ $(f \in M_a(\mathfrak{S})^*)$. This defines a linear isometric embedding of $M_a(\mathcal{S})$ into $M_a(\mathfrak{S})^{**}$. In particular, $F \odot \mu$, $\mu \odot F$ and $\mu \odot \nu$ make sense as elements of $M_a(\mathfrak{S})^{**}$ for all $\mu, \nu \in M_a(\mathfrak{S})$ and $F \in M_a(\mathfrak{S})^{**}$; moreover, $\mu \odot \nu = \mu * \nu$.

An element m in the second dual $M_a(\mathfrak{S})^{**}$ of $M_a(\mathfrak{S})$ is said to be a *mean* on $M_a(\mathfrak{S})^*$ if $\|m\| = m(u) = 1$, where $u \in M_a(\mathfrak{S})^*$ is defined by

$$u(\mu) = \mu(\mathfrak{S})$$

for all $\mu \in M_a(\mathfrak{S})$. The set of all means on $M_a(\mathfrak{S})^*$ is denoted by $P_1(M_a(\mathfrak{S})^{**})$. We say that a mean m on $M_a(\mathfrak{S})^*$ is $\mathcal{M}$*-inner invariant* if

$$m \odot \mu = \mu \odot m$$

for all $\mu \in \mathcal{M}$; or equivalently,

$$m(f\mu) = m(\mu f)$$

for all $\mu \in \mathcal{M}$ and $f \in M_a(\mathfrak{S})^*$, where

$$(\mu f)(\nu) = f(\nu * \mu)$$

for all $\nu \in M_a(\mathfrak{S})$; we also say that $\mathfrak{S}$ is $\mathcal{M}$*-inner amenable* if there exists an $\mathcal{M}$-inner invariant mean on $M_a(\mathfrak{S})^*$. Finally, recall that $\mathfrak{S}$ is called *foundation* if the set $\bigcup\{\mathrm{supp}(\mu) : \mu \in M_a(\mathfrak{S})\}$ is dense in $\mathfrak{S}$.

Proposition 2.2. *Let $\mathfrak{S}$ be a foundation semigroup with identity, and $\mathcal{M} \subseteq M(\mathfrak{S})$. If $\mathfrak{S}$ is $\mathcal{M}$-inner amenable, then any left Banach $\mathfrak{S}$-module $\mathfrak{X}$ has the $\mathcal{M}$-fixed point property.*

Proof. Let m be an $\mathcal{M}$-inner invariant mean on $M_a(\mathfrak{S})^*$. Since $\mathfrak{S}$ is a foundation semigroup with identity, it follows from [19] that $M_a(\mathfrak{S})$ can be considered as the predual of a C^*-algebra; see also [15]. Thus $P_1(M_a(\mathfrak{S}))$ is weak* dense in $P_1(M_a(\mathfrak{S})^{**})$; see Lemma 2.1 in [9]. Thus, there is a net (ν_β) in $P_1(M_a(\mathfrak{S}))$ with

$$\mu * \nu_\beta - \nu_\beta * \mu \longrightarrow 0$$

in the weak topology of $M_a(\mathfrak{S})$ for all $\mu \in \mathcal{M}$.

Now, let X be the locally convex space $\Pi\{M_a(\mathfrak{S}) : \mu \in \mathcal{M}\}$ under the product of the norm topology of $M_a(\mathfrak{S})$. Then the weak topology of X is the product of the weak topology of $M_a(\mathfrak{S})$. Following an idea due to Namioka [14], let $T : M_a(\mathfrak{S}) \longrightarrow X$ be defined by

$$T(\nu)(\mu) = \mu * \nu - \nu * \mu$$

for all $\nu \in M_a(\mathfrak{S})$ and $\mu \in \mathcal{M}$. Then T is well defined and linear. Since

$$T(\nu_\beta) \longrightarrow 0$$

in the weak topology of X, it follows that 0 lies in the weak closure of $T(P_1(M_a(\mathfrak{S})))$ in X. Now, the convexity of $T(P_1(M_a(\mathfrak{S})))$ implies that 0 lies in the closure of $T(P_1(M_a(\mathfrak{S})))$ in X with respect to the product of the norm topology of $M_a(\mathfrak{S})$. So there exists a net (μ_α) in $P_1(M_a(\mathfrak{S}))$ such that

$$\|(T(\mu_\alpha))(\mu)\| \longrightarrow 0$$

for all $\mu \in \mathcal{M}$. That is
$$\|\mu * \mu_\alpha - \mu_\alpha * \mu\| \longrightarrow 0$$
for all $\mu \in \mathcal{M}$. So, the result follows from Lemma 2.1. ■

Before we give the main result of this paper, let us remark that $M_a(\mathfrak{S})$ equipped with the map $(x, \mu) \longmapsto x.\mu$ defined by

$$x.\mu = \delta_x * \mu \qquad (\mu \in M_a(\mathfrak{S}), x \in \mathfrak{S}),$$

is a left Banach $\mathfrak{S}$-module; note that in this case we have

$$f.\mu = f\mu \quad \text{and} \quad \nu.F = \nu \odot F$$

for all $\mu \in M(\mathfrak{S}), \nu \in M_a(\mathfrak{S}),\ f \in M_a(\mathfrak{S})^*$, and $F \in M_a(\mathfrak{S})^{**}$.

