Harrison's criterion, Witt equivalence and reciprocity equivalence

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Abstract

Harrison's criterion characterizes the isomorphy of the Witt rings of two fields in terms of properties of these fields. In this article, we discuss about the existence of such characterizations for the isomorphism of Witt groups of hermitian forms over certain algebras with involution. In the cases where we consider the Witt group of a quadratic extension with its non-trivial automorphism or the Witt group of a quaternion division algebra with its canonical involution, such criteria are proved. In the framework of global fields, these criteria are reformulated in terms of properties involving certain real places of the considered fields.

1 Introduction

One of the basic questions in the algebraic theory of quadratic forms is to give necessary and sufficient conditions for two fields K_1 and K_2 to have isomorphic Witt rings: in this case, K_1 and K_2 are said to be *Witt equivalent*. In [6], Harrison expresses Witt equivalence in the following terms:

Theorem 1.1 (Harrison). *Let K_1 and K_2 be two fields of characteristic different from* 2. *Then the following are equivalent:*
(1) *K_1 and K_2 are Witt equivalent.*
(2) *There is a group isomorphism $t : {K_1}^*/{K_1}^{*2} \to {K_2}^*/{K_2}^{*2}$ with $t(-1) = -1$ such that the quadratic form $\langle x, y\rangle$ represents 1 over K_1 if and only if the quadratic form $\langle t(x), t(y)\rangle$ represents 1 over K_2 for all $x, y \in {K_1}^*$.*

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In the literature, the previous Theorem is known as "Harrison's criterion".

In [1], Baeza and Moresi study the possibilities to extend Harrison's criterion to fields K_1 and K_2 of characteristic 2. On the one hand, they show that the bilinear Witt rings $W(K_1)$ and $W(K_2)$ of K_1 and K_2 are isomorphic if and only if K_1 and K_2 are isomorphic in the case where $\dim_{{K_1}^2} K_1 = \dim_{{K_2}^2} K_2 > 2$, and they give a complete treatment of the cases where $\dim_{{K_1}^2} K_1 = \dim_{{K_2}^2} K_2 = 1$ or 2: see [1, Theorem 2.9, Proposition 2.10]. On the other hand, in [1, Theorem 3.1], they characterize the isomorphy of the quadratic Witt modules $W_q(K_1)$ and $W_q(K_2)$ in the following way:

Theorem 1.2 (Baeza-Moresi). *Let K_1 and K_2 be two fields of characteristic 2. Then the following are equivalent:*
(1) *There exist a ring isomorphism $\Phi : W(K_1) \to W(K_2)$ and a group isomorphism $\Psi : W_q(K_1) \to W_q(K_2)$ such that $\Psi(b.q) = \Phi(b).\Psi(q)$ for all $b \in W(K_1)$ and for all $q \in W_q(K_1)$.*
(2) *There exist groups isomorphisms*

$$t_1 : {K_1}^*/{K_1}^{*2} \to {K_2}^*/{K_2}^{*2}, \quad t_2 : K_1/\wp(K_1) \to K_2/\wp(K_2)$$

such that $t_1\big(D_{K_1}(\langle 1, a\rangle)\big) = D_{K_2}\big(\langle 1, t_1(a)\rangle\big)$, $t_2(D_{K_1}[1, b]) = D_{K_2}[1, t_2(b)]$ for all $a \in {K_1}^*$ and for all $b \in K_1$ (where $\wp(K_i) = \{a + a^2 \mid a \in K_i\}$, $i = 1, 2$).

Such criteria are very useful. For example, Theorem 1.1 is used by Mináč and Spira to connect the Witt equivalence of two fields K_1 and K_2 to the isomorphy of some groups G_{K_1} and G_{K_2} (called W-groups), G_{K_i} being the Galois group of a certain field extension ${K_i}^{(3)}$ of K_i for $i = 1, 2$: see [11]. Another consequence of Theorem 1.1 is the classification of Witt rings of order at most 32 up to Witt equivalence by their group structure: see [3, Theorem 7.1].

In this context, a natural question arises: is it possible to obtain such criteria for the Witt group of a central simple algebra with involution ? After recalling some notations and basic facts in Section 2, we explain how to obtain such criteria in two particular cases in Section 3. We first treat the case of the Witt group of a quadratic field extension equipped with its nontrivial automorphism:

Theorem 1.3. *Let K_1 and K_2 be two fields of characteristic different from 2. Let $L_1 = K_1(\sqrt{a_1})$ (resp. $L_2 = K_2(\sqrt{a_2})$) be a quadratic field extension of K_1 (resp. K_2) equipped with its non trivial automorphism σ_1 (resp. σ_2). Then, the following are equivalent:*
(1) *$W(L_1, \sigma_1) \simeq W(L_2, \sigma_2)$ as rings.*
(2) *There is a group isomorphism $t : {K_1}^*/N_{L_1/K_1}({L_1}^*) \to K_2^*/N_{L_2/K_2}({L_2}^*)$ with $t(-1) = -1$ such that the quadratic form $\langle\langle a_1, x, y\rangle\rangle$ is hyperbolic over K_1 if and only if the quadratic form $\langle\langle a_2, t(x), t(y)\rangle\rangle$ is hyperbolic over K_2 for all $x, y \in {K_1}^*$, where $N_{L_i/K_i}({L_i}^*)$ denotes the norm group of the extension L_i/K_i for $i = 1, 2$.*

Next, we consider the case of the Witt group of a quaternion division algebra endowed with its canonical involution. In this direction, we obtain Theorem 3.7 whose statement is similar to Theorem 1.2. The similarity of this result with Theorem 1.1 shows up by taking $K_1 = K_2 = K$:

Corollary 1.4. *Let $Q_1 = (a,b)_K$ (resp. $Q_2 = (c,d)_K$) be a quaternion division algebra over K endowed with its canonical involution γ_1 (resp. γ_2). Then, the following are equivalent:*
(1) $W(Q_1,\gamma_1) \simeq W(Q_2,\gamma_2)$ *as $W(K)$-modules.*
(2) *There is a group isomorphism $\tilde{t} : K^*/\mathrm{Nrd}_{Q_1/K}({Q_1}^*) \simeq K^*/\mathrm{Nrd}_{Q_2/K}({Q_2}^*)$ with $\tilde{t}(-1) = -1$ such that the quadratic form $\langle\langle a,b,u,v\rangle\rangle$ is hyperbolic over K if and only if the quadratic form $\langle\langle c,d,\tilde{t}(u),\tilde{t}(v)\rangle\rangle$ is hyperbolic over K for all $u,v \in K^*$, where $\mathrm{Nrd}_{Q_i/K}({Q_i}^*)$ denotes the group of reduced norms from the quaternion algebra Q_i for $i = 1,2$.*

In this framework, another interesting problem is to give necessary and sufficient conditions for two global fields to be Witt equivalent. This problem is now entirely solved. In [12, §3, §4], Perlis, Szymiczek, Conner and Litherland prove that two global fields K_1 and K_2 of characteristic different from 2 are Witt equivalent if and only if they are reciprocity equivalent (i.e. if there exist a group isomorphism t between their square class groups and a bijection T between their nontrivial places such that the Hilbert symbols $(x,y)_P$ and $(t(x),t(y))_{T(P)}$ are equal for any $x,y \in {K_1}^*/{K_1}^{*2}$ and for any non trivial place P over K_1).

