On close-to-star functions

Paweł Zaprawa

Abstract

For a given class A and a set D the sets $\bigcap_{f\in A} f(D)$ and $\bigcup_{f\in A} f(D)$ are called the Koebe set and the covering set for A over D, respectively. These sets are found for the class H of close-to-star functions f of the form $f(z) = \frac{z}{1-z^2}p(z)$, where $\operatorname{Re} p(z) > 0, p(0) = 1$. Analogous results concerning some other subclasses of close-to-star functions are established too.

1 Introduction

Let $\mathcal{A}$ denote the class of all functions f which are analytic in the unit disk Δ, normalized by $f(0) = f'(0) - 1 = 0$, and let A be a fixed non-empty subset of $\mathcal{A}$. In [6] the following definitions of the generalized Koebe set and the generalized covering set, both over a given set $D \subset \Delta$ containing 0, were introduced:

$$K_A(D) = \bigcap_{f\in A} f(D) \quad \text{and} \quad L_A(D) = \bigcup_{f\in A} f(D) .$$

The natural choice of D is $\Delta_r = \{z \in C : |z| < r\}$, $r \in (0, 1)$. In this case we are able to estimate the real and imaginary parts or modulus of level curves for functions in the class A.

The problem of determining such sets is usually easy when A is invariant under the rotation, i.e.

$$\forall f \in A \ \ \forall \varphi \in R \quad e^{-i\varphi} f(ze^{i\varphi}) \in A . \tag{1}$$

It is clear that if A satisfies (1) and $D = \Delta_r$, $r \in (0, 1)$, then

$$L_A(\Delta_r) = \Delta_{M(r)} , \quad \text{where} \quad M(r) = \max\{|f(z)| : f \in A, z \in \partial\Delta_r\} .$$

Received by the editors March 2008.
Communicated by F. Brackx.
2000 *Mathematics Subject Classification* : Primary 30C25, Secondary 30C45.
Key words and phrases : close-to-star functions, Koebe set, covering set.

If, additionally, each $f \in A$ is univalent in Δ_r then

$$K_A(\Delta_r) = \Delta_{m(r)}\,, \quad \text{where} \quad m(r) = \min\{|f(z)| : f \in A, z \in \partial\Delta_r\}\,.$$

Remark. If a function is not univalent then its level curves for sufficiently big $r \in (0,1)$ have "loops" directed inside the image of Δ_r under this function. This is the reason why the envelope of the level curves for functions in a given family A may be entirely included in a set $f(\Delta_r)$ for some $f \in A$, and in consequence, in the Koebe set for A over Δ_r.

The condition (1) is not fulfilled by classes of functions with real coefficients. Some results established in [6, 7, 8] were concerned with the class T of typically real functions, i.e. $T = \{f \in \mathcal{A} : \operatorname{Im} z \operatorname{Im} f(z) \geq 0, z \in \Delta\}$, and its subclass $T^{(2)}$ consisted of odd functions.

We want to turn to the class of functions for which coefficients are not real and (1) is not satisfied.

Denote by $CS^\star$ the class of functions $f \in \mathcal{A}$ for which there exist a real number $\beta \in (0,\pi)$ and a function g of the class $S^\star$ of normalized, starlike functions such that

$$\operatorname{Re}\left\{\frac{f(z)}{e^{i\beta}g(z)}\right\} \geq 0\,,\ z \in \Delta\,. \tag{2}$$

Because of the similarity to the definition of close-to-convex functions, the functions defined above are called close-to-star. Certainly, the full class $CS^\star$ satisfies (1), so in view of the inequalities

$$\frac{r(1-r)}{(1+r)^3} \leq |f(z)| \leq \frac{r(1+r)}{(1-r)^3} \quad \text{for} \quad r = |z|$$

which hold for $f \in CS^\star$ we obtain immediately that $L_{CS^\star}(\Delta_r) = \Delta_{\frac{r(1+r)}{(1-r)^3}}$. Analogous conclusion about the Koebe set is not so obvious because there are functions in $CS^\star$ which are not univalent. We have only $K_{CS^\star}(\Delta_r) = \Delta_{\frac{r(1-r)}{(1+r)^3}}$ for $r \leq r_S(CS^\star)$, where $r_S(CS^\star)$ means the radius of univalence of $CS^\star$. The number $r_S(CS^\star) = 2-\sqrt{3}$ was found by Sakaguchi in [9].

In this paper we are mainly interested in a special subclass of $CS^\star$. Similarly as in the class of close-to-convex functions, it is difficult to describe the subclass of $CS^\star$ consisting of functions with real coefficients. However, it is possible to establish other restrictions.

We take into account the class of functions f satisfying (2) with two additional assumptions: $\beta = 0$ and $g \in S^\star_R$, i.e. g is a starlike function with real coefficients. We denote the class defined in such a way by $CS^\star_R$. It it obvious that there are functions in $CS^\star_R$ which do not have real coefficients.