Theorem 2.3. *Let $\mathfrak{S}$ be a foundation semigroup with identity and $\mathcal{M} \subseteq M(\mathfrak{S})$. Then the following assertions are equivalent.*
(a) *$\mathfrak{S}$ is $\mathcal{M}$-inner amenable.*
(b) *Every left Banach $\mathfrak{S}$-module $\mathfrak{X}$ has the $\mathcal{M}$-fixed point property.*
(c) *$M_a(\mathfrak{S})$ has the $\mathcal{M}$-fixed point property.*

Proof. That (a) implies (b) follows from Proposition 2.2; also, (b) implies (c) trivially.

Now, suppose that (c) holds, and choose an element Λ of $\mathcal{P}(M_a(\mathfrak{S}), M_a(\mathfrak{S})^{**})$ such that
$$\Lambda_\mu \Lambda = \Lambda \Lambda_\mu$$
for all $\mu \in \mathcal{M}$. To prove (a), we suppose that (μ_α) is a net in $P_1(M_a(\mathfrak{S}))$ such that

$$\Lambda_{\mu_\alpha} \longrightarrow \Lambda$$

in the weak* operator topology of $\mathcal{B}(M_a(\mathfrak{S})^{**})$. Since $\mathfrak{S}$ is a foundation semigroup with identity, $M_a(\mathfrak{S})$ has a bounded approximate identity (e_γ) in $P_1(M_a(\mathfrak{S}))$; see [18]. Let E be a weak* cluster point of (e_γ) in $M_a(\mathfrak{S})^{**}$. Then E is a right identity for $M_a(\mathfrak{S})^{**}$ by the continuity properties of the first Arens product, and therefore for each $\mu \in \mathcal{M}$,

$$\begin{aligned} \mu * \mu_\alpha - \mu_\alpha * \mu &= (\mu * \mu_\alpha - \mu_\alpha * \mu) \odot E \\ &= (\mu.(\mu_\alpha.E)) - (\mu_\alpha.(\mu.E)) \\ &= \Lambda_\mu \Lambda_{\mu_\alpha} E - \Lambda_{\mu_\alpha} \Lambda_\mu E \\ &\longrightarrow 0 \end{aligned}$$

in the weak topology of $M_a(\mathfrak{S})$. So, any weak* cluster point of (μ_α) in $M_a(\mathfrak{S})^{**}$ is an $\mathcal{M}$-inner invariant mean on $M_a(\mathfrak{S})^*$. ■

Let $\delta_{\mathfrak{S}} := \{\delta_x : x \in \mathfrak{S}\}$. We say that a left Banach $\mathfrak{S}$-module $\mathfrak{X}$ has the *fixed point property* if it has the $\delta_{\mathfrak{S}}$-fixed point property; we also say that $\mathfrak{S}$ is *inner amenable* if there exists an inner invariant mean on $M_a(\mathfrak{S})^*$; that is, a $\delta_{\mathfrak{S}}$-inner invariant mean on $M_a(\mathfrak{S})^*$; see [13].

The study of inner amenability was initiated by Efros [7] and pursued by Akemann [1], H. Choda and M. Choda [3], M. Choda [4], Kaniuth and Markfort [8], Paschke [16], Pier [17], and Watatani [22] for discrete groups, Lau and Paterson [10], Losert and Rindler [12], Stokke [20], Takahashi [21], Yuan [23] for locally compact groups, and by Ling [11] for discrete semigroups.

Our last result is the following consequence of Theorem 2.3 which is due to Lau and Paterson [10] in the case of locally compact groups $\mathfrak{S}$.

Corollary 2.4. *Let $\mathfrak{S}$ be a foundation semigroup with identity. Then the following assertions are equivalent.*

(a) *$\mathfrak{S}$ is inner amenable.*

(b) *Every left Banach $\mathfrak{S}$-module $\mathfrak{X}$ has the fixed point property.*

(c) *$M_a(\mathfrak{S})$ has the fixed point property.*

We end this work with some examples.

Example 2.5. (a) Let $\mathfrak{S}$ be a locally compact commutative semigroup. Then $\mathfrak{S}$ is $M(\mathfrak{S})$-inner amenable, and $M_a(\mathfrak{S})$ has the $M(\mathfrak{S})$-fixed point property trivially; indeed, any element Λ of $\mathcal{P}(M_a(\mathfrak{S}), M_a(\mathfrak{S})^{**})$ satisfies

$$\Lambda_\mu \Lambda = \Lambda \Lambda_\mu$$

for all $\mu \in M(\mathfrak{S})$. So, it follows from Proposition 2.2 that every left Banach $\mathfrak{S}$-module $\mathfrak{X}$ has the fixed point property.

(b) Let $\mathcal{S}$ be the semigroup $[0,1]$ with the operation $xy = \min\{x, y\}$ for all $x, y \in [0,1]$. Then $\mathfrak{S}$ endowed with the topology induced from the real line is not a foundation semigroup; indeed,

$$M_a(\mathfrak{S}) = \mathbb{C}\,\delta_0.$$

Therefore,

$$\mathcal{P}(M_a(\mathfrak{S}), \mathfrak{X}^{**}) = \{\Lambda_{\delta_0}\}$$

for all left Banach $\mathfrak{S}$-module $\mathfrak{X}$, and for each $\mu \in M(\mathfrak{S})$,

$$\Lambda_\mu \Lambda_{\delta_0} = \Lambda_{\delta_0} = \Lambda_{\delta_0} \Lambda_\mu.$$

Acknowledgments. The authors thank the referee of the paper for a very careful reading and valuable comments. The second author thanks the Center of Excellence for Mathematics at the Isfahan University of Technology.

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