In Section 4, we obtain similar results for the two types of Witt groups mentioned above when the base fields are supposed to be global. We naturally adapt the notion of reciprocity equivalence in each of this two cases: the square class groups are replaced by norm class groups (resp. reduced norm class groups) and the role of the nontrivial places is played by the real places (resp. certain real places): see Definition 4.5 and Theorem 4.6. In the first case, we get:

Theorem 1.5. *Let K_1 and K_2 be two global fields of characteristic different from 2. Let $L_1 = K_1(\sqrt{a_1})$ (resp. $L_2 = K_2(\sqrt{a_2})$) be a quadratic field extension of K_1 (resp. K_2) equipped with its nontrivial automorphism σ_1 (resp. σ_2). Then, the following are equivalent:*
(1) $W(L_1,\sigma_1) \simeq W(L_2,\sigma_2)$ *as rings.*
(2) *There is an (a_1,a_2)-quadratic reciprocity equivalence between K_1 and K_2.*

2 Basic results and notations

From now on, all fields are supposed to be of characteristic different from 2.

2.1 Central simple algebras with involution

The general reference for the theory of central simple algebras with involution is [8]: see also [13, Chapter 8]

In this Section, K will be a field and D will denote a finite-dimensional division algebra over K. Then $\dim_K D = n^2$ for some $n \in \mathbb{N}$, and $n = \deg D$ is called the degree of D. Suppose that D is endowed with an involution σ. The map σ restricts to an involution of K and we can distinguish two cases: if $\sigma|_K$ is the identity, we say that σ is of *the first kind*, otherwise $\sigma|_K$ is of *the second kind*.

A central simple algebra D of degree 2 is called a *quaternion algebra*. As $\mathrm{char}(K) \neq 2$, every quaternion algebra has a quaternion basis $\{1, i, j, k\}$, that is a basis of the K-algebra Q subject to the relations

$$i^2 = a \in K^*, j^2 = b \in K^*, ij = k = -ji.$$

This algebra Q is then denoted by $Q = (a, b)_K$. Note also that every quaternion algebra has a canonical involution (usually denoted by γ) which is of the first kind and defined as follows:

$$\gamma(i) = -i, \gamma(j) = -j.$$

2.2 Hermitian forms

The standard reference for the theory of hermitian forms is [13, Chapter 7]. All vector spaces considered will be finite dimensional right vector spaces.

A *hermitian form* over (D, σ) is a pair (V, h) where V is a D-vector space and h is a map $h : V \times V \to D$ which is σ-sesquilinear in the first argument, D-linear in the second argument and which satisfies

$$\sigma\big(h(x, y)\big) = h(y, x) \text{ for any } x, y \in V.$$

If $D = K$ and $\sigma = \mathrm{id}_K$ then a hermitian form is a symmetric bilinear form which can be identified with a quadratic form as $\mathrm{char}(K) \neq 2$. All forms considered will be nondegenerate. Every hermitian form over (D, σ) can be diagonalized and such a diagonalization will be denoted by $\langle a_1, \cdots, a_n \rangle$ where $\sigma(a_i) = a_i$ for $i = 1, \cdots, n$.

If y is an element of D such that $h(x, x) = y$ for a certain $x \in V \setminus \{0\}$, then we say that *h represents y*. If h represents 0, we say that h is *isotropic*, *anisotropic* otherwise. If q is a quadratic form over K, denote by $D_K(q)$ the set of those elements of K^* that are represented by q.

Let (V, h) and (V', h') be two hermitian forms over (D, σ). If these forms are isometric then we write $h \simeq h'$ for short. Their *orthogonal sum* is denoted by $h \perp h'$.

2.3 The Witt group of a division algebra with involution

We refer to [13, Chapter 7, 10] for more details about the Witt group.

The orthogonal sum induces a commutative monoïd structure on the set of isometry classes of nondegenerate hermitian forms over (D, σ). The *Witt group* of (D, σ) is the quotient group of the Grothendieck group of this commutative monoïd by the subgroup generated by *hyperbolic forms* and is denoted by $W(D, \sigma)$. In the case where $D = K$, the tensor product can be used to define a structure of ring on $W(K, \sigma)$. If moreover $\sigma = \mathrm{id}_K$, this ring is called the *Witt ring* of K and is denoted by $W(K)$.

The tensor product gives a $W(K, \sigma|_K)$-module structure on $W(D, \sigma)$. The submodule generated by nondegenerate hermitian forms of even dimension is denoted by $I_1(D, \sigma)$ (or by $I(K)$ if $D = K$ and $\sigma = \mathrm{id}_K$). We write $I^n(K)$ for $\big(I(K)\big)^n$.

The ideal $I^n(K)$ is additively generated by the so-called n-fold Pfister forms

$$\langle 1, -a_1 \rangle \otimes \cdots \otimes \langle 1, -a_n \rangle := \langle\langle a_1, \cdots, a_n \rangle\rangle,$$

for $a_1, \cdots, a_n \in K^*$.

2.4 The (refined) discriminant of a hermitian form

We refer to [2, §2] for more general statements about this invariant.

Let (V, h) be a hermitian form over (D, σ) and suppose first that σ is an involution of the first kind. Let $\{e_1, \cdots, e_n\}$ be a D-basis of the right D-vector space V. Let M be the matrix of h with respect to this basis, $E = M_n(D)$ and $m = n \deg(D)$. We define the *signed discriminant* of (V, h) by

$$\mathrm{d}_\pm(h) = (-1)^{\frac{m(m-1)}{2}} \mathrm{Nrd}_{E/K}(M) \in K^*,$$

where $\mathrm{Nrd}_{E/K}$ denotes the usual reduced norm map: see [4, §22]. One can show that $\mathrm{d}_\pm$ induces a well-defined group homomorphism, again denoted by $\mathrm{d}_\pm$

$$\mathrm{d}_\pm : I_1(D, \sigma) \to K^*/K^{*2}.$$

More precisely, if $\mathrm{Nrd}_{D/K}(D^*)$ is the group of reduced norms from D, $\mathrm{d}_\pm$ induces a group homomorphism

$$\mathrm{Disc} : I_1(D, \sigma) \to \mathrm{Nrd}_{D/K}(D^*)/\mathrm{Nrd}_{D/K}(D^*)^2,$$

which is called the *refined discriminant*.