If $f \in CS^\star_R$ then it can be written in the form

$$f(z) = g(z)p(z)\,, \quad \text{where} \quad \operatorname{Re} p(z) > 0\,. \tag{3}$$

Due to the normalization of f and g we have $p(0) = 1$, so p is in the Caratheodory class P.

We claim that $CS^\star_R$ is not invariant under the rotation. Consider the function $f_0(z) = \frac{z(1+z)}{(1-z)^3}$ which is extremal for example in the result of Sakaguchi or in the distorsion problem, both mentioned above.

Denote by f_φ a function $e^{-i\varphi} f_0(ze^{i\varphi}) = \frac{z(1+ze^{i\varphi})}{(1-ze^{i\varphi})^3}$ for a fixed $\varphi \in R$. Then $f''_\varphi(0) = 8e^{i\varphi}$. Assuming $p(z) = 1 + b_1 z + b_2 z^2 + \cdots \in P$ and $g(z) = z + a_2 z^2 + \cdots \in S^\star_R$, we have $8e^{i\varphi} = 2(a_2 + b_1)$. Consequently, for φ such that $e^{i\varphi} \notin R$, there is $|b_1| = |4e^{i\varphi} - a_2| > 2$ because $-2 \le a_2 \le 2$. This contradicts the estimation $|b_1| \le 2$ for $p \in P$. It means $f_\varphi \notin CS^\star_R$ for suitably chosen φ. Hence (1) does not hold for $CS^\star_R$.

We concentrate our research on some subclasses of $CS^\star_R$. Choosing $g(z) = \frac{z}{(1-z)^2}$ or $g(z) = \frac{z}{1-z^2}$ in (3) we obtain the classes denoted by Q and H respectively. Therefore,

$$f \in Q \Leftrightarrow f(z) = \frac{z}{(1-z)^2} p(z)\,,\ p \in P\,, \tag{4}$$

and

$$f \in H \Leftrightarrow f(z) = \frac{z}{1-z^2} p(z)\,,\ p \in P\,. \tag{5}$$

The class Q will be helpful in determining the covering sets for $CS^\star_R$ over Δ_r. The class H is closely related to the class T of typically real functions. Recall that

$$f \in T \Leftrightarrow f(z) = \frac{z}{1-z^2} p(z)\,,\ p \in P_R\,, \tag{6}$$

where P_R means the set of all functions from P which have real coefficients. A similar generalization of the class of typically real functions was discussed by Hengartner and Schober in [3, 4].

From the above definition it follows that H is a proper superclass of T. Hence for a given set D

$$K_T(D) \supset K_H(D) \quad \text{and} \quad L_T(D) \subset L_H(D)\,.$$

In case $D = \Delta_r$ the sets $K_T(D)$ and $L_T(D)$ are known (see, [6]). We shall find analogous sets for the class H and compare these sets with with $K_T(D)$ and $L_T(D)$.

2 Basic tools

In this section we establish the general theorem which will be applied to obtain some particular results.
The following notation is useful: for a fixed $z_0 \in C$, $r \in R_+$, $\lambda \in R$ and for a given set D let $D(z_0, r)$ denote the disk $|z - z_0| < r$ and let λD denote the set $\{\lambda z : z \in D\}$. For a fixed $A \subset \mathcal{A}$ and $z \in \Delta$ let $\Omega_A(z) = \{f(z) : f \in A\}$ be the set of values for A at a point z. Since the region $\Omega_P(z)$ coincides with the disk $D(\frac{1+r^2}{1-r^2}, \frac{2r}{1-r^2})$ we conclude

Lemma 1. *If $z = re^{i\varphi} \in \Delta$, $z \neq 0$ and $g \in S^\star_R$ are fixed then for the class $A_g = \{g(z)p(z) : p \in P\}$ the region $\Omega_{A_g}(z)$ coincides with the disk $g(z) \cdot D(\frac{1+r^2}{1-r^2}, \frac{2r}{1-r^2})$.*

Each boundary point of this set corresponds to a suitable function $f_{g,\theta}(z) = g(z) \cdot \frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}$ *of the class* A_g.

By this lemma, if $g \in S_R^\star$ and $r \in (0,1)$ then

$$\max\{|f(z)| : f \in A_g, z \in \partial\Delta_r\} = \max\{|f_{g,\theta}(z)| : \theta \in R, z \in \partial\Delta_r\} = \max\{|f_{g,\theta}(re^{i\varphi})| : \theta, \varphi \in R\} \quad (7)$$

and

$$\min\{|f(z)| : f \in A_g, z \in \partial\Delta_r\} = \min\{|f_{g,\theta}(z)| : \theta \in R, z \in \partial\Delta_r\} = \min\{|f_{g,\theta}(re^{i\varphi})| : \theta, \varphi \in R\}\ . \quad (8)$$

Discuss the function

$$F(\theta,\varphi) \equiv g(re^{i\varphi}) \cdot \frac{1+re^{i(\varphi-\theta)}}{1-re^{i(\varphi-\theta)}} \quad , \quad \theta, \varphi \in R\ . \quad (9)$$

The boundaries of the Koebe set and the covering set for the class A_g over Δ_r are contained in the set $F(R \times R)$. For each interior point (θ_0, φ_0) of either the Koebe set or the covering set we have $J_F(\theta_0, \varphi_0) \neq 0$. Hence the boundaries of $K_{A_g}(\Delta_r)$ and $L_{A_g}(\Delta_r)$ are subsets of $\{F(\theta,\varphi) : J_F(\theta,\varphi) = 0\}$. This is the reason why both these sets can be derived simultaneously.