If σ is an involution of the second kind and if F is the fixed field of σ in K, the signed discriminant of (V, h) is defined by the formula above and induces a group homomorphism

$$\mathrm{d}_\pm : I_1(D, \sigma) \to F^*/N_{K/F}(K^*),$$

where $N_{K/F}(K^*)$ is the group of norms of K/F.

In both cases, the kernel of the signed discriminant homomorphism is denoted by $I_2(D, \sigma)$.

2.5 Chain equivalence

Throughout this Subsection we will use the notations of [9, Chapter I, §5] and will refer to it for more general statements.

Let D be a division algebra over K endowed with an involution σ (of arbitrary kind). Let $h = \langle a_1, \cdots, a_n \rangle$ and $h' = \langle a'_1, \cdots, a'_n \rangle$ be two hermitian forms over (D, σ). They are said to be *simply equivalent* if there exists indices $i, j \in \{1, \cdots, n\}$ such that $\langle a_i, a_j \rangle \simeq \langle b_i, b_j \rangle$ and $a_k = b_k$ for every k different from i and j (note that, if $i = j$, the expression $\langle a_i, a_j \rangle$ is understood to be $\langle a_i \rangle$). Two (diagonalized) hermitian forms h and h' over (D, σ) are *chain equivalent* if there is a sequence of diagonalized hermitian forms $f_0, \cdots, f_m$ over (D, σ) such that $h = f_0$, $h' = f_m$ and such that f_i is simply equivalent to f_{i+1} for $0 \leq i \leq m-1$. We immediately see that two chain equivalent forms are isometric. In fact, the converse is also true by "Witt's Chain equivalence Theorem":

Theorem 2.1 (Witt). *If h and h' are two (diagonalized) hermitian forms over (D, σ) and if h is isometric to h' then h and h' are chain equivalent.*

Proof. The proof can be easily adapted from [9, Chapter I, Theorem 5.2] by replacing usual squares by hermitian squares (that is, elements of the form $\sigma(x)x$). ∎

2.6 Further results

We state two results that are used several times in this paper.

The following result known as "Arason-Pfister Hauptsatz"gives a dimension-theoretic sufficient condition for a quadratic form to belong to $I^n(K)$.

Theorem 2.2 (Arason-Pfister). *Let q be a positive-dimensional anisotropic quadratic form over K. If $q \in I^n(K)$, then $\dim q \geq 2^n$.*

Proof. See [9, Chapter X, Hauptsatz 5.1] or [13, Chapter 4, Theorem 5.6]. ∎

Let L/K, $L = K(\sqrt{a})$, be a quadratic field extension endowed with its non trivial automorphism $-$ and $D = (a, b)_K$ be a quaternion algebra endowed with its canonical involution γ. We define the following usual transfer maps

$$\pi_L : \begin{cases} W(L, -) & \to & W(K) \\ [h] & \mapsto & [x \mapsto h(x, x)] \end{cases}, \quad \pi_D : \begin{cases} W(D, \gamma) & \to & W(K) \\ [h] & \mapsto & [x \mapsto h(x, x)] \end{cases}.$$

Theorem 2.3 (Jacobson). *With the above notations, the maps π_L and π_D are injective.*

Proof. See [13, Chapter 10, 1.1, 1.2, 1.7] or [7]. ∎

Moreover, $\mathrm{im}(\pi_L) = \langle\langle a \rangle\rangle W(K)$ and for any positive integer n, $\pi_L\big(I_1(L, -)^n\big) = \langle\langle a \rangle\rangle I^n(K)$.

3 Analogues of Harrison's criterion

In this Section, we prove isomorphy criteria for the Witt group of a quadratic field extension with its nontrivial automorphism and for the Witt group of a quaternion division algebra with its canonical involution, in analogy with Theorem 1.1.

3.1 The case of quadratic field extensions

Let us keep the same notations as in Theorem 1.3. First, we rephrase Theorem 1.1 by introducing another equivalent condition and it is this condition we will then generalize to the setting of hermitian forms.

Lemma 3.1. *Let K_1 and K_2 be two fields of characteristic different from 2. Then the following are equivalent:*
(1) K_1 and K_2 are Witt equivalent.
(2) There is a group isomorphism $t : {K_1}^*/{K_1}^{*2} \to {K_2}^*/{K_2}^{*2}$ with $t(-1) = -1$ and such that the quadratic form $\langle\langle x, y \rangle\rangle$ is hyperbolic over K_1 if and only if the quadratic form $\langle\langle t(x), t(y) \rangle\rangle$ is hyperbolic over K_2 for all $x, y \in {K_1}^*$.

Proof. The quadratic form $\langle x, y\rangle$ represents 1 over K_1 if and only if the 2-fold Pfister form $\langle\langle x, y\rangle\rangle$ is hyperbolic over K_1. The equivalence then follows from Theorem 1.1. ∎

In the proof of Theorem 1.3, we will need the following two lemmas.

Lemma 3.2. *For $i = 1, 2$, the signed discriminant induces a group isomorphism* $\mathrm{d}_\pm : I_1(L_i, \sigma_i)/I_2(L_i, \sigma_i) \simeq {K_i}^*/N_{L_i/K_i}({L_i}^*)$.

Proof. The kernel of $\mathrm{d}_\pm : I_1(L_i, \sigma_i) \to {K_i}^*/N_{L_i/K_i}({L_i}^*)$ is $I_2(L_i, \sigma_i)$. If $b \in {K_i}^*$, then $\mathrm{d}_\pm(\langle 1, -b\rangle) = b \mod N_{L_i/K_i}({L_i}^*)$, hence $\mathrm{d}_\pm$ is onto. ∎

As in the case of quadratic forms, the ideals I_1 and I_2 are related as follows:

Lemma 3.3. *We have* $\big(I_1(L_i, \sigma_i)\big)^2 = I_2(L_i, \sigma_i)$ *for* $i = 1, 2$.