Theorem 1. *For a fixed* $g \in S_R^\star$ *and* $r \in (0,1)$ *the jacobian of F given by (9) is zero in the set* $B = \left\{(\theta,\varphi) : \tan(\varphi-\theta) = \frac{1-r^2}{1+r^2} \cdot \frac{\operatorname{Im} T_g(re^{i\varphi})}{\operatorname{Re} T_g(re^{i\varphi})}\right\}$, *where* $T_g(z) = \frac{zg'(z)}{g(z)}$.

Remark. By starlikeness of g, there is $\operatorname{Re} T_g(z) > 0$ for $z \in \Delta$.

Proof.

The equation $J_F(\theta,\varphi) = 0$ is equivalent to

$$\begin{vmatrix} \frac{\partial \operatorname{Re} F}{\partial\theta} & \frac{\partial \operatorname{Re} F}{\partial\varphi} \\ \frac{\partial \operatorname{Im} F}{\partial\theta} & \frac{\partial \operatorname{Im} F}{\partial\varphi} \end{vmatrix} (\theta,\varphi) = 0\ ,$$

and further, to

$$\operatorname{Im}\left(\frac{\partial F}{\partial\theta} \cdot \overline{\frac{\partial F}{\partial\varphi}}\right)(\theta,\varphi) = 0\ . \quad (10)$$

Substituting $re^{i\varphi} = z$, $e^{-i\theta} = \zeta$ we can write

$$\begin{aligned} \frac{\partial F}{\partial\theta} &= g(z) \cdot \frac{2z}{(1-z\zeta)^2}(-i\zeta) \\ \frac{\partial F}{\partial\varphi} &= \left[g'(z) \cdot \frac{1+z\zeta}{1-z\zeta} + g(z) \cdot \frac{2\zeta}{(1-z\zeta)^2}\right] iz\ . \end{aligned}$$

Short calculation gives that (10) holds if and only if

$$\operatorname{Im}\left(\frac{zg'(z)}{g(z)} \cdot (\overline{z\zeta} - |z\zeta|^2 z\zeta)\right) = 0\ ,$$

which in terms of θ, φ becomes

$$-r(1+r^2)\sin(\varphi-\theta)\operatorname{Re} T_g(re^{i\varphi}) + r(1-r^2)\cos(\varphi-\theta)\operatorname{Im} T_g(re^{i\varphi}) = 0\ .$$

From this equation the assertion immediately follows. ∎

3 Koebe and covering sets for H

In order to determine the Koebe set for H over Δ_r, $r \in (0,1)$ we need to know the set of univalence, or at least the radius of univalence, for H. This number was derived by Koczan in [5] and is equal to $r_S(H) = \frac{1+\sqrt{5}}{2} - \sqrt{\frac{1+\sqrt{5}}{2}} = 0.346\ldots$. The related result was established by Bogowski and Burniak in [1].

Let functions F_K and F_L be defined as follows

$$F_K : \mathbb{R} \ni \varphi \mapsto \frac{re^{i\varphi}}{1-r^2e^{2i\varphi}} \frac{1-re^{i\alpha(\varphi)}}{1+re^{i\alpha(\varphi)}}, \tag{11}$$

$$F_L : \mathbb{R} \ni \varphi \mapsto \frac{re^{i\varphi}}{1-r^2e^{2i\varphi}} \frac{1+re^{i\alpha(\varphi)}}{1-re^{i\alpha(\varphi)}}, \tag{12}$$

where

$$\alpha : \mathbb{R} \ni \varphi \mapsto \arctan\left(\frac{\sin(2\varphi)}{m+1}\right) \quad \text{and} \quad m = \left(1/r^2 + r^2\right)/2\,. \tag{13}$$

From $\alpha(-\varphi) = -\alpha(\varphi)$ and $\alpha(\varphi+\pi) = \alpha(\varphi)$ it follows that $F_K(-\varphi) = \overline{F_K(\varphi)}$ and $F_K(\varphi+\pi) = -F_K(\varphi)$. Hence, $F_K(\pi-\varphi) = -\overline{F_K(\varphi)}$. It means that if F_K takes a value w (i.e. there is $\varphi_0 \in R$ such that $F_K(\varphi_0) = w$), then F_K takes also values: $\overline{w}$, $-w$ and $-\overline{w}$. It is still true if we replace F_K by F_L. Hence we have proved

Lemma 2. *The curves $F_K([0,2\pi])$ and $F_L([0,2\pi])$ are symmetric with respect to both axes of the complex plane.*

We describe other properties of F_K and F_L in the three following lemmas.