Proof. We obviously have $\big(I_1(L_i, \sigma_i)\big)^2 \subseteq I_2(L_i, \sigma_i)$. Conversely, suppose that $\phi \in I_2(L_i, \sigma_i)$ and that $\dim \phi = 2s$. We proceed by induction on s. When $s = 1$, ϕ is an hyperbolic plane hence $\phi \in \big(I_1(L_i, \sigma_i)\big)^2$. If $s = 2$ and $\phi \simeq \langle a, b, c, d\rangle$ then $d = abc \in {K_i}^*/N_{L_i/K_i}({L_i}^*)$ and $\phi \simeq \langle a\rangle \otimes \langle 1, ab\rangle \otimes \langle 1, ac\rangle$ thus $\phi \in \big(I_1(L_i, \sigma_i)\big)^2$. Suppose now that $s \geq 3$. Write $\phi = \langle a, b, c\rangle \perp \phi'$ with $\dim \phi' \geq 1$ and

$$\phi = \underbrace{\langle a, b, c, abc\rangle}_{\alpha} \perp \underbrace{(\phi' \perp \langle -abc\rangle)}_{\beta} \in W(L_i, \sigma_i).$$

As $\mathrm{d}_\pm(\phi) = 1$ and $\mathrm{d}_\pm(\alpha) = 1$, it follows that $\mathrm{d}_\pm(\beta) = 1$. By induction, $\beta \in \big(I_1(L_i, \sigma_i)\big)^2$ hence $\phi \in \big(I_1(L_i, \sigma_i)\big)^2$. ∎

Proof of Theorem 1.3: (1) $\Rightarrow$ (2) : let $\Phi : W(L_1, \sigma_1) \simeq W(L_2, \sigma_2)$ be a ring isomorphism. Since $I_1(L_i, \sigma_i)$ is the only ideal of index 2 in $W(L_i, \sigma_i)$, we must have $\Phi\big(I_1(L_1, \sigma_1)\big) = I_1(L_2, \sigma_2)$, and thus also $\Phi\big(I_2(L_1, \sigma_1)\big) = I_2(L_2, \sigma_2)$ by Lemma 3.3. By means of Lemma 3.2, Φ induces the following group isomorphism

$$t : \begin{cases} {K_1}^*/N_{L_1/K_1}({L_1}^*) & \to & {K_2}^*/N_{L_2/K_2}({L_2}^*) \\ c & \mapsto & \mathrm{d}_\pm\big(\Phi(\langle 1, -c\rangle)\big) \end{cases},$$

which obviously satisfies $t(-1) = -1$.

As Φ induces a factor ring isomorphism u from $I_1(L_1, \sigma_1)^2/I_1(L_1, \sigma_1)^3$ to $I_1(L_2, \sigma_2)^2/I_1(L_2, \sigma_2)^3$, we obtain the following commutative diagram:

$$\begin{array}{ccc} \big({K_1}^*/N_{L_1/K_1}({L_1}^*)\big) \times \big({K_1}^*/N_{L_1/K_1}({L_1}^*)\big) & \xrightarrow{\theta_{L_1}} & \big(I_1(L_1, \sigma_1)\big)^2/\big(I_1(L_1, \sigma_1)\big)^3 \\ \Big\downarrow (t, t) & & \Big\downarrow u \\ \big({K_2}^*/N_{L_2/K_2}({L_2}^*)\big) \times \big({K_2}^*/N_{L_2/K_2}({L_2}^*)\big) & \xrightarrow{\theta_{L_2}} & \big(I_1(L_2, \sigma_2)\big)^2/\big(I_1(L_2, \sigma_2)\big)^3 \end{array}$$

with $\theta_{L_i}(x, y) = \langle 1, -x\rangle \otimes \langle 1, -y\rangle \mod I_1(L_i, \sigma_i)^3$ for all $x, y \in {K_i}^*$ and for $i = 1, 2$. We claim that the hermitian form $\langle 1, -x, -y, xy\rangle$ is hyperbolic over (L_1, σ_1) if and only if it belongs to $I_1(L_1, \sigma_1)^3$. The "only if"part is clear. Conversely, let $\langle 1, -x, -y, xy\rangle \in I_1(L_1, \sigma_1)^3$. Using the notations of Subsection 2.6, we know that $\pi_{L_1}(\langle 1, -x, -y, xy\rangle) \in I^4(K_1)$. Applying successively 2.2 and 2.3, it follows that the hermitian form $\langle 1, -x, -y, xy\rangle$ is hyperbolic over (L_1, σ_1).

Lastly, the quadratic form $\langle\langle a_1, x, y\rangle\rangle = \pi_{L_1}(\langle 1, -x, -y, xy\rangle)$ is hyperbolic over K_1 if and only if $\langle 1, -x, -y, xy\rangle \in I_1(L_1, \sigma_1)^3$ by the claim and Theorem 2.3. By commutativity of the previous diagram, this is equivalent to the fact that $\langle 1, -t(x), -t(y), t(xy)\rangle \in I_1(L_2, \sigma_2)^3$ which in turn is equivalent to the hyperbolicity of the quadratic form $\langle\langle a_2, t(x), t(y)\rangle\rangle = \pi_{L_2}(\langle 1, -t(x), -t(y), t(xy)\rangle)$ over K_2.

$(2) \Rightarrow (1)$: we define a map Φ on diagonal forms by

$$\Phi(\langle b_1, \cdots, b_n\rangle) = \langle t(b_1), \cdots, t(b_n)\rangle.$$

We first show that this definition does not depend on the chosen diagonalization. If $n = 1$, this is clear. If $n = 2$, suppose that $\langle u, v\rangle \simeq \langle u', v'\rangle$ as hermitian forms over (L_1, σ_1). By taking the signed discriminant on both sides, we have $uv = u'v' \in K_1{}^*/N_{L_1/K_1}(L_1{}^*)$. If we let the one-dimensional hermitian form $\langle u\rangle$ act on both sides, it follows that the hermitian form $\langle 1, -uu', -uv', u'v'\rangle$ is hyperbolic over (L_1, σ_1). As a consequence, the hermitian forms $\langle t(u), t(v)\rangle$ and $\langle t(u'), t(v')\rangle$ are isometric over (L_2, σ_2). If $n > 2$, the result comes from Theorem 2.1 and from the fact that the property holds for $n = 2$. As $t(-1) = -1$, Φ preserves hyperbolicity and induces a well-defined map between $W(L_1, \sigma_1)$ and $W(L_2, \sigma_2)$. Besides, Φ is additive and multiplicative (Φ being multiplicative over rank one forms which generate additively $W(L_1, \sigma_1)$) and t^{-1} provides an inverse for Φ which is thus a ring isomorphism. ∎

In Theorem 1.3, we can show that the condition $t(-1) = -1$ is not a consequence of the other two conditions of Assertion (2):

Example 3.4. Let $K_1 = \mathbb{Q}_3$ and $K_2 = \mathbb{Q}_5$. Then $K_1{}^*/K_1{}^{*2}$ (resp. $K_2{}^*/K_2{}^{*2}$) consists of four elements, represented by $1, -1, 3, -3$ (resp. $1, 2, 5, 10$). For a field K, denote by $u(K)$ the u-invariant of K (see [9, Chapter XI, §6]). Then, $u(K_1) = u(K_2) = 4$, and the unique anisotropic quadratic form of dimension 4 over K_1 (resp. over K_2) is $\langle 1, 1, -3, -3\rangle$ (resp. $\langle 1, -2, -5, 10\rangle$) (see [9, Chapter VI, Theorem 2.2]). Let $L_1 = K_1(\sqrt{3})$ and $L_2 = K_2(\sqrt{2})$. It is easy to show that $|K_1{}^*/D_{K_1}(\langle 1, -3\rangle)| = 2 = |K_2{}^*/D_{K_2}(\langle 1, -2\rangle)|$ and that we have a group isomorphism defined by

$$\begin{array}{rcl} t : K_1{}^*/D_{K_1}(\langle 1, -3\rangle) & \to & K_2{}^*/D_{K_2}(\langle 1, -2\rangle) \\ 1 & \mapsto & 1 \\ -1 & \mapsto & 5 \end{array}$$

As $u(K_1) = u(K_2) = 4$, the quadratic form $\langle\langle 3, x, y\rangle\rangle$ (resp. $\langle\langle 2, t(x), t(y)\rangle\rangle$) is hyperbolic over K_1 (resp. over K_2) for all $x, y \in K_1{}^*$. Finally, $\langle 1, -2\rangle$ clearly represents -1 over K_2 and $t(-1) \neq -1 = 1 \in K_2{}^*/D_{K_2}(\langle 1, -2\rangle)$.

3.2 The case of quaternion division algebras

In this Subsection, $Q_1 = (a, b)_{K_1}$ (resp. $Q_2 = (c, d)_{K_2}$) will denote a quaternion division algebra over K_1 (resp. over K_2) with its canonical involution γ_1 (resp. γ_2).

The following two examples show that the group structure of the Witt ring is not sufficient to classify fields up to Witt equivalence as in Theorem 1.1 thus motivating our choice of the module structure in Theorem 3.7 and Corollary 1.4. In the first example, the cardinality of the Witt rings is infinite and in the second, it is finite.

Examples 3.5. (1) One can find this example in [12, §7]. If $K_1 = \mathbb{Q}(\sqrt[3]{2})$ and $K_2 = \mathbb{Q}$, one can show that $W(K_1) \simeq W(K_2)$ as groups. But, by [12, §4, Corollary 2], $W(K_1)$ and $W(K_2)$ are not isomorphic as rings.
(2) One can find this example in [3, Example 7.2]. The construction is based on [5, §II.1] which was obtained in 1965 by Gross and Fischer. We choose $K_1 = \mathbb{Q}_2(\sqrt{d})$ where $d \in \mathbb{Q}_2^* \setminus \pm \mathbb{Q}_2^{*2}$. Then, we have $|{K_1}^*/{K_1}^{*2}| = 16$ (see [9, Chapter VI, Corollary 2.23]). By [3, Theorem 4.5], there exists a field K_2 with $|{K_2}^*/{K_2}^{*2}| = 8$ and such that $W(K_1) \simeq W(K_2) \simeq C_4 \times C_4 \times C_2 \times C_2$ as groups. But $W(K_1)$ and $W(K_2)$ are not isomorphic as rings by Theorem 1.1 as we have $|{K_1}^*/{K_1}^{*2}| \neq |{K_2}^*/{K_2}^{*2}|$.

In order to simplify the statement of Theorem 3.7, we define:

Definition 3.6. Two fields K_1 and K_2 of characteristic different from 2 are said to be (Q_1, Q_2)-*equivalent* if there is a group homomorphism $t : {K_1}^*/{K_1}^{*2} \to {K_2}^*/{K_2}^{*2}$ with $t(-1) = -1$ such that, if the quadratic form $\langle\langle x, y\rangle\rangle$ is hyperbolic over K_1, then the quadratic form $\langle\langle t(x), t(y)\rangle\rangle$ is hyperbolic over K_2 for all $x, y \in {K_1}^*$ and which induces a group isomorphism $\tilde{t} : {K_1}^*/D_{K_1}(\langle\langle a, b\rangle\rangle) \simeq {K_2}^*/D_{K_2}(\langle\langle c, d\rangle\rangle)$. The pair $(t, \tilde{t})$ is called a (Q_1, Q_2)-*equivalence*.

Theorem 3.7. *The following are equivalent:*
(1) *There exist a ring homomorphism $\Phi : W(K_1) \to W(K_2)$ sending one-dimensional forms to one-dimensional forms and a group isomorphism $\Psi : W(Q_1, \gamma_1) \to W(Q_2, \gamma_2)$ such that $\Psi(\langle 1\rangle) = \langle 1\rangle$ and $\Psi(q.h) = \Phi(q).\Psi(h)$, for all $q \in W(K_1)$, $h \in W(Q_1, \gamma_1)$.*
(2) *There is a (Q_1, Q_2)-equivalence $(t, \tilde{t})$ between K_1 and K_2 such that the hermitian forms $\langle u, v\rangle$ and $\langle u', v'\rangle$ are isometric over (Q_1, γ_1) if and only if the hermitian forms $\langle \tilde{t}(u), \tilde{t}(v)\rangle$ and $\langle \tilde{t}(u'), \tilde{t}(v')\rangle$ are isometric over (Q_2, γ_2) for all $u, v, u', v' \in {K_1}^*$.*
(3) *There is a (Q_1, Q_2)-equivalence $(t, \tilde{t})$ between K_1 and K_2 such that the quadratic form $\langle\langle a, b, u, v\rangle\rangle$ is hyperbolic over K_1 if and only if the quadratic form $\langle\langle c, d, \tilde{t}(u), \tilde{t}(v)\rangle\rangle$ is hyperbolic over K_2 for all $u, v \in {K_1}^*$.*

First, we need to prove the following lemma:

Lemma 3.8. *Let Q be a quaternion division algebra over a field K with norm form N_Q. Let $u, v, u', v' \in K^*$. Suppose that the quadratic form $N_Q \otimes \langle u, v, -u', -v'\rangle$ is hyperbolic over K. Then $uvu'v'$ is represented by N_Q.*