Lemma 3. *For a fixed $r \in (0,1)$ the function $|F_K|$ decreases on $[0,\pi/2]$ and the function $\arg F_K$ increases on $[0,\pi/2]$.*

Proof.
Define a function $g(\varphi) = \log(F_K(\varphi))$, $\varphi \in [0,\pi/2]$. After rather long but not complicated calculation we obtain

$$g'(\varphi) = \frac{m+1}{(m-\cos 2\alpha(\varphi))[(m+1)^2 + (\sin 2\varphi)^2]} \times \left(m + \cos 2\varphi - \frac{2\sqrt{2(m+1)}\cos 2\varphi}{\sqrt{(m+1)^2+(\sin 2\varphi)^2}}\right)\left(-\sin 2\varphi + i\sqrt{m^2-1}\right).$$

It is easy to check that the expression $m + \cos 2\varphi - \frac{2\sqrt{2(m+1)}\cos 2\varphi}{\sqrt{(m+1)^2+(\sin 2\varphi)^2}}$ is positive for all $\varphi \in [0,\pi/2]$. This means that for all $\varphi \in (0,\pi/2)$

$$\frac{d}{d\varphi}\operatorname{Re} g(\varphi) < 0 \quad \text{and} \quad \frac{d}{d\varphi}\operatorname{Im} g(\varphi) > 0\,,$$

which proves the assertion. ■

Hence, taking into account that $\arg F_K(0) = 0$ and $\arg F_K(\pi/2) = \pi/2$, we get

$$F_K([0,2\pi]) \cap \{w : \operatorname{Re} w \geq 0, \operatorname{Im} w \geq 0\} = F_K([0,\pi/2])\,. \tag{14}$$

Lemma 4. *Let $r_1 = 0.455\ldots$ be the only solution of the equation $r^6 - r^4 - 8r^3 - r^2 + 1 = 0$ in $[0,1]$. Then*

1. *If $0 < r \le r_1$ then $|F_L|$ decreases on $[0,\pi/2]$ and $\arg F_L$ increases on $[0,\pi/2]$.*
2. *If $r_1 < r < 1$ then there exists a number $\varphi_0 \in (0,\pi/2)$ such that*
   - *(a) $|F_L|$ decreases on $[0,\varphi_0]$ and increases on $[\varphi_0,\pi/2]$,*
   - *(b) $\arg F_L$ increases on $[0,\varphi_0]$ and decreases on $[\varphi_0,\pi/2]$.*

Proof.
Analogously to the previous proof we discuss a function $h(\varphi) = \log(F_L(\varphi))$, $\varphi \in [0,\pi/2]$ and get

$$h'(\varphi) = \frac{m+1}{(m-\cos 2\alpha(\varphi))[(m+1)^2+(\sin 2\varphi)^2]} \times \left(m + \cos 2\varphi + \frac{2\sqrt{2(m+1)}\cos 2\varphi}{\sqrt{(m+1)^2+(\sin 2\varphi)^2}}\right)\left(-\sin 2\varphi + i\sqrt{m^2-1}\right) .$$

Observe that the equation $h'(\varphi) = 0$ has only one solution in $[0,\pi/2]$. Indeed, this equation is equivalent to

$$m + x + \frac{2x\sqrt{2(m+1)}}{\sqrt{(m+1)^2+1-x^2}} = 0\,, \quad \text{where} \quad x = \cos 2\varphi\,. \tag{15}$$

It obviously has no solutions for $x \in [0,1]$. For $-1 \le x < 0$ the equation (15) takes the form

$$(m-x)P_m(x) = 0\,,$$

where $P_m(x) = (m+x)^3 + 6(m+1)(m+x) - 4m(m+1)$. Consequently, the only solution of (15) is given by

$$x_0 = \sqrt[3]{2(m+1)(m+\sqrt{(m+1)^2+1})} - \frac{2(m+1)}{\sqrt[3]{2(m+1)(m+\sqrt{(m+1)^2+1})}} - m\,. \tag{16}$$

If $x_0 \in [-1,0)$ then there exists a corresponding $\varphi_0 \in [0,\pi/2]$ satisfying $\cos 2\varphi_0 = x_0$. It is possible only when the right hand side of (16) is not less then -1, i.e. if $m^3 - m^2 - m - 7 \le 0$ or equivalently $r^6 - r^4 - 8r^3 - r^2 + 1 \ge 0$.