Proof. As the quadratic forms $q = \langle u, v, -u', -v'\rangle$ and $q' = \langle 1, -uvu'v'\rangle$ have the same signed discriminant, $q \perp (-q')$ belongs to $I^2(K)$. Thus,

$$N_Q \otimes \langle u, v, -u', -v'\rangle \equiv N_Q \otimes \langle 1, -uvu'v'\rangle \mod I^4(K).$$

By assumption and by Theorem 2.2, the quadratic form $N_Q \otimes \langle 1, -uvu'v'\rangle$ is hyperbolic over K and it follows that $uvu'v' \in D_K(N_Q)$. ∎

Proof of Theorem 3.7: $(3) \Rightarrow (2)$: let $u, v, u', v' \in {K_1}^*$ be such that $\langle u, v\rangle \simeq \langle u', v'\rangle$ as hermitian forms over (Q_1, γ_1). By Theorem 2.3, this is equivalent to the hyperbolicity of the quadratic form $\langle\langle a, b\rangle\rangle \otimes \langle u, v, -u', -v'\rangle$ over K_1. By Lemma 3.8, $uvu'v' \in D_{K_1}(\langle\langle a, b\rangle\rangle)$ and $t(uvu'v') \in D_{K_2}(\langle\langle c, d\rangle\rangle)$. Now we also have $\langle 1, uv\rangle \simeq \langle uu', uv'\rangle$ as hermitian forms over (Q_1, γ_1) and, since $uvu'v' \in D_{K_1}(\langle\langle a, b\rangle\rangle)$, it follows that $\langle 1, u'v'\rangle \simeq \langle vv', vu'\rangle$ as hermitian forms over (Q_1, γ_1). This is equivalent to the hyperbolicity of the quadratic form $\langle\langle a, b, vv', vu'\rangle\rangle$ over K_1 and it follows from Assertion (3) that the quadratic form $\langle\langle c, d, \tilde{t}(vv'), \tilde{t}(vu')\rangle\rangle$ is hyperbolic over K_2. Now, the hermitian forms $\langle \tilde{t}(u), \tilde{t}(v)\rangle$ and $\langle \tilde{t}(u'), \tilde{t}(v')\rangle$ are isometric over (Q_2, γ_2). The converse is similar.
$(2) \Rightarrow (1)$: let $(t, \tilde{t})$ be a (Q_1, Q_2)-equivalence between K_1 and K_2 satisfying the conditions of Assertion (2). Mimicking the first part of the proof of Theorem 1.3, one can define a group homomorphism $\Phi : W(K_1) \to W(K_2)$ sending a one-dimensional form to a one-dimensional form. We define Ψ in the following way

$$\Psi : \begin{cases} W(Q_1, \gamma_1) & \to & W(Q_2, \gamma_2) \\ \langle a_1, \cdots, a_n\rangle & \mapsto & \langle \tilde{t}(a_1), \cdots, \tilde{t}(a_n)\rangle \end{cases}.$$

As in the proof of Theorem 1.3, by using Theorem 2.1, we can show that Ψ is a well-defined map which induces a group homomorphism, and that the inverse of $\tilde{t}$ induces an inverse for Ψ. Finally, the compatibility relation between Φ and Ψ is easily proved.
$(1) \Rightarrow (3)$: let us suppose the existence of Φ and Ψ as in Assertion (1). As $\Phi(I(K_1)) \subset I(K_2)$, Φ induces the following group homomorphism

$$t : \begin{cases} {K_1}^*/{K_1}^{*2} & \to & {K_2}^*/{K_2}^{*2} \\ a & \mapsto & \mathrm{d}_{\pm}\left(\Phi(\langle 1, -a\rangle)\right) \end{cases}$$

and t satisfies the other properties stated in Definition 3.6 by Theorem 1.1. We are going to show that

$$D_{K_1}(\langle\langle a, b\rangle\rangle)/{K_1}^{*2} = t^{-1}(D_{K_2}(\langle\langle c, d\rangle\rangle)/{K_2}^{*2}). \tag{1}$$

Let $\overline{u} \in D_{K_1}(\langle\langle a, b\rangle\rangle)/{K_1}^{*2}$. Then $\Psi(\langle u\rangle) = \Psi(\langle 1\rangle) = \langle 1\rangle$ on the one hand, and $\Psi(\langle u\rangle) = \Phi(\langle u\rangle).\langle 1\rangle$ on the other hand (note that $\Phi(\langle u\rangle)$ is a quadratic form over K_2 whereas $\Psi(\langle u\rangle)$ is a hermitian form over (Q_2, γ_2)). Denote $\Phi(\langle u\rangle) = \langle x\rangle$. Then, we easily see that $t(\overline{u}) = x$ and that $x \in D_{K_2}(\langle\langle c, d\rangle\rangle)$ hence $t(\overline{u}) \in D_{K_2}(\langle\langle c, d\rangle\rangle)/{K_2}^{*2})$.

Let $\overline{v} \in D_{K_2}(\langle\langle c, d\rangle\rangle)/{K_2}^{*2}$ be such that $t(\overline{y}) = \overline{v}$ for a $y \in {K_1}^*$. As $\Phi(\langle y\rangle) = \langle v\rangle$,

$$\Psi(\langle y\rangle) = \Phi(\langle y\rangle).\langle 1\rangle = \langle v\rangle.\langle 1\rangle = \langle 1\rangle.$$

By injectivity of Ψ, it follows that $\langle y\rangle \simeq \langle 1\rangle$ and (1) holds.

Hence, t induces a unique injective group homomorphism

$$\tilde{t} : \begin{cases} {K_1}^*/D_{K_1}(\langle\langle a, b\rangle\rangle) & \to & {K_2}^*/D_{K_2}(\langle\langle c, d\rangle\rangle) \\ x & \mapsto & t(x) \end{cases}$$

Now, we show that $\tilde{t}$ is onto. Let $\overline{w} \in {K_2}^*/D_{K_2}(\langle\langle c, d\rangle\rangle)$. Ψ being surjective, there is a hermitian form h over (Q_1, γ_1) such that $\Psi(h) = \langle w\rangle = \Phi(q).\langle 1\rangle$ where

$h = q.\langle 1\rangle$ and q is a quadratic form over K. Without loss of generality, one can suppose that $h = \langle a_1, \cdots, a_n\rangle$ and $\Phi(q) = \langle b_1, \cdots, b_n\rangle$ with $a_1, \cdots, a_n, b_1, \cdots, b_n \in {K_1}^*$ (note that n is odd). By taking the refined signed discriminant on both sides of the previous equality, we obtain

$$\prod_{i=1}^{n} {b_i}^2 = w^2 \mod D_{K_2}(\langle\langle c, d\rangle\rangle)^2.$$

Consequently, there is a $\delta \in {Q_2}^*$ such that $\big(\prod_{i=1}^{n} {b_i}^2\big).\mathrm{Nrd}_{Q_2/K_2}(\delta)^2 = w^2$ and

$$w = \pm\big(\prod_{i=1}^{n} b_i\big).\mathrm{Nrd}_{Q_2/K_2}(\delta). \tag{2}$$

An easy calculation and Equation (2) show that

$$\tilde{t}\big(\pm\prod_{i=1}^{n} a_i\big) = \pm\prod_{i=1}^{n} b_i \mod D_{K_2}(\langle\langle c, d\rangle\rangle) = \overline{w},$$

hence $\tilde{t}$ is a group isomorphism.