We conclude from the above that if $0 < r \le r_1$ then for all $\varphi \in (0,\pi/2)$

$$\frac{d}{d\varphi}\operatorname{Re} h(\varphi) < 0 \quad \text{and} \quad \frac{d}{d\varphi}\operatorname{Im} h(\varphi) > 0\,.$$

Moreover, if $r_1 < r < 1$ then

$$\frac{d}{d\varphi}\operatorname{Re} h(\varphi) \begin{cases} < 0 & \text{for} \quad \varphi \in (0,\varphi_0), \\ > 0 & \text{for} \quad \varphi \in (\varphi_0,\pi/2) \end{cases}$$

and

$$\frac{d}{d\varphi}\,\mathrm{Im}\,h(\varphi)\begin{cases} > 0 & \text{for} \quad \varphi \in (0,\varphi_0), \\ < 0 & \text{for} \quad \varphi \in (\varphi_0,\pi/2)\,. \end{cases}$$

From this the assertion follows. ∎

Furthermore, from (12) it immediately follows that $\mathrm{Im}\,F_L(\varphi) = 0$ iff $\sin\varphi = 0$. Taking into account this fact and $\arg F_L(0) = 0$, $\arg F_L(\pi/2) = \pi/2$, we obtain

$$F_L([0,2\pi]) \cap \{w : \mathrm{Re}\,w \geq 0, \mathrm{Im}\,w \geq 0\} = \begin{cases} F_L([0,\pi/2]) & \text{for} \quad 0 < r \leq r_1\,, \\ F_L([0,\varphi_1]) & \text{for} \quad r_1 < r < 1\,, \end{cases} \tag{17}$$

where φ_1 is the only solution of $\mathrm{Re}\,F_L(\varphi) = 0$ in $(0,\pi/2)$.
This equation can be written in the form

$$r(1-r^2)^2\cos\varphi - 2r^2(1+r^2)\sin\varphi\sin\alpha(\varphi) = 0\,.$$

Hence $\varphi = \pi/2$ or

$$\frac{2x}{m-1} = \sqrt{\frac{(m+1)^2 + 4x(1-x)}{2(m+1)}}\,, \quad \text{where} \quad x = \sin^2\varphi\,.$$

Therefore, if $m^3 - m^2 - m - 7 \leq 0$ then

$$x_1 = \frac{(m-1)\left(m-1+\sqrt{(m-1)^2+(m+1)^2(m^3+3)}\right)}{2(m^2+3)}\,. \tag{18}$$

is the only solution of the above equation, and $x_1 \in [0,1]$. Hence there exists $\varphi_1 \in (0,\pi/2)$ such that

$$\cos\varphi_1 = x_1\,. \tag{19}$$

Lemma 5. *$F_K([0,2\pi]) \cap F_L([0,2\pi]) = \emptyset$ for a fixed $r \in (0,1)$.*

Proof.
From (11-13)

$$|F_K(\varphi)|^2 = \frac{1}{2(m-\cos 2\varphi)}\frac{M-\cos\alpha(\varphi)}{M+\cos\alpha(\varphi)} \quad \text{and} \quad |F_L(\varphi)|^2 = \frac{1}{2(m-\cos 2\varphi)}\frac{M+\cos\alpha(\varphi)}{M-\cos\alpha(\varphi)}\,,$$

where $M = \sqrt{(m+1)/2} = (1/r + r)/2$.

By Lemma 2, Lemma 3 and (14)

$$\max\left\{|F_K(\varphi)|^2 : \varphi \in [0,2\pi]\right\} = |F_K(0)|^2 = \frac{1}{2(m-1)}\frac{M-1}{M+1}\,.$$

For $\varphi \in [0,2\pi]$ we have $\cos\alpha(\varphi) \geq \frac{2M^2}{\sqrt{4M^4+1}} > \frac{2}{\sqrt{5}}$. Therefore, $\frac{M+\cos\alpha(\varphi)}{M-\cos\alpha(\varphi)} > \frac{m-\cos 2\varphi}{m+1}$ and then

$$|F_L(\varphi)|^2 > \frac{1}{2(m+1)}\,.$$

Since $\frac{1}{2(m-1)}\frac{M-1}{M+1} < \frac{1}{2(m+1)}$ we have eventually proved that

$$|F_K(\phi)| < |F_L(\psi)| \quad , \quad \text{for all} \quad \phi,\psi \in [0,2\pi]\,,$$

which completes the proof. ∎

Theorem 2. *Let $r_1 = 0.455\ldots$ be defined in Lemma 4 and φ_1 be given by (19) and (18). Then*

1. *The Koebe domain $K_H(\Delta_r)$, $r \in \left(0, \frac{1+\sqrt{5}}{2} - \sqrt{\frac{1+\sqrt{5}}{2}}\right)$ is symmetric with respect to both axes and bounded. Its boundary is the curve $F_K([0, 2\pi])$.*
2. *The covering domain $L_H(\Delta_r)$, $r \in (0, 1)$ is symmetric with respect to both axes and bounded. Its boundary in the first quadrant of the complex plane is $F_L([0, \pi/2])$ for $0 < r \le r_1$ and $F_L([0, \varphi_1])$ for $r_1 < r < 1$.*

Proof.
Let K and L denote the Koebe set and the covering set for H over Δ_r respectively. It is easily seen that $p \in P$ if and only if $p(-z) \in P$ and $p \in P$ if and only if $\overline{p(\overline{z})} \in P$. Consequently, $f \in H$ if and only if $-f(-z) \in H$ and $f \in H$ if and only if $\overline{f(\overline{z})} \in H$. From this K and L are symmetric with respect to both axes. It is a reason why we can derive the boundaries of K and L only in the first quadrant.

For $g(z) = \frac{z}{1-z^2}$ we have $T_g(z) = \frac{1+z^2}{1-z^2}$. By Theorem 1, the jacobian of F given by (9), with $z = re^{i\varphi}$, is zero if

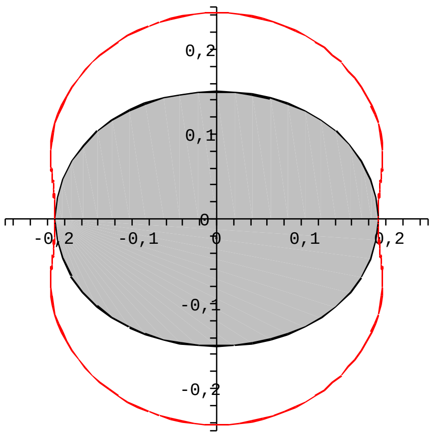
$$\tan(\varphi - \theta) = \frac{2r^2 \sin 2\varphi}{(1 + r^2)^2} . \tag{20}$$

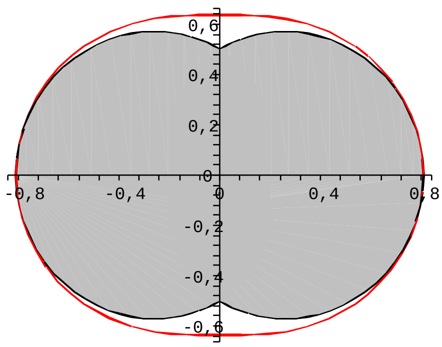
Hence ∂K and ∂L are included in the set $\{F(\theta, \varphi) : (\theta, \varphi)$ satisfy (20)$\}$, i.e. in $\{F_K(\varphi) : \varphi \in R\} \cup \{F_L(\varphi) : \varphi \in R\}$, where F_K and F_L are defined by (11) and (12). In fact, the condition $\varphi \in R$ can be replaced by $\varphi \in [0, 2\pi]$.

By Lemma 5, the closed curves $F_K([0, 2\pi])$ and $F_L([0, 2\pi])$ are disjoint. Since

$$F_K(0) = \frac{r}{(1 + r)^2} < \frac{r}{(1 - r)^2} = F_L(0)$$

we conclude that $\partial K \subset F_K([0, 2\pi])$ and $\partial L \subset F_L([0, 2\pi])$. The proof is completed by applying the radius of univalence for H and the properties of F_K and F_L described in the above lemmas. ∎





The Koebe sets and the covering sets for H and T over Δ_r, $r = \frac{1+\sqrt{5}}{2} - \sqrt{\frac{1+\sqrt{5}}{2}}$;
$K_H(\Delta_r) \subset K_T(\Delta_r)$ $L_H(\Delta_r) \supset L_T(\Delta_r)$

4 Koebe and covering sets for $H^{(2)}$

Let $H^{(2)}$ be the class of functions $f \in H$ which are odd. Similarly to (5) we have the representation

$$f \in H^{(2)} \Leftrightarrow f(z) = \frac{z}{1-z^2} p(z^2)\,,\ p \in P\,. \tag{21}$$

It is a consequence of (5) and the representation of even functions from P. Namely,

$$\{p \in P : p(-z) = p(z)\} = \{p(z^2) : p \in P\}\,.$$

Obviously, $H^{(2)}$ is closely related to $T^{(2)}$, i.e. the class of typically real odd functions. In fact, if $f \in T^{(2)}$ then $f \in H^{(2)}$.

In order to determine both the Koebe and the covering sets we need information about univalence and the set of values at z for $H^{(2)}$.

Lemma 6. $r_S(H^{(2)}) = \sqrt{2} - 1$.

Proof.
Let $f \in H^{(2)}$. Then $f(z) = \frac{z}{1-z^2} p(z^2)\,,\ p \in P$ and

$$\frac{zf'(z)}{f(z)} = \frac{1+z^2}{1-z^2} + \frac{2z^2 p'(z^2)}{p(z^2)}\,.$$

Hence

$$\mathrm{Re}\,\frac{zf'(z)}{f(z)} \geq \frac{1-r^2}{1+r^2} - \frac{4r^2}{1-r^4}\,,$$

with equality for $p_0(z) = \frac{1+z}{1-z}$ and $z = ir$. If $r \leq \sqrt{2} - 1$ then $\mathrm{Re}\,\frac{zf'(z)}{f(z)} \geq 0$, which means that f is starlike, hence univalent, in $\Delta_{\sqrt{2}-1}$. The extremal function is $f_0(z) = \frac{z(1+z^2)}{(1-z^2)^2}$ and $f_0 \in T^{(2)}$. It is known (see for example [2]) that f_0 is univalent in the set $\{z \in \Delta : |1+z^2| > 2|z|\}$, called the Goluzin lens. The greatest disk contained in this lens has the radius $\sqrt{2} - 1$. Hence the number $\sqrt{2} - 1$ cannot be increased. ■

Note that we have actually proved that $\sqrt{2} - 1$ is the radius of starlikeness for $H^{(2)}$.