Lastly, let $u, v \in {K_1}^*$ be such that the quadratic form $\langle\langle a, b, u, v\rangle\rangle$ is hyperbolic over K_1. By Theorem 2.3 we have

$$0 = \Psi(\langle 1, -u, -v, uv\rangle) = \big(\Phi(\langle 1, -u\rangle) \otimes \Phi(\langle 1, -v\rangle)\big).\langle 1\rangle \in W(Q_1, \gamma_1).$$

By definition of t and $\tilde{t}$, we then have

$$0 = \Psi(\langle 1, -u, -v, uv\rangle) = \langle 1, -\tilde{t}(u), -\tilde{t}(v), \tilde{t}(u)\tilde{t}(v)\rangle \in W(Q_1, \gamma_1).$$

It follows that the quadratic form $\langle\langle c, d, \tilde{t}(u), \tilde{t}(v)\rangle\rangle$ is hyperbolic over K_2. Conversely, if the quadratic form $\langle\langle c, d, \tilde{t}(u), \tilde{t}(v)\rangle\rangle$ is hyperbolic over K_2 then the quadratic form $\langle\langle a, b, u, v\rangle\rangle$ is hyperbolic over K_1 by Theorem 2.3 and by injectivity of Ψ. ∎

In the particular case where $K_1 = K_2 = K$, Theorem 3.7 readily implies Corollary 1.4 stated in the Introduction.

Remark 3.9. In [10], Leep and Marshall construct a surjective map between $\mathrm{Aut}\big(W(K_1)\big)$ and the set of the so-called "Harrison maps"(i.e. satisfying Assertion (2) of Theorem 1.1) and describe the kernel of this map. To prove their results, they used the fact that every $\rho \in \mathrm{Hom}_{\mathrm{ring}}\big(W(K_1), W(K_2)\big)$ induces an element $\overline{\rho} \in \mathrm{Hom}_{\mathrm{ring}}\big(W(K_1), W(K_2)\big)$ respecting the dimension of quadratic forms and characterized by

$$\overline{\rho}(q) \equiv \rho(q) \mod \big(I(K_2)\big)^2$$

for all $q \in W(K_1)$. It might be interesting to check if such properties hold for hermitian forms over a quaternion division algebra.

4 Reciprocity equivalence

In this Section we recall some basic results about global fields and reciprocity equivalence and refer to [12] for a complete treatment of reciprocity equivalence. Finally, we define the notion of quadratic reciprocity equivalence and prove Theorem 1.5.

4.1 Preliminaries

A *global field* is either an algebraic number field (i.e. a finite field extension of $\mathbb{Q}$) or an algebraic function field in one variable (i.e. a finite field extension of a field of the form $\mathbb{F}_q(X)$ where $\mathbb{F}_q$ is the finite field with q elements for some prime power q and X is an indeterminate).

Let K be a global field, P be a nontrivial place of K and K_P be a completion of K at P. Let Ω_K be the set of nontrivial places of K and set $\Omega_K^r = \{\text{real places of } K\}$.

Let K be a global field and suppose $P \in \Omega_K^r$. Then, there is a topological isomorphism $\phi : K_P \simeq \mathbb{R}$. Via ϕ, K_P is an ordered field, real closed and euclidean, with unique ordering ${K_P}^2$ (see [13, Chapter 3, Theorem 1.1.4]). We thus say that an element $a \in K^*$ is positive at P if $a \in K_P^{*2}$, and we write $a >_P 0$, negative otherwise. If $a \in K^*$, we introduce the notation

$$\Omega_K^a = \{P \in \Omega_K^r \mid a <_P 0\}.$$

The following result will be useful in Subsection 4.3:

Lemma 4.1. *Let K be a global field and $P \in \Omega_K^r$. A n-fold Pfister form $q = \langle\langle a_1, \cdots, a_n\rangle\rangle$ is anisotropic over K_P if and only if $a_i <_P 0$ for $i = 1, \cdots, n$.*

Proof. The Lemma follows from the well-known properties of quadratic and Pfister forms over real-closed fields. ∎

4.2 Reciprocity equivalence

Throughout this Subsection, K_1 and K_2 will denote global fields of characteristic different from 2. The notion of reciprocity equivalence between such fields has been defined in [12, §1]:

Definition 4.2. A *reciprocity equivalence* between K_1 and K_2 is a pair of maps (t, T), where t is a group isomorphism $t : {K_1}^*/{K_1}^{*2} \to {K_2}^*/{K_2}^{*2}$ and T is a bijection $T : \Omega_{K_1} \to \Omega_{K_2}$ such that (t, T) respects Hilbert symbols, i.e.

$$(x, y)_P = (tx, ty)_{TP}$$

for all $x, y \in {K_1}^*/{K_1}^{*2}$ and for all $P \in \Omega_{K_1}$.

Remark 4.3. As $(x, y)_P = 1$ if and only if the 2-fold Pfister form $\langle\langle x, y\rangle\rangle$ is hyperbolic over $(K_1)_P$, one can replace the condition concerning Hilbert symbols in Definition 4.2 by $\langle\langle x, y\rangle\rangle$ being hyperbolic over $(K_1)_P$ if and only if the quadratic form $\langle\langle t(x), t(y)\rangle\rangle$ is hyperbolic over $(K_2)_{T(P)}$.

The main Theorem of [12] says that:

Theorem 4.4. *K_1 and K_2 are Witt equivalent if and only if they are reciprocity equivalent.*

The proof of the "if"-part of this Theorem very much relies on Theorem 1.1. The proof of the converse is much more difficult and is based on a particular description of the 2-torsion of the Brauer group. We refer to[12, §3, 4] for more details.

4.3 Quadratic Reciprocity equivalence

The purpose of this Subsection is to give a proof of Theorem 1.5. Let us keep the same notations.