The set $\{p(z^2) : p \in P\}$ coincides with the disk $D(\frac{1+r^4}{1-r^4}, \frac{2r^2}{1-r^4})$. Thus for a fixed $z = re^{i\varphi} \in \Delta, z \neq 0$ we have $\Omega_{H^{(2)}}(z) = \frac{z}{1-z^2} D(\frac{1+r^4}{1-r^4}, \frac{2r^2}{1-r^4})$. Each boundary point of this set corresponds to a suitable function $f_\theta(z) = \frac{z}{1-z^2} \cdot \frac{1+z^2 e^{-i\theta}}{1-z^2 e^{-i\theta}}$ of the class $H^{(2)}$.

Let functions G_K and G_L be defined as follows

$$G_K : \mathbb{R} \ni \varphi \mapsto \frac{re^{i\varphi}}{1-r^2 e^{2i\varphi}} \frac{1-r^2 e^{i\beta(\varphi)}}{1+r^2 e^{i\beta(\varphi)}}\,, \tag{22}$$

$$G_L : \mathbb{R} \ni \varphi \mapsto \frac{re^{i\varphi}}{1-r^2 e^{2i\varphi}} \frac{1+r^2 e^{i\beta(\varphi)}}{1-r^2 e^{i\beta(\varphi)}}\,, \tag{23}$$

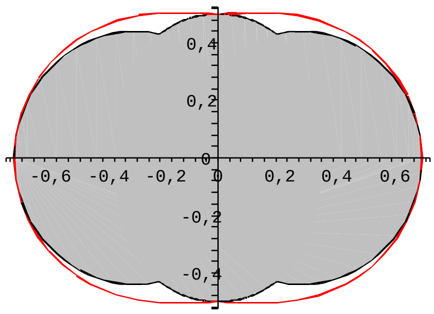
where

$$\beta : \mathbb{R} \ni \varphi \mapsto \arctan\left(\frac{\sin(2\varphi)}{m}\right) \quad \text{and} \quad m = \left(1/r^2 + r^2\right)/2\,. \tag{24}$$

In a similar way to the one used in proving Theorem 2 we can obtain

Theorem 3. *Let* $r_2 = (\sqrt{3}-1)/\sqrt{2} = 0.517\ldots$ *and* φ_2 *be the only solution of the equation* $\sin^2\varphi = \frac{m^2(m-1)}{2(m^2-2m+2)}$ *in* $(0, \pi/2)$. *Then*

1. *The Koebe domain* $K_{H^{(2)}}(\Delta_r)$, $r \in (0, \sqrt{2}-1)$ *is symmetric with respect to both axes and bounded. Its boundary is of the form* $G_K([0, 2\pi])$.
2. *The covering domain* $L_{H^{(2)}}(\Delta_r)$, $r \in (0,1)$ *is symmetric with respect to both axes and bounded. Its boundary in the first quadrant of the complex plane is* $G_L([0, \pi/2])$ *for* $0 < r \le r_2$ *and* $G_L([0, \varphi_2])$ *for* $r_2 < r < 1$.



The covering sets for $H^{(2)}$ and $T^{(2)}$ over Δ_r, $r = \sqrt{2}-1$;
$L_{H^{(2)}}(\Delta_r) \supset L_{T^{(2)}}(\Delta_r)$

5 Koebe and covering sets for Q and $CS^\star_R$

As it was said in Introduction, the radius of univalence in the class of close-to-star functions was found by Sakaguchi [9] in 1964 and is equal to $2-\sqrt{3}$. In fact, he proved that this number is the radius of starlikeness of this class. The extremal function $f(z) = \frac{z+z^2}{(1-z)^3}$ belongs to Q, and then to $CS^\star_R$. Hence $2-\sqrt{3}$ is also the radius of univalence as well as the radius of starlikeness in both classes Q and $CS^\star_R$.

By Lemma 1, for $z = re^{i\varphi} \in \Delta$, $z \neq 0$ we have $\Omega_Q(z) = \frac{z}{(1-z)^2} \cdot D(\frac{1+r^2}{1-r^2}, \frac{2r}{1-r^2})$. Each boundary point of this set corresponds to a suitable function $f_\theta(z) = \frac{z}{(1-z)^2} \cdot \frac{1+ze^{-i\theta}}{1-ze^{-i\theta}}$ of the class Q.

Let functions H_K and H_L be defined as follows

$$H_K : \mathbb{R} \ni \varphi \mapsto \frac{re^{i\varphi}}{(1-re^{i\varphi})^2} \frac{1-re^{i\gamma(\varphi)}}{1+re^{i\gamma(\varphi)}}\,, \tag{25}$$

$$H_L : \mathbb{R} \ni \varphi \mapsto \frac{re^{i\varphi}}{(1-re^{i\varphi})^2} \frac{1+re^{i\gamma(\varphi)}}{1-re^{i\gamma(\varphi)}}\,, \tag{26}$$

where

$$\gamma : \mathbb{R} \ni \varphi \mapsto \arctan\left(\frac{\sin(\varphi)}{M}\right) \quad \text{and} \quad M = (1/r + r)\,/2\,. \tag{27}$$