Definition 4.5. An (a_1, a_2)-*quadratic reciprocity equivalence* between K_1 and K_2 is a pair of maps (t, T) where $t : {K_1}^*/N_{L_1/K_1}({L_1}^*) \to {K_2}^*/N_{L_2/K_2}({L_2}^*)$ is a group isomorphism with $t(-1) = -1$ and where T is a bijection $T : \Omega_{K_1}^{a_1} \to \Omega_{K_2}^{a_2}$ such that the quadratic form $\langle\langle a_1, x, y\rangle\rangle$ is hyperbolic over $(K_1)_P$ if and only if the quadratic form $\langle\langle a_2, t(x), t(y)\rangle\rangle$ is hyperbolic over $(K_2)_{T(P)}$ for all $x, y \in {K_1}^*/N_{L_1/K_1}({L_1}^*)$ and for all $P \in \Omega_{K_1}^{a}$.

Proof of Theorem 1.5: $(2) \Rightarrow (1)$: by Theorem 1.3, it suffices to show that the quadratic form $\langle\langle a_1, x, y\rangle\rangle$ is hyperbolic over K_1 if and only if the quadratic form $\langle\langle a_2, t(x), t(y)\rangle\rangle$ is hyperbolic over K_2. Note that, for all $P \in \Omega_{K_1} \backslash \Omega_{K_1}^{a_1}$ (resp. for all $Q \in \Omega_{K_2} \backslash \Omega_{K_2}^{a_2}$) and for all $x, y \in {K_1}^*$, the quadratic form $\langle\langle a_1, x, y\rangle\rangle$ (resp. $\langle\langle a_2, t(x), t(y)\rangle\rangle$) is hyperbolic over $(K_1)_P$ (resp. over $(K_2)_Q$). This fact is obvious if the place is complex or if $a_1 >_P 0$ (resp. if $a_2 >_Q 0$). If P (resp. Q) is finite, it comes from the fact that $(K_1)_P$ (resp. $(K_2)_Q$) is a local field with $u\big((K_1)_P\big) = 4 = u\big((K_2)_Q\big)$ (see [9, Chapter XI, Example 6.2(4)]). If the quadratic form $\langle\langle a_1, x, y\rangle\rangle$ is hyperbolic over K_1, then $\phi = \langle\langle a_2, t(x), t(y)\rangle\rangle$ is hyperbolic over $(K_2)_Q$ for all $Q \in \Omega_{K_2}^{b}$ hence ϕ is hyperbolic over K_2 by the Hasse-Minkowski-Principle (see [9, Chapter VI, Hasse-Minkowski-Principle 3.1]). The converse is similar.
$(1) \Rightarrow (2)$: take t as in Theorem 1.3. For $x, y \in {K_1}^*$, the Pfister form $\langle\langle a, -x, -y\rangle\rangle$ is isotropic over $(K_1)_P$ for any nonreal place P on K_1. By the Hasse-Minkowski-Principle, it is anisotropic over K_1 if and only if it is anisotropic over $(K_1)_P$ for some $P \in \Omega_{K_1}^{r}$ if and only if $x, y >_P 0$ for some $P \in \Omega_{K_1}^{a}$ by Lemma 4.1. Similarly, the form $\langle\langle b, -t(x), -t(y)\rangle\rangle$ is anisotropic over K_2 if and only if $t(x), t(y) >_Q 0$ for some $Q \in \Omega_{K_2}^{b}$. Therefore:

$$\exists P \in \Omega_{K_1}^{a} : x, y >_P 0 \iff \exists Q \in \Omega_{K_2}^{b} : t(x), t(y) >_Q 0 \tag{3}$$

Since K_1 is global, $\Omega_{K_1}^{r}$ is finite. For every $P \in \Omega_{K_1}^{a}$ the Weak Approximation Theorem gives $x_P \in {K_1}^*$ which is positive with respect to P and negative with respect to any other places in $\Omega_{K_1}^{a}$.

For $P \in \Omega_{K_1}^{a}$, Equivalence (3) with $y = 1$ shows that there exists $Q \in \Omega_{K_2}^{b}$ with $t(x_P) >_Q 0$. We choose such Q and denote it by $T(P)$.

Next we claim that if $P, P' \in \Omega^a_{K_1}$ and $t(x_P) >_{T(P')} 0$, then $P = P'$. Indeed, (3) with $x = x_P$ and $y = x_{P'}$ yields $P'' \in \Omega^a_{K_1}$ with $x_P, x_{P'} >_{P''} 0$. Necessarily $P = P'' = P'$.

It follows from the claim that T is injective. Hence $|\Omega^a_{K_1}| \leq |\Omega^b_{K_2}|$ and by symmetry, $|\Omega^a_{K_1}| = |\Omega^b_{K_2}|$ so T is in fact bijective.

Further, if $x >_P 0$ for some $P \in \Omega^a_{K_1}$, then (3) yields $Q \in \Omega^b_{K_2}$ satisfying $t(x), t(x_P) >_Q 0$. By the surjectivity of T and the claim, $Q = T(P)$, and therefore $t(x) >_{T(P)} 0$. Consequently, if $\langle\langle a, -x, -y\rangle\rangle$ is anisotropic over $(K_1)_P$ then $\langle\langle b, -t(x), -t(y)\rangle\rangle$ is anisotropic over $(K_2)_{T(P)}$. By symmetry, this is actually an equivalence. ■

In the case of quaternion algebras, we can prove similarly:

Theorem 4.6. *Let K be a global field of characteristic different from 2. Let $Q_1 = (a, b)_K$ (resp. $Q_2 = (c, d)_K$) be a quaternion algebra over K endowed with its canonical involution γ_1 (resp. γ_2). For $\alpha, \beta \in K^*$, denote by $\Omega_K^{(\alpha,\beta)}$ the set of real places at which α and β are negative. Then, the following are equivalent:*
(1) $W(Q_1, \gamma_1) \simeq W(Q_2, \gamma_2)$ as $W(K)$-modules.
(2) There exists a pair of maps (t, T) where t is a group isomorphism $t : K^*/\mathrm{Nrd}_{Q_1/K}({Q_1}^*) \simeq K^*/\mathrm{Nrd}_{Q_2/K}({Q_2}^*)$ with $t(-1) = -1$ and where T is a bijection $T : \Omega_K^{(a,b)} \to \Omega_K^{(c,d)}$ such that the quadratic form $\langle\langle a, b, x, y\rangle\rangle$ is hyperbolic over K_P if and only if the quadratic form $\langle\langle c, d, t(x), t(y)\rangle\rangle$ is hyperbolic over $K_{T(P)}$ for all $x, y \in K^*/\mathrm{Nrd}_{Q_1/K}({Q_1}^*)$ and for all $P \in \Omega_K^{(a,b)}$.

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