In a similar way to the one used in proving Theorem 2 we can obtain

Theorem 4. *Let* $r_3 = \frac{1+\sqrt{5}}{2} - \sqrt{\frac{1+\sqrt{5}}{2}} = 0.346\ldots$ *and* φ_3 *be the only solution of the equation* $M\cos\varphi - 1 = \frac{\sin^2\varphi}{\sqrt{M^2+\sin^2\varphi}}$ *in* $(0, \pi/2)$*. Then*

1. *The Koebe domain* $K_Q(\Delta_r)$*,* $r \in (0, 2-\sqrt{3})$ *is symmetric with respect to the real axis and bounded. Its boundary coincides with* $H_K([0, 2\pi])$*.*
2. *The covering domain* $L_Q(\Delta_r)$*,* $r \in (0,1)$ *is symmetric with respect to the real axis and bounded. Its boundary in the first quadrant of the complex plane is* $H_L([0, \pi/2])$ *for* $0 < r \le r_3$ *and* $H_L([0, \varphi_3])$ *for* $r_3 < r < 1$*.*

In [6] it was proved that $L_T(\Delta_r) = k_1(\Delta_r) \cup k_{-1}(\Delta_r)$, where $k_1(z) = \frac{z}{(1-z)^2}$, $k_{-1}(z) = \frac{z}{(1+z)^2}$. From the properties of covering domains it follows that $L_{S_R^\star}(\Delta_r) \subset L_T(\Delta_r)$ because $S_R^\star \subset T$. Since k_1 and k_{-1} are starlike, there is $L_{S_R^\star}(\Delta_r) = k_1(\Delta_r) \cup k_{-1}(\Delta_r)$ and consequently for each $g \in S_R^\star$:

$$g(\Delta_r) \subset k_1(\Delta_r) \cup k_{-1}(\Delta_r).$$

It leads to

$$g(\Delta_r) \cap \{w : \operatorname{Re} w \ge 0\} \subset k_1(\Delta_r) \cap \{w : \operatorname{Re} w \ge 0\}\,.$$

From this we conclude that for a fixed $a \in [0, 2\pi]$

$$\begin{aligned}\max\{|f(z)| : f(z) = g(z)p(z), g \in S_R^\star, p \in P, z \in \Delta_r, \arg f(z) = a\} = \\ \max\{|f(z)| : f(z) = k_1(z)p(z), p \in P, z \in \Delta_r, \arg f(z) = a\}\end{aligned} \tag{28}$$

and then

$$L_{S_R^\star}(\Delta_r) \cap \{w : \operatorname{Re} w \ge 0\} = L_Q(\Delta_r) \cap \{w : \operatorname{Re} w \ge 0\}\,.$$

We have proved

Theorem 5. *Let* r_3 *and* φ_3 *be the same as in Theorem 4. Then the covering domain* $L_{CS_R^\star}(\Delta_r)$*,* $r \in (0,1)$ *is symmetric with respect to both axes and bounded. Its boundary in the first quadrant of the complex plane is of the form* $H_L([0, \pi/2])$ *for* $0 < r \le r_3$ *and* $H_L([0, \varphi_3])$ *for* $r_3 < r < 1$*.*

References

[1] Bogowski F., Burniak C., On the domain of local univalence and starlikeness in a certain class of holomorphic functions, *Demonstr. Math.* 20, No.3/4, 519-536 (1987).

[2] Golusin, G., On Typically-Real Functions, (Russian), *Mat. Sb.* 27(69) (1950), 201-218.

[3] Hengartner W., Schober G., On schlicht mappings to domains convex in one direction, *Comment. Math. Helv.* 45, 303-314 (1970).

[4] Hengartner W., Schober G., A remark on level curves for domains convex in one direction, *Appl. Analysis* 3, 101-106 (1973).

[5] Koczan, L., Promienie jednolistności w pewnych klasach wypukłych funkcji holomorficznych, (Polish) *Prace Inst. Mat., Fiz. i Chemii Politechniki Lubelskiej Sect.C* No.1, 25-28 (1979). Vol. LII.2, 95-101 (1998).

[6] Koczan, L., Zaprawa, P., On covering problems in the class of typically real functions, *Ann. Univ. Mariae Curie Sklodowska Sect. A* Vol. LIX, 51-65 (2005).

[7] Koczan, L., Zaprawa, P., Koebe domains for the class of typically real odd functions, *Tr. Petrozavodsk. Gos. Univ., Ser. Mat.* 12, 51-70 (2005).

[8] Koczan, L., Zaprawa, P., Covering domains for the class of typically real odd functions, *Ann.Univ.Mariae Curie Sklodowska Sect. A* Vol. LX, 23-30 (2006).

[9] Sakaguchi, K., A variational method for functions with positive real part, *J. Math. Soc. Japan* 16, 287-297 (1964).

Department of Applied Mathematics,
Lublin University of Technology,
Nadbystrzycka 38D, 20-618 Lublin, Poland
email : p.zaprawa@pollub.pl